Finite Element Heterogeneous Multiscale Method for the Wave Equation: Long-Time Effects

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Abstract

For limited time the propagation of waves in a highly oscillatory medium is well-described by the non-dispersive homogenized wave equation. With increasing time, however, the true solution deviates from the classical homogenization limit, as a large secondary wave train develops unexpectedly. Here, we propose a new finite element heterogeneous multiscale method (FE-HMM), which captures not only the short-time macroscale behavior of the wave field but also those secondary long-time dispersive effects.

1 Long-Time Wave Propagation

Let $\Omega \subset \mathbb{R}^d$ be a domain and $T > 0$. We consider the wave equation

$$\begin{cases}
\partial_t u^\varepsilon - \nabla \cdot (a^\varepsilon \nabla u^\varepsilon) = F & \text{in } \Omega \times (0,T), \\
\partial_t u^\varepsilon(x,0) = f(x) & \text{in } \Omega, \\
\partial_t u^\varepsilon(x,0) = g(x) & \text{in } \Omega,
\end{cases}$$

(1)

where $a^\varepsilon(x) \in (L^\infty(\Omega))^{d \times d}$ is symmetric, uniformly elliptic, and bounded. Here $\varepsilon > 0$ represents a small scale in the problem, which characterizes the multiscale nature of the tensor $a^\varepsilon(x)$. We set either homogeneous Dirichlet or periodic boundary conditions to uniquely determine the solution for every $\varepsilon > 0$.

1.1 Classical homogenization

According to classical homogenization theory, $u^\varepsilon$ converges to the solution $u^0$ of the “homogenized” wave equation as $\varepsilon \to 0$,

$$\partial_t u^0 - \nabla \cdot (a^0 \nabla u^0) = F,$$

where the homogenized tensor (or squared velocity field) $a^0$ can only rarely be computed explicitly. Thus, $u^0$ approximates $u^\varepsilon$ but only for short times. For longer times $T \sim \varepsilon^{-2}$, the homogenized solution becomes increasingly inadequate, since it neglects microscopic dispersive effects that accumulate over time, as shown in Figure 1. Here we consider (1) in $\Omega = (-1,1)$ with periodic boundary conditions, let $u(x,0)$ be a Gaussian pulse with zero initial velocity

and set

$$a^\varepsilon = \sqrt{2} + \sin\left(\frac{2\pi x}{\varepsilon}\right) \quad \text{with } \varepsilon = \frac{1}{50}.$$  

(2)

The reference solution of (1)–(2) corresponds to a direct numerical simulation (DNS), where the microscale is fully resolved. After one revolution ($T = 2$), the homogenized and the DNS solution coincide. After fifty revolutions ($T = 100$), however, the DNS displays dispersive effects, which the homogenized solution fails to capture.

1.2 Effective dispersive equation

Various formal asymptotic arguments were derived to elucidate that peculiar inherently dispersive long-time behavior of waves propagating through a strongly heterogeneous periodic medium [1]. An effective equation that captures those dispersive effects was recently derived in [2] for the one-dimensional case when $a^\varepsilon$ is $\varepsilon$-periodic:

$$\partial_t (u^{\text{eff}} - \varepsilon^2 b \partial_{xx} u^{\text{eff}}) - a^0 \partial_{xx} u^{\text{eff}} = F.$$  

(3)

Again, $a^0$ denotes the homogenized effective coefficient from classical homogenization theory and $b > 0$. As shown in Figure 1, $u^\varepsilon$ and $u^{\text{eff}}$ essentially coincide both at early and later times.

2 FE Heterogeneous Multiscale Method

In [3], the FE-HMM for elliptic [4] was extended to the time dependent wave equation. It was shown to...
converge to \( u^0 \) at finite times, yet it failed to capture long-time dispersive effects in the true solution. To incorporate those dispersive effects, we not only need an effective bilinear form but we add a correction to the \( L^2 \) inner product, akin to the weak formulation of (3). Similarly to the computation of the bilinear form, the correction relies on numerical solutions of micro problems on sampling domains \( K_\delta \) of size \( \delta \) proportional to \( \varepsilon \). An alternative HMM scheme, based on the finite difference approximation of an effective flux, was proposed in [5].

We now give a description of the algorithm: First, we generate a macro triangulation \( T_H \) and choose an appropriate macro FE space \( S(\Omega, T_H) \). By macro we mean that \( H \gg \varepsilon \) is allowed. Within each macro element \( K \in T_H \) we choose a quadrature formula \( \{ x_{K,j}, \omega_{K,j} \} \). The FE-HMM solution \( u_H \) is given by the following variational problem:

\[
\text{Find } u_H : [0, T] \to S(\Omega, T_H) \text{ such that } \begin{cases} (\partial_t u_H, v_H)_Q + B_H(u_H, v_H) = (F, v_H) \\ \text{for all } v_H \in S(\Omega, T_H) \text{ and, } \\ u_H(0) = f_H, \quad \partial_t u_H(0) = g_H \text{ in } \Omega, \end{cases} \tag{4}
\]

where the initial data \( f_H \) and \( g_H \) are suitable approximations of \( f \) and \( g \) in \( S(\Omega, T_H) \). The effective bilinear form \( B_H \) and \((\cdot, \cdot)_Q \) are defined as follows:

\[
B_H(v_H, w_H) = \sum_{K,j} \omega_{K,j} \int_{K_\delta} a^\varepsilon(x) \nabla v_h(x) \cdot \nabla w_h(x) dx,
\]

and

\[
(v_H, w_H)_Q = (v_H, w_H)+ \sum_{K,j} \omega_{K,j} \int_{K_\delta} (v_h(x) - v_{H,\text{lin}}(x))(w_h(x) - w_{H,\text{lin}}(x)) dx.
\]

In the above, the micro solution \( v_h \) (resp. \( w_h \)) is given by

\[
\text{Find } v_h \text{ such that } (v_h - v_{H,\text{lin}}(x)) \in S(K_\delta, T_h) \text{ and } \int_{K_\delta} a^\varepsilon(x) \nabla v_h(x) \cdot \nabla z_h(x) dx = 0,
\]

for all \( z_h \in S(K_\delta, T_h) \).

Here \( S(K_\delta, T_h) \) is a micro FE space on the sampling domain \( K_\delta \) with micro triangulation \( T_h \), and \( v_{H,\text{lin}} \) denotes the linearization of \( v_H \) at the quadrature point \( x_{K,j} \). Since \( B_H \) is elliptic and bounded and \( (\cdot, \cdot)_Q \) is a true inner product, the FE-HMM is well-defined for all \( H, h > 0 \). It can be shown that the correction of the \( L^2 \) inner product is of order \( \varepsilon^2 \) in agreement with (3).

3 Numerical Experiments

We again apply our FE-HMM, defined in (4), to (1)–(2) as in Figure 1. We use cubic FE at the macro- and the micro-scale, with mesh sizes \( H = 1/75 \) and \( h = \varepsilon/20 = 1/1000 \). Note that linear or quadratic finite elements could also be used. For time-stepping we use a standard Leap-Frog scheme, with \( \Delta t = H/10 \).

As shown in Figure 2, the new FE-HMM succeeds in capturing, the long-time effects in the true solution. In contrast, the solution of the FE-HMM of [3] without correction is unable to capture those dispersive effects, since this solution was proven to converge to the homogenized solution, \( u^0 \), as \( \varepsilon \to 0 \).

References


