Nonrelativistic limit of the Nonlinear Klein-Gordon equation: dynamics over long times

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Outline

1 Summary

2 Formal theory and general ideas

3 NLKG

4 Hamiltonian approach

5 Dynamics

6 Longer time estimates: $M = R^d$

7 Longer time estimates: $M = [0, \pi]$
The problem: Nonrelativistic limit of the Klein Gordon equation

\[ \frac{1}{c^2} u_{tt} - \Delta u + c^2 u = -\lambda u^3 , \quad x \in M , \quad c \to \infty \]

\( M \) compact manifold or \( \mathbb{R}^d \).

Heuristic Discussion

- Formal limit
- Formal limit, how to estimate the error
Summary

• **The problem:** Nonrelativistic limit of the Klein Gordon equation

\[ \frac{1}{c^2} u_{tt} - \Delta u + c^2 u = -\lambda u^3, \quad x \in M, \quad c \to \infty \]

where \( M \) is a compact manifold or \( \mathbb{R}^d \).

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  - Formal limit
  - Formal limit, how to estimate the error

• **Rigorous results**
  - Masmoudi and Nakanishi
  - Faou and Schratz
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- Hamiltonian approach
  - Hamiltonian formulation
  - Normal Form: standard approach/Galerkin Averaging

- Dynamics
  - Approximate solutions and short time estimates
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Longer time estimates
- On \( \mathbb{R}^3 \) by dispersive estimates
- On \([0, \pi]\) a preliminary result by extension of BNF for semilinear PDEs
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The operator and the complex variables $v := \frac{\dot{u}}{c^2}$

\[
\langle \nabla \rangle_c := (c^2 - \Delta)^{1/2} = c - \frac{1}{2c} \Delta + O\left(\frac{\Delta^2}{c^3}\right)
\]
The operator and the complex variables $v := \frac{\dot{u}}{c^2}$

$$\langle \nabla \rangle_c := (c^2 - \Delta)^{1/2} = c - \frac{1}{2c} \Delta + O\left(\frac{\Delta^2}{c^3}\right)$$

$$\psi = \frac{1}{\sqrt{2}} \left[ \left(\frac{\langle \nabla \rangle_c}{c}\right)^{1/2} u - i \left(\frac{c}{\langle \nabla \rangle_c}\right)^{1/2} v \right] = \frac{u - iv}{\sqrt{2}} + h.o.t.$$  

Structure of the equations:

$$\frac{\dot{\psi}}{i} = c \langle \nabla \rangle_c \psi + N(\psi) \quad (1)$$

with

$$N(\psi) := +\lambda \left(\frac{c}{\langle \nabla \rangle_c}\right)^{1/2} \left(\frac{\psi + \psi^*}{\sqrt{2}}\right)^3$$

nonlinear term.
A model problem

\[ \dot{\psi} = ic \langle \nabla \rangle_c \psi + \tilde{N}(\psi) , \quad \tilde{N}(\psi) := \lambda \left( \frac{c}{\langle \nabla \rangle_c} \right)^{1/2} |\psi|^3 \]
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Approximate:

\[ -i\dot{\psi}_a = c^2 \psi_a - \frac{1}{2} \Delta \psi_a + \lambda |\psi_a|^3 \quad (2) \]

Gauge transform: \( \psi_a(t) := e^{ic^2t} \phi(t) \) then

\[ -i\dot{\phi} = -\frac{1}{2} \Delta \phi + \lambda |\phi|^3 \quad (3) \]
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A solution of (2) solves actually

\[ \dot{\psi}_a = ic \langle \nabla \rangle_c \psi_a + \tilde{N}(\psi_a) + \frac{1}{c^2} R(t), \]

\[ \frac{R(t)}{c^2} = (ic^2 \psi_a - i \frac{1}{2} \Delta \psi_a + ic \langle \nabla \rangle_c \psi_a) + \left( \lambda |\psi_a|^3 - \tilde{N}(\psi_a) \right). \]
Estimates of the error

- Equation for the error
  \[
  \delta := \psi - \psi_a, \quad \dot{\delta} = i c \langle \nabla \rangle_c \delta + \tilde{N}(\psi_a + \delta) - \tilde{N}(\psi_a) - \frac{R(t)}{c^2}
  \]

- Duhamel
  \[
  \delta(t) = \int_0^t e^{i c(t-s) \langle \nabla \rangle_c} [d\tilde{N}(\psi_a(s))] \delta(s) ds + O(\delta^2) + O\left(\frac{1}{c^2}\right)
  \]

- Solution \[\|\delta(t)\| \leq c^{-2} \text{ for } |t| \lesssim 1\]
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- How to improve times?
Estimates of the error

**Equation for the error**

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**Solution**

\[ ||\delta(t)|| \leq c^{-2} \text{ for } |t| \lesssim 1 \]

**How to improve times?**

**One should improve the trivial estimate of the solution of**

\[ \dot{\delta} = i c \langle \nabla \rangle_c \delta + [d \tilde{N}(\psi_a(s))] \delta \]
Estimates of the error

- Equation for the error

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- solution \( \|\delta(t)\| \leq c^{-2} \) for \( |t| \lesssim 1 \)

- How to improve times?

- One should improve the trivial estimate of the solution of

\[ \dot{\delta} = ic \langle \nabla \rangle_c \delta + [d\tilde{N}(\psi_a(s))] \delta \quad \Rightarrow \quad \|\delta(t)\| \leq e^{at} \delta_0 \]
Dispersive estimates

In $\mathbb{R}^3$

$$\dot{\psi} = i\Delta \psi \implies \psi(t) = \frac{c}{t^{3/2}} \int_{\mathbb{R}^3} e^{\frac{|x-y|^2}{it}} \psi_0(y) dy$$
Dispersive estimates

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2. Strichartz estimates:

\[ \|e^{it\Delta} \psi_0\|_{L^2_t L^6_x} \lesssim \|\psi_0\|_{L^2_x} \]

3. Stable under perturbation: They hold for

\[ -i\dot{\psi} = -\Delta \psi + \epsilon A(t) \psi \]

for $A$ in suitable classes.
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Rigorous results (nonlinear case), \( M = \mathbb{R}^d \)

- With loss of regularity, Tsutsumi 1984, Najman 1990,
- In energy space: series of papers by Masmoudi, Machimara, Nakanishi, Ozawa, around 2000.
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**Theorem (Masmoudi Nakanishi 2002)**

Let $\psi_c^0 \to \phi_0$ in $H^{1/2}$. Consider the solution $\phi(t)$ of NLS with $\phi(0) = \phi^0$, and let $T^*$ be its maximal existence time. Let $\psi_c(t)$ be the solution of NLKG and let $T_{c}^*$ be its maximal existence time, then

$$\liminf_{c \to \infty} T_{c}^* \geq T^*$$

and

$$\psi_c - e^{ic^2 t} \phi \to 0 \quad \text{in} \quad C([0, T^*), H^{1/2})$$

Use of adapted Strichartz estimates in $c$ dependent Besov spaces.
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Use of adapted Strichartz estimates in $c$ dependent Besov spaces.
- Realistic models Mauser and collaborators around 2002
Theorem (Faou Schratz 2014)

Fix $T < T^*$, then for any $s$ there exists $s_1$, s.t. if

$$\| \psi_0^c - \phi^0 \|_{H^{s+s_1}} \lesssim c^{-1},$$

then

$$\| \psi_c - \phi \|_{C([0,T];H^s)} \lesssim c^{-1}$$
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Proof through modulated Fourier expansion (variant of averaging/normal form theory).
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\]

Proof through modulated Fourier expansion (variant of averaging/normal form theory).
Actually the result is stronger: Error=\( O(c^{-r}) \), but not longer times!
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The Hamiltonian

\[ H = \int_M c \left| \langle \nabla \rangle_{\sqrt{c}}^{1/2} \psi \right|^2 + \frac{\lambda}{4} \left[ \left( \frac{c}{\langle \nabla \rangle_{\sqrt{c}}} \right)^{1/2} \frac{\psi + \bar{\psi}}{\sqrt{2}} \right]^4 dX \]
The Hamiltonian

\[ H = \int_M \left( c \left| \langle \nabla \rangle_c^{1/2} \psi \right|^2 + \frac{\lambda}{4} \left[ \left( \frac{c}{\langle \nabla \rangle_c} \right)^{1/2} \frac{\psi + \bar{\psi}}{\sqrt{2}} \right]^4 \right) \, dx \]

Expansion

\[ H = \int_M c^2 |\psi|^2 \, dx + \int_M \left( \frac{1}{2} |\nabla \psi|^2 + \frac{\lambda}{2} (\psi + \bar{\psi})^4 \right) \, dx \]

+ singular h.o.t.

Rescale time: \( \tau := c^2 t \)

\[ H = \int_M |\psi|^2 \, dx + \frac{1}{c^2} \int_M \left( \frac{1}{2} |\nabla \psi|^2 + \frac{\lambda}{2} (\psi + \bar{\psi})^4 \right) \, dx + h.o.t. \]
The Hamiltonian

\[ H = \int_M c \left| \langle \nabla \rangle_c^{1/2} \psi \right|^2 + \frac{\lambda}{4} \left[ \left( \frac{c}{\langle \nabla \rangle_c} \right)^{1/2} \psi + \bar{\psi} \right]^4 \, dx \]

Expansion

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Structure: \( \epsilon := c^{-2} \)

\[ H \sim h_0 + \sum_{k \geq 1} \epsilon^k h_k + \sum_{k \geq 1} \epsilon^k F_k \, . \]

with \( h_0 \) generating a periodic flow \( \Phi^t \).
Formal theory: \( \forall r \geq 0, \exists T(r) \) formal canonical transformation s.t.

\[
H \circ T^{(r)} = h_0 + \epsilon (h_1 + \langle F_1 \rangle) + \sum_{k=2}^{r} \epsilon^r Z_r + O(\epsilon^{r+1}) ,
\]

with

\[
\{ h_0; Z_r \} = 0 , \quad \langle F_1 \rangle (\psi) := \frac{1}{2\pi} \int_0^{2\pi} F_1(\Phi^t \psi) dt .
\]
Formal theory: \( \forall r \geq 0, \exists \mathcal{T}^{(r)} \) formal canonical transformation s.t.

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\]

the method is constructive, e.g. (for NLKG): \( h_0 + \epsilon(h_1 + \langle F_1 \rangle) \equiv \text{NLS}, \)

\[
\epsilon^2(h_2 + Z_2) = \frac{1}{c^4} \int \left[ \frac{1}{8} \psi \cdot \Delta^2 \psi \psi - 17 \lambda^2 |\psi|^6 + \frac{3}{2} \lambda |\psi|^2 (\bar{\psi} \Delta \psi + \psi, \Delta \bar{\psi}) \right] dx
\]
Rigorous theory: Galerkin averaging

- Topology: \( H^s(M), \ s \) large. Assume \( X_{h_j} \in C^\infty(H^{s+2j}, H^s) \), \( X_{F_j} \in C^\infty(H^{s+2(j-1)}, H^s) \), (true for NLKG)
- Cutoff operator \( \Pi_N := \text{spectral projector of } -\Delta \), on eigenvalues smaller then \( N^2 \)
- Strategy:
  - cutoff the Hamiltonian: \( H_N := H \circ \Pi_N \): the error is small as an operator loosing many derivatives
  - Put in normal form \( H_N \), choose \( N \) and the loss of smoothness in a suitable way.
A BNF theorem

Let \( B_s(R) = \text{Ball of radius } R \) and center 0 in \( H^s(M) \).

**Theorem**

Consider the Klein Gordon equation; fix \( r \geq 1 \) and \( s \gg 1 \). If \( \epsilon \equiv c^{-2} \ll 1 \), then there exists \( \mathcal{T}^{(r)} : B_{4r^2 + s}(1) \to B_{4r^2 + s}(2) \) analytic, s.t.

\[
H \circ \mathcal{T}^{(r)} = h_0 + \epsilon (h_1 + \langle F_1 \rangle) + \sum_{k=2}^{r} \epsilon^k Z_r + \epsilon^{r+1/2} \mathcal{R}
\]

Furthermore, on \( B_{s+4r^2}(1) \) one has

\[
\|X_{Z_r}\|_s, \|X_{\mathcal{R}}\|_s \leq C
\]

What about the dynamics?
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Simplified system: \( H_{\text{simp}} := h_0 + \epsilon (h_1 + \langle F_1 \rangle) + \sum_{k=2}^{r} \epsilon^r Z_r \)
Dynamics: general (reduction to Gronwall lemma)

- Simplified system: $H_{\text{simp}} := h_0 + \epsilon(h_1 + \langle F_1 \rangle) + \sum_{k=2}^{r} \epsilon^r Z_r$
- Let $\psi_s(\tau)$ be a solution of
  \[ \dot{\psi}_s = X_{H_{\text{simp}}}(\psi_s) = \text{NLS} + \text{h.o. normalised corrections} \]

Then $\psi_a(t) := T^{(r)}(\psi_s(c^2 t))$ solves

\[ \dot{\psi}_a = ic \langle \nabla \rangle_c \psi_a + N(\psi_a) - \frac{1}{c^{2r}} T^{(r)*} R(\psi_a) = \text{NLKG} + \mathcal{O}(\epsilon^{-2r}) \]
Dynamics: general (reduction to Gronwall lemma)

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- the remainder is evaluated on the **approximate solution**.
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- the remainder is evaluated on the approximate solution.
- If $\psi$ is a solution of NLKG, then the error $\delta := \psi - \psi_a$ fulfills

\[ \dot{\delta} = ic \langle \nabla \rangle_c \delta + [N(\psi_a + \delta) - N(\psi_a)] + \frac{1}{c^{2r}} \mathcal{T}^{(r)*} R(\psi_a(t)) \]

or (rescaling time to $t' = ct$)

\[ \delta(t) = \frac{1}{c} \int_0^t e^{i(t-s)\langle \nabla \rangle_c} dN(\psi_a(s)) \delta(s) ds + \mathcal{O}(\delta^2) + \mathcal{O}(\frac{1}{c^{2r+1}}) \]
Corollary

Fix $s$, assume $\|\psi_0\|_{4r^2+s} \leq 1/2$ and

$$\exists T \text{ s.t. } \|\psi_s(t)\|_{4r^2+s} \leq 1, \quad |t| \leq T$$

(4)

(non rescaled time) then

$$\|\delta(t)\|_s \leq c^{-2r}, \quad |t| \leq T.$$
Gronwall Lemma

**Corollary**

Fix $s$, assume $\|\psi_0\|_{4r^2+s} \leq 1/2$ and

$$\exists T \text{ s.t. } \|\psi_s(t)\|_{4r^2+s} \leq 1 , \quad |t| \leq T$$

(non rescaled time) then

$$\|\delta(t)\|_s \leq c^{-2r} , \quad |t| \leq T .$$

- This is essentially Faou Schratz result on general $M$. 

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$$\exists T \text{ s.t. } \|\psi_s(t)\|_{4r^2+s} \leq 1, \quad |t| \leq T$$  \hspace{1cm} (4)

(non rescaled time) then

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- This is essentially Faou Schratz result on general $M$.
- Problem: the correction of second order become effective after a time $\mathcal{O}(c)$, so that they are here invisible.
- In the focusing case, the second order correction are defocusing, so they are expected to change qualitatively the dynamics of the normalized equation. What about the original system?
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Case $M = \mathbb{R}^3$: use Strichartz estimates to estimate sol of

$$\delta(t) = \frac{1}{c} \int_0^t e^{i(t-s)\langle \nabla \rangle_c} dN(\psi_a(s))\delta(s)ds :$$

Strichartz estimates typically persist under perturbations! (In suitable classes.)
Strichartz estimates

- Standard estimates: let \((p, q)\) be a Schrödinger admissible pair, then

\[
\left\| e^{i\langle \nabla \rangle t} \psi \right\|_{L_t^p L_x^q} \lesssim \left\| \langle \nabla \rangle^{\frac{1}{p} - \frac{1}{q} + \frac{1}{2}} \psi \right\|_{L_x^2}, \quad \langle \nabla \rangle := \sqrt{1 - \Delta}
\]

\[
\left\| e^{i\Delta t} \psi \right\|_{L_t^p L_x^q} \lesssim \left\| \psi \right\|_{L_x^2}
\]

- Difficulty: in NLKG there is a change of smoothness, not in the limit equation NLS. No trivial uniform estimate is possible!
Strichartz estimates

- Standard estimates: let \((p, q)\) be a Schrödinger admissible pair, then

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\| e^{i\langle \nabla \rangle t} \psi \|_{L^p_t L^q_x} \lesssim \| \langle \nabla \rangle \frac{1}{p} - \frac{1}{q} + \frac{1}{2} \psi \|_{L^2_x}, \quad \langle \nabla \rangle := \sqrt{1 - \Delta}
\]

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\| e^{i\Delta t} \psi \|_{L^p_t L^q_x} \lesssim \| \psi \|_{L^2_x}
\]

- Difficulty: in NLKG there is a change of smoothness, not in the limit equation NLS. No trivial uniform estimate is possible!

- Solution: scaling from the standard estimate by D’Ancona-Fanelli.

Lemma

For any Schrödinger admissible pair \((p, q)\) and any \(k \geq 0\), one has

\[
\| e^{i\langle \nabla \rangle_c t} \psi \|_{L^p_t W^{k,q}_x} \lesssim c^{\frac{1}{q} - \frac{1}{2}} \| \langle \nabla \rangle_c \frac{1}{p} - \frac{1}{q} + \frac{1}{2} \psi \|_{H^k}.
\] (5)
Lemma

Take an initial datum such that the solution $\psi_s$ of the normalized equation exists for all times and has the structure

$$
\psi_s(x, t) = \psi_{rad}(x, t) + \sum_{l=1}^{N} \eta_l(x - v_l t)
$$

with some $\eta_l \in S$, $v_l \in \mathbb{R}^3$ and $\psi_{rad} \in L^p_t W_x^{k,q}$ with $(p, q)$ any Schrödinger admissible pair, then the flow map of

$$
i \langle \nabla \rangle_c + \frac{1}{c} dN(\psi_s(t))
$$

fulfills the estimate (5).
Main dispersive tool

Lemma

Take an initial datum such that the solution $\psi_s$ of the normalized equation exists for all times and has the structure

$$\psi_s(x, t) = \psi_{rad}(x, t) + \sum_{l=1}^{N} \eta_l(x - v_l t)$$  \hspace{1cm} (6)

with some $\eta_l \in \mathcal{S}$, $v_l \in \mathbb{R}^3$ and $\psi_{rad} \in L^p_t W^{k,q}_x$ with $(p, q)$ any Schrödinger admissible pair, then the flow map of

$$i \langle \nabla \rangle_c + \frac{1}{c} dN(\psi_s(t))$$

fulfills the estimate (5).

Use this lemma to estimate

$$\left\| \frac{\langle \nabla \rangle_c^{1/2}}{c^{1/2}} \delta \right\|_{L^\infty_t H^k_x}$$

$\langle \nabla \rangle_c = (c^2 - \Delta)^{1/2}$
A theorem

Theorem

Fix $k \geq 0$ and a large $r$, then there exists a large $k_*$, with the following property: take an initial datum such that the solution $\psi_s$ of the normalized equation exists for all times and has the structure (6) and in particular $\psi_{rad} \in L^p_t W^{k_* q}$; denote $\psi_a(t) := \mathcal{T}^{(r)}(e^{ic^2 t} \psi_s(t))$. Let $\psi(t)$ be the sol of NLKG with the corresponding initial datum, then one has

$$\|\psi_a(t) - \psi(t)\|_{H^k} \lesssim \frac{1}{c}, \quad |t| \lesssim c^r, \quad c \gg 1$$
A theorem

Theorem

Fix $k \geq 0$ and a large $r$, then there exists a large $k_*$, with the following property: take an initial datum such that the solution $\psi_s$ of the normalized equation exists for all times and has the structure (6) and in particular $\psi_{rad} \in L_t^p W_x^{k*,q}$; denote $\psi_a(t) := T^{(r)}(e^{ic^2t} \psi_s(t))$. Let $\psi(t)$ be the sol of NLKG with the corresponding initial datum, then one has

$$\|\psi_a(t) - \psi(t)\|_{H^k} \lesssim \frac{1}{c}, \quad |t| \lesssim c^r, \quad c \gg 1$$

- Difficulty: do there exist solutions with the above property? Soliton resolution conjecture! Something is known, but not too much.
- The Theorem says that in this case the soliton dynamics and the dynamics of the radiation is well described by the approximate equation.
Summary

Formal theory and general ideas

NLKG

Hamiltonian approach

Dynamics

Longer time estimates: $M = \mathbb{R}^d$

Longer time estimates: $M = [0, \pi]$
A modified problem

\[
\frac{1}{c^2} u_{tt} - u_{xx} + V \ast u + c^2 u = \lambda u.
\]

\[V(x) = \sum_{k > 0} \frac{V_k}{k^2} \cos(kx), \ V_k \in [-1/2, 1/2] \text{ iid} \ . \ V \text{ corresponding probability space endowed by the product measure.} \]
Compact theorem

**Theorem**

There exists $\mathcal{A} \subset (\mathcal{V} \times \mathbb{R})$ with

$$|\mathcal{A} \cap ([N, N + 1] \times \mathcal{V})| = 1$$

s.t. the following holds true. Fix $\alpha > 0$ and $r \gg 1$ and take $(c, V) \in \mathcal{A}$, then there exists $s_*$ with the property that $\forall s > s_*$ there exist $c_*, K_1, K_2, K_3$, s.t. for $c > c_*$

$$\left\|\psi_0\right\| \leq \frac{K_1}{c^{\alpha}} \implies \left\|\psi(t)\right\| \leq \frac{2K_1}{c^{\alpha}} , \text{ for } |t| \leq K_2c^r .$$

For the same times one has

$$\sum_k k^{2s} \left| \left| \psi_k(t) \right|^2 - \left| \psi_k(0) \right|^2 \right| \leq \frac{K_3}{c^{4\alpha}} .$$

- Small initial data
- Longer time description, but the limit equation is not identified.
THANK YOU