Discrete Mechanics and Optimal Control: Structure preserving integration for the optimal control of mechanical systems

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Motivation

A Optimal control problem

\[ \min_{x,u} J(x, u) \quad s.t. \quad \dot{x}(t) = f(x(t), u(t)), \quad x(0) = x^0 \]

- well established numerical methods
  - indirect: solve necessary optimality conditions
  - direct: discretize problem *in a clever way* and solve optimization problem
Motivation

A Optimal control problem
\[ \min_{x,u} J(x,u) \quad s.t. \quad \dot{x}(t) = f(x(t), u(t)), \quad x(0) = x^0 \]

- well established numerical methods
  indirect: solve necessary optimality conditions
  direct: discretize problem \textit{in a clever way} and solve optimization problem

B Structure preserving integration
approximate solution \( x(t) \) of \( \dot{x}(t) = f(x(t)) \) via a discrete solution \( \{x_k\}_{k=0}^N \) with e.g. \( x_{k+1} = \Psi_h(x_k) \)

- preserve conservation properties in discrete solution
- geometric integrators, symplectic integrators, here: variational integrators
Motivation

$A + B$ optimal control problem

$+ \text{ structure preserving integration}$

$= \textbf{?}$
Motivation

\[ A + B \text{ optimal control problem} + \text{structure preserving integration} = ? \]

?  

▶ What is preservation in presence of control \( u(t) \)?  
▶ Which properties can be derived from integration theory?  
▶ What do we gain for optimal control problems?
Outline

- Optimal control: problem formulation
- Discrete Mechanics and Optimal Control
- Preservation properties and approximation error briefly:
  - Variational approach to multirate integration

related works:

Betsch, Bock, Bonnans, Diehl, Gerdts, Hager, Kobilarov, Leok, Leyendecker, Marsden, Ortiz, von Stryk, West, Chyba, Hairer, Vilmart, ...
Optimal Control Problem

\[ \min_{x(\cdot), u(\cdot), (T)} J(x, u) = \int_0^T C(x(t), u(t)) \, dt + \Phi(x(T)) \]

subject to

system equations \[ \dot{x}(t) = f(x(t), u(t)), \]
initial value \[ x(0) = x_0, \]
path constraints \[ 0 \leq h(x(t), u(t)), \]
final point constraint \[ 0 = r(x(T)). \]

\( x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}, J : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}, C : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}, \)
\( \Phi : \mathbb{R}^{n_x} \to \mathbb{R}, f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}, h : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_h}, r : \mathbb{R}^{n_x} \to \mathbb{R}^{n_r}. \)
Optimal Control Problem

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\min_{x(\cdot), u(\cdot), (T)} \ J(x, u) = \int_0^T C(x(t), u(t)) \, dt + \Phi(x(T))
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\( x \in \mathbb{R}^{n_x}, \ u \in \mathbb{R}^{n_u}, \ J : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}, \ C : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}, \)

\( \Phi : \mathbb{R}^{n_x} \to \mathbb{R}, \ f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}, \ h : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_h}, \ r : \mathbb{R}^{n_x} \to \mathbb{R}^{n_r} \).

Necessary optimality conditions:

Pontryagin maximum principle
Theorem (Pontryagin Maximum Principle)

Let \((x^*, u^*)\) be an optimal solution. Then, there exists a function \(\lambda : [0, T] \to \mathbb{R}^{n_x}\) and a vector \(\alpha \in \mathbb{R}^{n_r}\) such that

\[
\mathcal{H}(x^*(t), u^*(t), \lambda(t)) = \max_{u(t) \in U} \mathcal{H}(x(t), u(t), \lambda(t)) \quad \forall t \in [0, T],
\]

\[
\dot{x}^*(t) = \nabla_{x} \mathcal{H}(x^*(t), u^*(t), \lambda(t)), \quad x^*(0) = x_0,
\]

\[
\dot{\lambda}(t) = -\nabla_{x} \mathcal{H}(x^*(t), u^*(t), \lambda(t)),
\]

\[
\lambda(T) = \nabla_{x} \Phi(x^*(T)) - \nabla_{x} r(x^*(T)) \alpha
\]

with the Hamiltonian

\[
\mathcal{H}(x(t), u(t), \lambda(t)) = -C(x(t), u(t)) + \lambda^T(t) \cdot f(x(t), u(t)).
\]
Necessary optimality conditions

**Theorem (Pontryagin Maximum Principle)**

Let \((x^*, u^*)\) be an optimal solution. Then, there exists a function \(\lambda : [0, T] \rightarrow \mathbb{R}^{n_x}\) and a vector \(\alpha \in \mathbb{R}^{n_r}\) such that

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\]

**State**

\[
\dot{x}^*(t) = \nabla_\lambda \mathcal{H}(x^*(t), u^*(t), \lambda(t)), \quad x^*(0) = x_0,
\]

**Adjoint**

\[
\dot{\lambda}(t) = -\nabla_x \mathcal{H}(x^*(t), u^*(t), \lambda(t)),
\]

\[
\lambda(T) = \nabla_x \Phi(x^*(T)) - \nabla_x r(x^*(T)) \alpha.
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with the Hamiltonian

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\mathcal{H}(x(t), u(t), \lambda(t)) = -C(x(t), u(t)) + \lambda^T(t) \cdot f(x(t), u(t)).
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Necessary optimality conditions

**Theorem (Pontryagin Maximum Principle)**

Let \((x^*, u^*)\) be an optimal solution. Then, there exists a function \(\lambda : [0, T] \rightarrow \mathbb{R}^{nx}\) and a vector \(\alpha \in \mathbb{R}^{nr}\) such that

\[
\mathcal{H}(x^*(t), u^*(t), \lambda(t)) = \max_{u(t) \in U} \mathcal{H}(x(t), u(t), \lambda(t)) \quad \forall t \in [0, T],
\]

\[
\begin{align*}
\dot{x}^*(t) &= \nabla_\lambda \mathcal{H}(x^*(t), u^*(t), \lambda(t)), \quad x^*(0) = x_0, \\
\dot{\lambda}(t) &= -\nabla_x \mathcal{H}(x^*(t), u^*(t), \lambda(t)), \\
\lambda(T) &= \nabla_x \Phi(x^*(T)) - \nabla_x r(x^*(T)) \alpha.
\end{align*}
\]

with the Hamiltonian

\[
\mathcal{H}(x(t), u(t), \lambda(t)) = -C(x(t), u(t)) + \lambda^T(t) \cdot f(x(t), u(t)).
\]
Mechanical system: variational formulation

- \( n \)-dimensional configuration manifold \( Q \)
- Lagrangian \( L : TQ \rightarrow \mathbb{R}, \; L(q, \dot{q}) = K(q, \dot{q}) - V(q) \)
- Lagrangian control force \( f_L : TQ \times U \rightarrow T^*Q \)

Lagrange-d’Alembert principle

\[
\delta \int_0^T L(q(t), \dot{q}(t)) \, dt + \int_0^T f_L(q(t), \dot{q}(t), u(t)) \cdot \delta q(t) \, dt = 0
\]

\( \Rightarrow \) Euler-Lagrange equations

\[
\frac{d}{dt} \frac{\partial}{\partial \dot{q}} L(q(t), \dot{q}(t)) - \frac{\partial}{\partial q} L(q(t), \dot{q}(t)) = f_L(q(t), \dot{q}(t), u(t))
\]
Mechanical system: variational formulation

- $n$-dimensional configuration manifold $Q$
- Lagrangian $L : TQ \rightarrow \mathbb{R}$, $L(q, \dot{q}) = K(q, \dot{q}) - V(q)$
- Lagrangian control force $f_L : TQ \times U \rightarrow T^*Q$

Lagrange-d’Alembert principle

\[
\delta \int_{0}^{T} L(q(t), \dot{q}(t)) \, dt + \int_{0}^{T} f_L(q(t), \dot{q}(t), u(t)) \cdot \delta q(t) \, dt = 0
\]

$\Rightarrow$ Euler-Lagrange equations

\[
\frac{d}{dt} \frac{\partial}{\partial \dot{q}} L(q(t), \dot{q}(t)) - \frac{\partial}{\partial q} L(q(t), \dot{q}(t)) = f_L(q(t), \dot{q}(t), u(t))
\]

$\Rightarrow \dot{x}(t) = f(x(t), u(t))$ with $x = (q, \dot{q})$
What are the relevant preservation properties?

Noether Theorem (no forcing)

Invariance of the Lagrangian under a group action of a Lie group $G$ leads to the **preservation of momentum maps** along the trajectory.

1. Lagrangian $L : TQ \to \mathbb{R}$, Lie group $G$, left action
   
   $\Psi_g : Q \to Q$, $g \in G$

2. invariance of the Lagrangian: $L \circ \Psi_g^{TQ} = L$

3. left action induces momentum map $J$ with $J \circ F_L^t = J$ for all $t$ with Lagrangian flow $F_L^t : TQ \to TQ$
What are the relevant preservation properties?

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- invariance of the Lagrangian: $L \circ \Psi_{TQ}^g = L$
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Symplecticity (no forcing)

Lagrangian flow is symplectic: $(F_L^t)^*(\Omega) = \Omega$. 
Discrete variational principle

Replace

- state space $TQ$ with $Q \times Q$
- curve $q(t) \in Q$ with sequence $q_d = \{q_k\}_{k=0}^N$
- curve $u(t) \in U$ with sequence $u_d = \{u_k\}_{k=0}^{N-1}$
Discrete variational principle

Replace

- state space $TQ$ with $Q \times Q$
- curve $q(t) \in Q$ with sequence $q_d = \{q_k\}_{k=0}^{N}$
- curve $u(t) \in U$ with sequence $u_d = \{u_k\}_{k=0}^{N-1}$

Approximate Lagrangian and virtual work:

- discrete Lagrangian $L_d : Q \times Q \rightarrow \mathbb{R}$

\[ L_d(q_k, q_{k+1}) \approx \int_{kh}^{(k+1)h} L(q(t), \dot{q}(t)) dt \]

\[ f_k^- \cdot \delta q_k + f_k^+ \cdot \delta q_{k+1} \approx \int_{kh}^{(k+1)h} f_L(q(t), \dot{q}(t), u(t)) \cdot \delta q(t) \, dt \]
Discrete variational principle

- Discrete Lagrange-d’Alembert principle
  \[ \sum_{k=0}^{N-1} \delta L_d(q_k, q_{k+1}) + \sum_{k=0}^{N-1} [f_k^- \cdot \delta q_k + f_k^+ \cdot \delta q_{k+1}] = 0 \]

- Discrete Euler-Lagrange equations (DEL)
  \[ D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) + f_k^- + f_{k-1}^+ = 0 \]

- Variational integrator for forward simulation
  - symplectic (preservation of symplectic form) \( \Rightarrow \) good long-time energy behavior
  - preserves momentum maps, if \( L_d \) is invariant under group action
    - order given by order of discrete Lagrangian and forces

- Boundary conditions: continuous boundary conditions on \( TQ \) are transformed in discrete boundary conditions on \( Q \times Q \) via the discrete Legendre transformation

Hairer & Lubich (2004)
Marsden & West (2001)
Discrete optimal control problem

\[ \min_{q_d, u_d, (h)} J_d(q_d, u_d) = \sum_{k=0}^{N-1} C_d(q_k, q_{k+1}, u_k) + \Phi_d(q_{N-1}, q_N, u_{N-1}) \]

s.t. \[ \text{DEL} \quad D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) + f^+_k + f^-_k = 0 \]

initial conditions \[ s_d(q_0, q_1, u_0, q^0, \dot{q}^0) = 0 \]

path constraints \[ h_d(q_k, q_{k+1}, u_k) \geq 0 \]

final constraint \[ r_d(q_{N-1}, q_N, u_{N-1}, q^T, \dot{q}^T) = 0 \]
Discrete optimal control problem

\[ \text{(DMOC: OB, Junge & Marsden (2008))} \]

\[
\min_{q_d, u_d, (h)} J_d(q_d, u_d) = \sum_{k=0}^{N-1} C_d(q_k, q_{k+1}, u_k) + \Phi_d(q_{N-1}, q_N, u_{N-1})
\]

s.t. \( \text{DEL} \quad D_2L_d(q_{k-1}, q_k) + D_1L_d(q_k, q_{k+1}) + f^+_{k-1} + f^-_k = 0 \)

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final constraint \( r_d(q_{N-1}, q_N, u_{N-1}, q^T, \dot{q}^T) = 0 \)

\[ \Downarrow \]

restricted optimization problem

\[ \min_\xi \varphi(\xi) \quad \text{s. t.} \quad a(\xi) = 0, \; b(\xi) \geq 0 \]
Discrete optimal control problem

\[
\begin{align*}
\text{(DMOC: OB, Junge & Marsden (2008))} \\
\min_{q_d, u_d, (h)} J_d(q_d, u_d) &= \sum_{k=0}^{N-1} C_d(q_k, q_{k+1}, u_k) + \Phi_d(q_{N-1}, q_N, u_{N-1}) \\
s.t. \quad &D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) + f^+_k - f^-_k = 0 \\
&\text{initial conditions } s_d(q_0, q_1, u_0, q^0, \dot{q}^0) = 0 \\
&\text{path constraints } h_d(q_k, q_{k+1}, u_k) \geq 0 \\
&\text{final constraint } r_d(q_{N-1}, q_N, u_{N-1}, q_T, \dot{q}_T) = 0
\end{align*}
\]

\[\downarrow\]

restricted optimization problem

\[
\begin{align*}
\min_\xi \varphi(\xi) \quad \text{s. t. } a(\xi) &= 0, \quad b(\xi) \geq 0
\end{align*}
\]

\[\downarrow\]

necessary optimality conditions

Karush-Kuhn-Tucker: \[
\nabla \varphi(\xi) - \lambda^T \nabla a(\xi) - \mu^T \nabla b(\xi) = 0
\]

with Lagrange multipliers \( \lambda \) und \( \mu \).
Structure preservation in presence of forcing

Consequence from Noether:

- evolution of momentum map exactly determined by control forces
- preservation of momentum map if control forces act orthogonal to group action ($\langle f_L, \xi_Q \rangle = 0$)
Structure preservation in presence of forcing

**Consequence from Noether:**
- Evolution of momentum map exactly determined by control forces
- Preservation of momentum map if control forces act orthogonal to group action ($\langle f_L, \xi_Q \rangle = 0$)

**Example:** control force acts orthogonal to relative velocity $\Rightarrow$ exact preservation of velocity’s absolute value
The adjoint system

Optimal control problem

Optimization

State and adjoint system

Discretization

Discrete state and adjoint system

Discretization

Optimization problem

Discrete state and adjoint system

Order of adjoint scheme is in general NOT the same as for state scheme


But: state and adjoint scheme coincide if a variational integrator is used
The adjoint system

Optimal control problem

Optimization

Discretization

State and adjoint system

Discretization

Order of approximation?

Discrete state and adjoint system

Optimization

Discrete state and adjoint system

RK methods: order of adjoint scheme is in general NOT the same as for state scheme

[HAGER 2000], [BONNANS, LAURENT-VARIN 2006]
The adjoint system

Optimal control problem

- Optimization
  - State and adjoint system
    - Discretization
      - Discrete state and adjoint system
    - Optimization
      - Discrete state and adjoint system

- Discretization
  - Optimization problem

 RK methods: order of adjoint scheme is in general NOT the same as for state scheme

[Hager 2000], [Bonnans, Laurent-Varin 2006]

BUT: using the right scheme for the state system, state and adjoint scheme coincide (e.g. special class of VI)
Properties of DMOC: order of approximation

Proof strategy:

1. VI of order $r$ given by

$$L_d = h \sum_{i=1}^{s} b_i L(Q_i, \dot{Q}_i)$$

⇔

symplectic partitioned Runge-Kutta scheme of order $r$

[Suris 1990], [Marsden, West 2001]
Properties of DMOC: order of approximation

Proof strategy:

1. VI of order $r$ given by $L_d = h \sum_{i=1}^{s} b_i L(Q_i, \dot{Q}_i)$

   $\Updownarrow$

   symplectic partitioned Runge-Kutta scheme of order $r$

   [Suris 1990], [Marsden, West 2001]

2. determine approximation order $\kappa$ of adjoint system (as for RK methods [Hager 2000]):
   - necessary optimality conditions for continuous system (Pontryagin)
   - necessary optimality conditions for discrete system (KKT)
1 Hamiltonian formulation

- Lagrangian \((q, \dot{q})\) can be rewritten as Hamiltonian \((q, p)\)
- partitioned Runge-Kutta scheme [Marsden, West 2001]

\[
q_1 = q_0 + h \sum_{j=1}^{s} b_j^q \dot{Q}_j, \quad p_1 = p_0 + h \sum_{j=1}^{s} b_j^p \dot{P}_j,
\]

\[
Q_i = q_0 + h \sum_{j=1}^{s} a_{ij}^q \dot{Q}_j, \quad P_i = p_0 + h \sum_{j=1}^{s} a_{ij}^p \dot{P}_j
\]

\[
P_i = \frac{\partial L}{\partial \dot{q}}(Q_i, \dot{Q}_i), \quad \dot{P}_i = \frac{\partial L}{\partial q}(Q_i, \dot{Q}_i) + f_L(Q_i, \dot{Q}_i, U_i)
\]

\(i = 1, \ldots, s\), with internal stages \((Q_i, P_i)\), control samples \(U_i = u_d(t_0 + c_i h)\), symplectic if \(b_i^q a_{ij}^p + b_j^p a_{ji}^q = b_i^q b_j^p, b_i^q = b_i^p\).
2 Continuous adjoint system

- substitute optimal control \( u(q(t), p(t), \lambda^q(t), \lambda^p(t)) \)
  obtained by solving minimization problem
- two-point boundary problem

\[
\begin{align*}
\dot{q}(t) &= \nu(q(t), p(t)), \quad q(0) = q^0, \\
\dot{p}(t) &= \eta(q(t), p(t), \lambda^q(t), \lambda^p(t)), \quad p(0) = p^0, \\
\dot{\lambda}^q(t) &= \phi^q(q(t), p(t), \lambda^q(t), \lambda^p(t)), \quad \lambda^q(T) = \Psi^q(q(T), p(T)), \\
\dot{\lambda}^p(t) &= \phi^p(q(t), p(t), \lambda^q(t), \lambda^p(t)), \quad \lambda^p(T) = \Psi^p(q(T), p(T)).
\end{align*}
\]
2 Discrete adjoint system (I)

\[
q_{k+1} = q_k + h \sum_{i=1}^{s} b_i \nu(Q_{ki}, P_{ki}), \quad q_0 = q^0, \\
p_{k+1} = p_k + h \sum_{i=1}^{s} b_i \eta(Q_{ki}, P_{ki}, \chi_{ki}^q, \chi_{ki}^p), \quad p_0 = p^0, \\
\lambda_{k+1}^q = \lambda_k^q + h \sum_{i=1}^{s} b_i \phi_q(Q_{ki}, P_{ki}, \chi_{ki}^q, \chi_{ki}^p), \quad \lambda_N^q = \Psi^q(q_N, p_N), \\
\lambda_{k+1}^p = \lambda_k^p + h \sum_{i=1}^{s} b_i \phi_p(Q_{ki}, P_{ki}, \chi_{ki}^q, \chi_{ki}^p), \quad \lambda_N^p = \Psi^p(q_N, p_N),
\]
\( Q_{ki} = q_k + h \sum_{j=1}^{s} a_{ij}^q \nu(Q_{kj}, P_{kj}), \)

\( P_{ki} = p_k + h \sum_{j=1}^{s} a_{ij}^p \eta(Q_{kj}, P_{kj}, \chi^q_{kj}, \chi^p_{kj}), \)

\( \chi_{ki}^q = \lambda^q_k + h \sum_{j=1}^{s} \bar{a}_{ij}^q \phi^q(Q_{kj}, P_{kj}, \chi^q_{kj}, \chi^p_{kj}), \)

\( \chi_{ki}^p = \lambda^p_k + h \sum_{j=1}^{s} \bar{a}_{ij}^p \phi^p(Q_{kj}, P_{kj}, \chi^q_{kj}, \chi^p_{kj}), \)

\( \bar{a}_{ij}^q = \frac{b_i b_j - b_j a_{ji}^q}{b_i}, \quad \bar{a}_{ij}^p = \frac{b_i b_j - b_j a_{ji}^p}{b_i}. \)
2 Order of approximation

Theorem (Order of approximation)

If the partitioned symplectic Runge-Kutta discretization of the state system is of order $\kappa$ and $b_i > 0$ for each $i$, then the scheme for the adjoint system is again a partitioned symplectic Runge-Kutta scheme of the same order (in particular we obtain the same schemes for $(q, p)$ and $(\lambda^p, \lambda^q)$).

Proof: With the symplecticity condition $a^{p}_{ij} = \frac{b_i b_j - b_j a^q_{ji}}{b_i}$ it holds

\[
\begin{align*}
\bar{a}_q^{ij} &= a^p_{ij}, \\
\bar{a}_p^{ij} &= a^q_{ij}.
\end{align*}
\]

[OB, Junge, Marsden 2009]
2 Discrete adjoint system (II)

\[
Q_{ki} = q_k + h \sum_{j=1}^{s} a_{ij}^q \nu(Q_{kj}, P_{kj}),
\]

\[
P_{ki} = p_k + h \sum_{j=1}^{s} a_{ij}^p \eta(Q_{kj}, P_{kj}, \chi_{kj}^q, \chi_{kj}^p),
\]

\[
\chi_{ki}^q = \lambda_{ki}^q + h \sum_{j=1}^{s} \bar{a}_{ij}^q \phi^q(Q_{kj}, P_{kj}, \chi_{kj}^q, \chi_{kj}^p),
\]

\[
\chi_{ki}^p = \lambda_{ki}^p + h \sum_{j=1}^{s} \bar{a}_{ij}^p \phi^p(Q_{kj}, P_{kj}, \chi_{kj}^q, \chi_{kj}^p),
\]

\[
\bar{a}_{ij}^q = \frac{b_i b_j - b_j a_{ji}^q}{b_i}, \quad \bar{a}_{ij}^p = \frac{b_i b_j - b_j a_{ji}^p}{b_i}.
\]
Properties of DMOC: convergence rates

- two-link pendulum optimally (minimal control effort) controlled from lower to upper equilibrium
- same convergence rates for configuration, control and adjoint
...to give answers to?

\( A + B \) optimal control problem

\[ + \text{ structure preserving integration} \]

\[ = \]

- inherited from integration scheme (\textit{var. formulation}): evolution of momentum maps due to symmetries determined by control forces

- gain for optimal control problems (\textit{symplecticity}): order of adjoint system = order of state system
...to give answers to?

\[
A+B \quad \text{optimal control problem} \\
\quad + \quad \text{structure preserving integration}
\]

\[
= \\
\quad \text{inherited from integration scheme (var. formulation): evolution of momentum maps due to symmetries determined by control forces}
\]

\[
\quad \text{gain for optimal control problems (symplecticity): order of adjoint system} = \text{order of state system}
\]

\[
\text{extension to (holonomic) constrained systems}
\]
Optimal control of multi-body dynamics (DMOCC)

[Leyendecker, OB, Marsden, Ortiz 08]

- constraint manifold \( \mathcal{C} = \{ q \mid q \in Q, g(q) = 0 \} \subset Q \)
- discrete version of Lagrange-d’Alembert principle with augmented Lagrangian \( \bar{L}(q, \dot{q}, \mu) = L(q, \dot{q}) - g^T(q) \cdot \mu \)
- Forced constrained discrete EL equations

\[
\begin{bmatrix}
D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) - hG^T(q_k) \cdot \mu_k + f_{k-1}^+ + f_k^- \\
\end{bmatrix} = 0
\]

\( g(q_{k+1}) = 0 \)
Optimal control of multi-body dynamics (DMOCC)

[Leyendecker, OB, Marsden, Ortiz 08]

- constraint manifold \( C = \{ q \mid q \in Q, g(q) = 0 \} \subset Q \)
- discrete version of Lagrange-d’Alembert principle with augmented Lagrangian \( \bar{L}(q, \dot{q}, \mu) = L(q, \dot{q}) - g^T(q) \cdot \mu \)
- Forced constrained discrete EL equations

\[
P^T(q_k) \left[ D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) - hG^T(q_k) \cdot \mu_k + f^+_k + f^-_k \right] = 0
\]

\[
g(q_{k+1}) = 0
\]

- dimension reduction via discrete nullspace matrix, relative reparametrization of configurations and forces
  - reduced scheme with minimal dimension
  - symplectic and momentum consistent
  - exact constraint fulfillment
Example (DMOCC): satellite reorientation

[Leyendecker, OB, Marsden, Ortiz 08]

- satellite actuated via rotors attached at main body
- Goal: attitude control with minimal control effort
- preservation of total angular momentum
Example (DMOCC): satellite reorientation

[Leyendecker, OB, Marsden, Ortiz 08]

- satellite actuated via rotors attached at main body
- Goal: attitude control with minimal control effort
- preservation of total angular momentum

![Graph showing energy and angular momentum over time]
Variational multirate integration

(joint work with S. Leyendecker)

- mechanical systems with dynamics on different time scales
- Lagrangian

\[ L(q, \dot{q}) = T(\dot{q}) - U(q), \quad U(q) = V(q) + W(q) \]

with \( W(q) = \frac{1}{\varepsilon} \bar{W}(q), \quad \varepsilon \ll 1. \)

- assumption: separation in slow and fast variables
  \( q = (q^s, q^f) \)
- fast potential \( W \) only dependent on fast variables \( q^f \)
- Lagrangian

\[ L(q, \dot{q}) = T(\dot{q}) - V(q) - W(q^f) \]

- aim: resolve slow and fast dynamics but with less function evaluations and smaller set of variables
Multirate: discrete approximation

- consider two time grids

macro time grid

\[ t_k = k \Delta T \mid k = 0, \ldots, N \]

micro time grid

\[ t_k^m = k \Delta T + m \Delta t \mid k = 0, \ldots, N - 1, m = 0, \ldots, p \]

- discrete slow variables \( \{ q_s^k \}_{k=0}^N \) with \( q_s^k \approx q_s(t_k) \)

- discrete fast variables \( \{ \{ q_f^{k,m} \}_{m=0}^p \}_{k=0}^{N-1} \) with \( q_f^{k,p} = q_f^{k+1} \) and \( q_f^{k,m} \approx q_f(t_k^m) \)
Discrete Lagrangian

\[ \bar{L}_d(q^s_k, q^s_{k+1}, q^f_{k}, \ldots, q^f_{k+p}) = \]

\[ p-1 \sum_{m=0}^{p-1} L_d(q^s_k, q^s_{k+1}, q^f_{m}, q^f_{k+m+1}) = \]

\[ = \sum_{m=0}^{p-1} \left[ T_d(q^s_k, q^s_{k+1}, q^f_{m}, q^f_{k+m+1}) - V_d(q^s_k, q^s_{k+1}, q^f_{m}, q^f_{k+m+1}) - W_d(q^f_{m}, q^f_{k+m+1}) \right] \]

with \( L_d = T_d - V_d - W_d \)

macro time grid

\[ \cdots \quad t_{k-1} \quad \ldots \quad t_k \quad \ldots \quad t_{k+1} \quad \Delta T \]

\[ t^0_{k-1} \quad \cdots \quad t^m_{k-1} \quad \cdots \quad t^m_{k} \quad \cdots \quad t^m_{k-1} \quad \cdots \quad t^m_{k+1} \quad \cdots \quad t^p_{k} \quad \Delta t \]

micro time grid
Discrete variational principle

- discrete action

\[ \mathcal{S}_d \left( \{q_k^s\}_{k=0}^N, \{\{q_k^{f,m}\}_{m=0}^p\}_{k=0}^{N-1} \right) = \sum_{k=0}^{N-1} \sum_{m=0}^{p-1} L_d(q_k^s, q_{k+1}^s, q_k^{f,m}, q_k^{f,m+1}) \]

- stationary discrete action

\[ \delta \mathcal{S}_d = \delta \sum_{k=0}^{N-1} \sum_{m=0}^{p-1} L_d(q_k^s, q_{k+1}^s, q_k^{f,m}, q_k^{f,m+1}) \]

\[ = \sum_{k=0}^{N-1} \sum_{m=0}^{p-1} \left( D_1 L_d \cdot \delta q_k^s + D_2 L_d \cdot \delta q_{k+1}^s \right) \]

\[ + \sum_{k=0}^{N-1} \sum_{m=0}^{p-1} \left( D_3 L_d \cdot \delta q_k^{f,m} + D_4 L_d \cdot \delta q_k^{f,m+1} \right) = 0 \]

gives discrete Euler-Lagrange equations for fast and slow dynamics
Triple pendulum in 3D

- different quadrature rules lead to different schemes
- invariance of discrete Lagrangian: preserved momentum maps
- discrete symplectic form preserved

configuration  

angular momentum  

energy ($\Delta T = 0.08, p = 10$)
Conclusion and future directions

Conclusion

▶ **fully discrete variational formulation** for the optimal control of mechanical systems

variational: structure preserving

symplectic: approximation order of adjoint system

= approximation order of state system

→ order of adjoint system also for extended DMOC versions (with holonomic constraints)

▶ **variational multirate integration**

▶ unified variational approach for the simulation of mechanical systems with different time scales

▶ structure preserving

→ analysis: comparison to existing methods, long-time behavior, efficiency

→ use for optimal control