Diploma Thesis

# A NEW EXPONENTIAL INTEGRATOR APPLIED TO THE DIFFUSION EQUATION ON EVOLVING DOMAINS

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# Contents

# Introduction

This thesis is the attempt to theoretically work out a test example for a new numerical time integrator, first proposed in [10]. This exponential integrator is constructed to approximate time integration of linear initial value problems with time dependent operators. Parallel to the efforts of Marlis Hochbruck and Alexander Ostermann to proof the convergence, I wrote a numerical program to verify the expected quality of the integrator. It implements a full-discretization of the diffusion equation on evolving domains.

The theoretical framework as well as the spatial discretization of the diffusion equation on evolving domains is inspired by [3], which deals with applying a finite element method to the diffusion equation on an evolving surfaces.

For the construction the exponential integrator, we use the theory for evolution problems from [15].

### Overview

Chapter 1 begins with the derivation of our example problem. We try to give some insights into some specialties of partial differential equations on evolving domains and give an existence and uniqueness result of a weak solution.

In Chapter 2, we prove that linear initial values problems with time dependent operators have unique solutions. This detailed insight into the construction of the solution was chosen, since the numerical integrator uses the same ansatz.

Chapter 3 deals with the spatial discretization of the problem we derived in Chapter 1. We present the general idea behind the spatial discretization and after showing some approximation results, we prove its convergence.

The construction of the integrator is done in Chapter 4. We then continue with the application of the integrator to our semi-discretized problem. It will be necessary to check some assumptions, such that we can be sure that the integrator works.

Finally, in Chapter 5, we are going to discuss the implementation of our test program, as well as the results of our numerical tests.

# Chapter 1

# The Diffusion Equation on Evolving Domains

Problems governed by partial differential equations (PDE) on a deformable domains which change in time arise in science and engineering. A classification of such PDEs would distinguish between:

- Problems where the movement of each of the points in the domain is a priori given.
- Free-boundary problems, where the boundary moves free according to a given law. These problems arise when modeling fluid-structure interactions and are much more challenging to treat theoretically and solve numerically.

We need a PDE with time dependent effects to solve it numerically and to test numerical integrators. Since differential equations with time dependent coefficients are mathematically almost completely studied and free-boundary PDEs are technically too elaborate for an exemplary problem, we decided the diffusion equation on a given evolving domain to be the proper test example.

Generally, the approach we choose is strongly motivated by [3], which presents a method for spatial discretization of diffusion equations on evolving surfaces. Nevertheless, we will indicate the parts and results, which are transferred from [3] to our problem, separately.

## 1.1 Derivation of the Diffusion Equation on Evolving Domains

We want to describe the behavior of a scalar quantity

$$u : \mathcal{N}_T \to \mathbb{R}$$

defined on a time-space domain

$$\mathcal{N}_T := \bigcup_{t\in[0,T]} \{t\} \times \Omega_t \,.$$

We consider $\Omega_t \subset \mathbb{R}^2$ as an evolving domain, which will be explained below. The evolution of the domain is associated with the motion of material points in the domain which transport material quantities such as heat, mass or concentration. As mentioned before, for our purpose we assume that the evolution of the domain is a priori given and not dependent on the solution.

### 1.1.1 The Evolving Domain

An evolving domain can be given in several ways, but since we want to model material flows we need to specify the exact motion of each point in the domain: Let $\Omega_0$ be some bounded initial domain in $\mathbb{R}^2$. We define the mapping

$$\Phi : [0, T] \times \Omega_0 \to \mathbb{R}^2 \,,$$

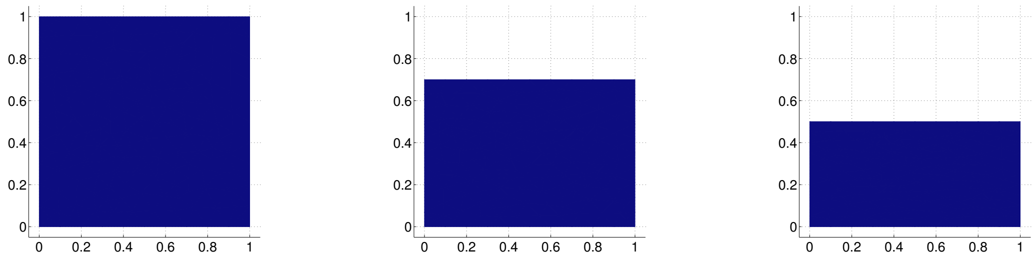


Figure 1.1: A basic example for an evolving domain: The unit square $[0.1] \times [0, 1] \subset \mathbb{R}^2$ is scaled along the $y$ axis.

which describes the trajectories of the points from $\Omega_0$ and set

$$\Omega_t := \Phi(t, \Omega_0) \tag{1.1}$$

for $t \in [0, T]$. The mapping $\Phi(t, \cdot)$ is supposed to be a diffeomorphism. By that we can assure that $\Omega_t$ is somehow topologically similar to $\Omega_0$. Two very basic linear examples of evolving domains are given in Figure 1.2 and Figure 1.1.

Considering $\Phi(t, \cdot)$ a diffeomorphism its inverse exists. The same is denoted by

$$\Phi^{-1}(t, \cdot) : \Omega_t \to \Omega_0 \, .$$

Observe that $\Phi^{-1}$ will always denote the inverse of $\Phi$ with respect to the space variable. The velocity $\underline{v}$ of a point in $\Omega_t$ is defined as the time derivative of its evolution,

$$\underline{v}(t, \Phi(t, \cdot)) := \partial_t \Phi(t, \cdot) \, . \tag{1.2}$$

Thus

$$\underline{v} : \mathcal{N}_T \to \mathbb{R}^2, \ (t, x) \mapsto \partial_t \Phi(t, \Phi^{-1}(t, x)) \, .$$

We need the following assumptions to be satisfied:

**Assumption 1.1.** *Let $\Omega_0$ be an open and bounded domain in $\mathbb{R}^2$ with Lipschitz boundary. The transformation of the domain $\Omega_0$ is given by the family of bijective mappings*

$$\Phi(t, \cdot) : \Omega_0 \to \Omega_t \, .$$

*so that $\Phi(t, \cdot)$ is a diffeomorphism for each $t \in [0, T]$. Thus $\Phi(t, \cdot)$ is continuously differentiable and has a continuously differentiable inverse $\Phi^{-1}(t, \cdot)$. By requiring $\Phi(0, \cdot) = \mathrm{Id}$, we assure that* $\min_{t \in [0,T], y \in \Omega_0} \det(J_x \Phi(t, y)) > 0$. *Moreover,*

$$\Phi \in C^1([0, T], C^1(\Omega_0)) \cap C^0([0, T], C^2(\Omega_0))$$

*so that the velocity $\underline{v} : \mathcal{N}_T \to \mathbb{R}^2$ of the evolving domain suffices*

$$\underline{v} \in C^0([0, T], C^1(\Omega_0)) \, .$$

It is difficult to define a time derivative for a function $f : \mathcal{N}_T \to \mathbb{R}$ at a fixed time and point: if we choose $x \in \Omega_t \setminus \Omega_{t+\tau}$ with $t, t + \tau \in [0, T]$ then $f(t, x)$ is defined and $f(t + \tau, x)$ is not, thus making it difficult to even write down a difference quotient. In fact we want a derivative that vanishes for functions that stay constant when the evolution of the domain is excluded. Therefore:

**Definition 1.2.** *(Material Derivative) Let $f : \mathcal{N}_T \to \mathbb{R}$. We define the material derivative $D_t$ of the function $f$ by*

$$D_t f(t, x) := \partial_t f(t, x) + \nabla f(t, x) \cdot \underline{v}(t, x) \, . \tag{1.3}$$

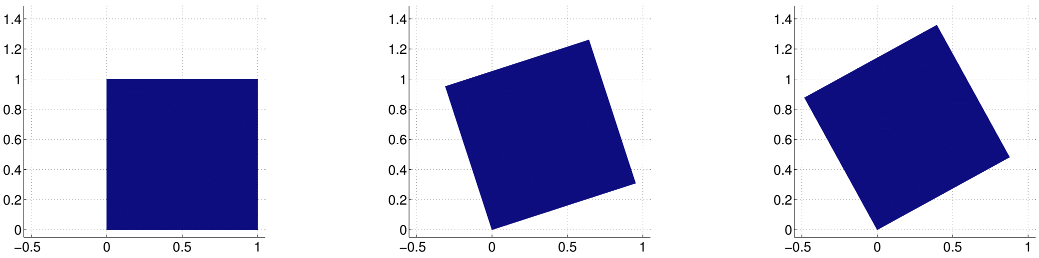


Figure 1.2: Another example for an evolving domain: The unit square $[0,1] \times [0,1] \subset \mathbb{R}^2$ rotates about 0.

Note that $D_t f$ depends on the evolving domain $\Omega_t$. The material derivative of $f$

$$D_t f : \mathcal{N}_t \to \mathbb{R}^2$$

satisfies

$$D_t f(t, \Phi(t,y)) = \frac{d}{dt}\Big[f(t, \Phi(t,y))\Big] \tag{1.4}$$

for $(t,y) \in [0,T] \times \Omega_0$. Hence the material derivative coincides with the derivative along the particle trajectories described by $\Phi$. The characterization (1.4) is in actually the motivation for (1.3).

### 1.1.2 Modeling Diffusive Behavior

This subsection is concerned with the derivation of a PDE that models diffuse behavior of some scalar quantity in an evolving domain. The deductions are not mathematically rigorous, but this is not necessary. We want to obtain some weak formulation and will in later sections define our own notion of a solution.

In the following we will consider arbitrary subsets $M_t \subset \Omega_t$ that have a sufficiently smooth boundary $\partial M_t$ with a well defined outer unit normal $\hat{n} : \partial M_t \to \mathbb{R}^2$.

*Remark* 1.3. All volume and surface integrals exclude the time parameter $t$, i.e.

$$\int_{M_t} u \, dx = \int_{M_t} u(t,\cdot) \, dx \, .$$

For a vector valued function $q : \partial M_t \to \mathbb{R}^2$, we denote surface integrals by

$$\int_{\partial M_t} q \cdot dS = \int_{\partial M_t} q(t,\cdot) \cdot \hat{n} \, dx$$

where the right-hand side can be considered as the integral over a parametrization of $\partial M_t$.

Since we want $u$ to be a physical quantity, let us start by describing the physical meaning of some mathematical quantities:

We consider

$$\int_{M_t} u \, dx$$

as the mass of $u$ in $M_t$.

Let $q$ be some material flux, which transports $u$. Note that $q$ takes values in $\mathbb{R}^2$. The mass of $u$ that flows through a surface $S$ is

$$-\int_S q \cdot \, dS \, .$$

To model the diffusion character of $u$ we choose the flow

$$q = -\alpha \nabla u$$

for some $\alpha > 0$. Since $-\nabla u$ points in the direction of steepest descent, this causes a flow of mass of $u$ from regions where $u$ is in general larger to regions where $u$ is smaller.

We require $u$ to obey mass conservation: The change of mass of $u$ in a set $M_t$ is equal to the amount of mass of $u$ flowing through its boundary $\partial M_t$ is then translated to

$$\frac{d}{dt}\int_{M_t} u\,dx = \int_{\partial M_t} \alpha\nabla u\cdot dS\ . \tag{1.5}$$

Convince yourself that the sign on the right-hand side is correct. An application of the divergence theorem yields

$$\begin{aligned}\frac{d}{dt}\int_{M_t} u\,dx &= \int_{\partial M_t} \alpha\nabla u\cdot dS\\ &= \alpha\int_{M_t} \operatorname{div}[\nabla u]\,dx\\ &= \alpha\int_{M_t} \Delta u\,dx\,.\end{aligned}$$

We need the following result to handle time derivatives of integrals over time dependent domains:

**Lemma 1.4.** *(Leibniz Formula) Let $\Omega_t$ be an evolving domain and $f$ be a function defined on $\mathcal{N}_T$ such that all of the following quantities exist. Then*

$$\frac{d}{dt}\int_{\Omega_t} f\,dx = \int_{\Omega_t} D_t f + f\operatorname{div}\underline{v}\,dx\,. \tag{1.6}$$

*Proof.* We can interchange integration and differentiation with respect to time if the integral is independent of time and the integrand is differentiable and integrable. Using the mapping $\Phi(t,\cdot):\Omega_0\to\Omega_t$ to switch the time dependency of the domain into the integrand, we then find

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} f\,dx &= \frac{d}{dt}\int_{\Omega_0} f\circ\Phi(t,\cdot)\det(J_x\Phi(t,\cdot))\,dx = \int_{\Omega_0}\frac{d}{dt}\Big[f\circ\Phi(t,\cdot)\det(J_x\Phi(t,\cdot))\Big]\,dx\\ &= \int_{\Omega_0}\frac{d}{dt}[f\circ\Phi(t,\cdot)]\det(J_x\Phi(t,\cdot)) + f\circ\Phi(t,\cdot)\frac{d}{dt}[\det(J_x\Phi(t,\cdot))]\,dx\end{aligned}$$

Hence, by using (1.4) and

$$\frac{d}{dt}[\det(J_x\Phi(t,\cdot))] = [\operatorname{div}\underline{v}]\circ\Phi(t,\cdot)\det(J_x\Phi(t,\cdot))$$

with the latter being found in [6, (2.8)], we get (1.6). □

This Lemma can be directly applied to the above integral equation and we get

$$\int_{M_t} D_t u + u\operatorname{div}\underline{v}\,dx = \alpha\int_{M_t}\Delta u\,dx\,. \tag{1.7}$$

Observe that the evolution of $M_t$ is given by the same family of diffeomorphism $\Phi(t,\cdot)$ as $\Omega_t$ and thus $\underline{v}$ in the above equation can be considered as in (1.2).

If (1.7) holds for each $M_t\subset\Omega_t$, we know that $u$ satisfies

$$D_t u(t,x) + u(t,x)\operatorname{div}\underline{v}(t,x) = \alpha\Delta u(t,x) \qquad t\in[0,T],\ x\in\Omega_t\,. \tag{1.8}$$

For each $t\in[0,T]$, (1.8) can be considered as a function equation in some function space that lies in $\{f:\Omega_t\to\mathbb{R}\}$. Later on we will specify this function space, however since $\Delta u$ and $D_t u$ need to exist in some sense it is obvious that we have to choose a function space which guarantees some kind of differentiability.

A PDE is only well-posed if we impose boundary conditions (BC). For simplicity we consider homogeneous Dirichlet BC, i.e. we claim that $u$ suffices

$$u(t,x) = 0 \qquad x \in \partial\Omega_t \, . \tag{1.9}$$

If $u$ is not a point wise defined function, we understand the BC in the $H^1$ trace operator sense. In the case of homogeneous Dirichlet BC, this is more simple: We choose a function space that adheres to our BC.

The problem we consider describes the evolution of a solution through time. A unique solution is only possible if we choose an initial value for $u$

$$u(0,\cdot) = u_0 \tag{1.10}$$

for some $u_0 : \Omega_0 \to \mathbb{R}$. In the later analysis of our problem we need to impose additional regularity assumptions on $u_0$.

Collecting (1.8), (1.9) and (1.10) we obtain our first PDE:

$$\begin{cases} D_t u + u \operatorname{div} \underline{v} = \alpha \Delta u & \text{in } \mathcal{N}_T \\ u(t,x) = 0 & x \in \partial\Omega_t,\ t \in [0,T] \\ u(0,\cdot) = u_0 & \text{in } \Omega_0 \end{cases} \tag{1.11}$$

Observe that (1.11) is similar to the diffusion equation with an advection-term in fixed domains: If we choose $\Phi(t,\cdot) = \mathrm{Id}$ the partial differential equation in (1.11) reduces to the well-known diffusion equation

$$\partial_t u = \alpha \Delta u \, .$$

Obviously, (1.11) is a generalization of the diffusion equation and the material derivative performs as the time derivative on the fixed domain.

Later we want to apply the finite element method (FEM) to our problem, thus we need a variational formulation of (1.11): Multiplying the partial differential equation in (1.11) with a smooth test function $\psi : \mathcal{N}_T \to \mathbb{R}$, $\psi(t,\cdot) \in C_0^\infty(\Omega_t)$ and integrating over $\Omega_t$, we get

$$\int_{\Omega_t} D_t u \, \psi \, dx + \int_{\Omega_t} u \, \psi \operatorname{div} \underline{v} \, dx = \alpha \int_{\Omega_t} \Delta u \, \psi \, dx \, .$$

*Remark* 1.5. Being aware of ambiguity of $D_t u \, \psi$, we always indicate the differentiation of products by square brackets, i.e.

$$D_t u \, \psi := D_t[u] \, \psi \, .$$

The above integral equation has to be satisfied for all test functions $\psi$ from some function space $F$, which is yet to be specified. Using the divergence theorem, we find

$$\begin{aligned} \int_{\Omega_t} \Delta u \, \psi \, dx = \int_{\Omega_t} \operatorname{div}[\nabla u] \, \psi \, dx &= \int_{\partial\Omega_t} \psi \nabla u \cdot dS - \int_{\Omega_t} \nabla u \cdot \nabla \psi \, dx \\ &= - \int_{\Omega_t} \nabla u \cdot \nabla \psi \, dx \end{aligned}$$

where the surface integral over $\partial\Omega_t$ vanishes since $\psi(t,x) = 0$ for $x \in \partial\Omega_t$. Now we have our first variational formulation:

$$\begin{cases} \displaystyle\int_{\Omega_t} D_t u \, \psi \, dx + \int_{\Omega_t} u \, \psi \operatorname{div} \underline{v} \, dx + \alpha \int_{\Omega_t} \nabla u \cdot \nabla \psi \, dx = 0 & \forall \psi \in F \\ u(t,x) = 0 & x \in \partial\Omega_t \\ u(0,\cdot) = u_0 & \text{in } \Omega_0 \end{cases} \tag{1.12}$$

Consider this application of the Leibniz formula (1.6) for two functions $u$ and $\psi$ defined on $\mathcal{N}_T$

$$\frac{d}{dt}\int_{\Omega_t} u\,\psi\,dx = \int_{\Omega_t} D_t[u\,\psi] + u\,\psi\,\mathrm{div}\,\underline{v}\,dx = \int_{\Omega_t} D_t u\,\psi + u\,D_t\psi + u\,\psi\,\mathrm{div}\,\underline{v}\,dx\,.$$

We use the above equation on (1.12) and get our second variational formulation:

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_{\Omega_t} u\,\psi\,dx + \alpha\int_{\Omega_t} \nabla u\cdot\nabla\psi\,dx = \int_{\Omega_t} u\,D_t\psi\,dx & \forall\psi\in F \\ u(t,x) = 0 & x\in\partial\Omega_t \\ u(0,\cdot) = u_0 & \text{in } \Omega_0 \end{cases} \tag{1.13}$$

The three formulations (1.11), (1.12) and (1.13) are the starting point for further analysis and, at this point, they can be understood as three different problems. In the following sections we consider (1.11) as the main problem, whereas the variational formulations will be used to define weak solutions.

## 1.2 Analysis of the Diffusion Equation on Evolving Domains

We need a mathematical framework to handle the PDE (1.11) and the same will be presented in this section. Moreover, to treat a problem numerically its well-posedness is crucial, therefore the main part of this section will be considered with proving the existence of a unique solution.

### 1.2.1 Analytic Framework and Notation

In this thesis $C$ and $c$ always denote generic constants. Nevertheless, we want to emphasize, that we watched to only merge constants into $C$ and $c$ that only depend on quantities that are a priori known to be bounded.

#### Absolute Value and Vectors

First note, that - with one unimportant exception - we always use real and not complex quantities. The absolute value of some $x\in\mathbb{R}$ is denoted by $|x|$, as well as the 2-norm for some vector $q\in\mathbb{R}^2$, $|q| := |q| = \sqrt{q_1^2+q_2^2}$. The scalar product of two vectors $q_1, q_2\in\mathbb{R}^2$ is denoted by $q_1\cdot q_2$.

Matrices $A\in\mathbb{R}^{2\times 2}$ are indicated by capital letters and the norm $|A|$ is the from the 2-norm induced matrix norm.

In later chapters, we will encounter vectors and matrices that result from semi-discretization. The different character of those quantities will be indicated by a bold notation, e.g. $\mathbf{y}\in\mathbb{R}^d$.

#### Lebesgue and Sobolev Spaces

To use the tools of the standard PDE theory, we need Lebesgue and Sobolev spaces. We assume the reader to be familiar with the basics of the Sobolev space theory and the notion of weak derivatives.

Let $\Omega\subset\mathbb{R}^2$ be some bounded and open set. We denote the norm of a function $f:\Omega\to\mathbb{R}$ with respect to some function space $F$ as $\|f\|_F$, e.g. for $F = L^2(\Omega)$

$$\|f\|^2_{L^2(\Omega)} = \int_\Omega f^2\,dx$$

where we omit, now and in the following, the integration variable in the arguments of the integrands.

Let $k\in\mathbb{N}$. The norm of a function $f$ from the Hilbert space $H^k(\Omega)$ with weak derivatives

$$\partial^\alpha f = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2} f$$

is defined as

$$\|f\|^2_{H^k(\Omega)} = \int_\Omega \sum_{|\alpha|\le k} \left|\partial^\alpha f\right|^2 \, dx$$

where $\alpha = (\alpha_1, \alpha_2)$ denotes the typical multiindex and $|\alpha| := \alpha_1 + \alpha_2$ its order. Moreover, we set

$$|f|^2_{k,\Omega} = \int_\Omega \sum_{|\alpha|=k} \left|\partial^\alpha f\right|^2 \, dx \, .$$

We will only use Sobolev spaces of order $k \le 2$. Note that $H^0(\Omega) = L^2(\Omega)$.

It is inherent to our problem that the domain of our functions varies a lot. Hence the typical norm abbreviations do not suffice. Instead we choose a combined notation, which indicates both the domain of integration as well as the order of the Sobolev space:

$$\|f\|_{k,\Omega} := \|f\|_{H^k(\Omega)} \ .$$

Given that $H^k(\Omega)$ is a Hilbert space for $k \ge 0$, we denote the inner product accordingly by $(\cdot,\cdot)_{k,\Omega}$. Observe

$$\|f\|^2_{1,\Omega} = \|f\|^2_{0,\Omega} + \|\nabla f\|^2_{0,\Omega}$$

where $\nabla$ denotes the gradient

$$\nabla f(x) = \left(\partial_{x_1} f(x), \partial_{x_2} f(x)\right)$$

We consider $\nabla f \in \mathbb{R}^{1\times 2}$ as a row vector. The Jacobain matrix of a function $g : \mathbb{R}^2 \to \mathbb{R}^2$ is denoted by $J_x g : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$.

At last, we need the function space $H^1_0(\Omega)$. Assuming $\Omega$ to be a Lipschitz domain, $H^1_0(\Omega)$ consists of functions that suffice homogeneous Dirichlet BC on $\partial\Omega$ in a trace operator sense:

$$f = 0 \qquad \text{on } \partial\Omega$$

if $f \in H^1_0(\Omega)$. An introduction into Sobolev spaces and trace operators can be found in [5].

### 1.2.2 Weak Solution

Our analytical approach is strongly motivated by [3]. All results and proofs of this section are transferred from the evolving-surface-case in [3, Section 4].

As already mentioned that we need the variational formulations of (1.11) to apply the FEM, hence we introduce the notion of a weak solution.

**Definition 1.6** (Weak Solution, [3, Definition 4.1])**.** *Let* $\mathcal{N}_T = \bigcup_{t\in[0,T]}\{t\}\times\Omega_t$ *and* $\alpha > 0$*. Moreover, let*

$$F := \{\psi : \mathcal{N}_t \to \mathbb{R} \mid \psi(t,\cdot) \in H^1_0(\Omega_t),\ D_t\psi(t,\cdot) \in L^2(\Omega_t)\} \, . \tag{1.14}$$

*A function* $u \in H^1(\mathcal{N}_T)$ *with* $u(t,\cdot) \in H^1_0(\Omega_t)$ *is a weak solution of* (1.11)*, if for every* $\psi \in F$

$$\frac{d}{dt}\int_{\Omega_t} u\,\psi \, dx + \alpha \int_{\Omega_t} \nabla u \cdot \nabla \psi \, dx = \int_{\Omega_t} u \, D_t \psi \, dx \qquad \textit{almost everywhere in } [0,T]. \tag{1.15}$$

Observe, that the homogeneous Dirichlet BC for $u$ are encoded in $u(t,\cdot) \in H^1_0(\Omega_t)$. Calling (1.15) a weak solution of (1.11) is justified by the relation between (1.11) and (1.13), which was explained in the previous section.

*Remark* 1.7. Observe that our notion of a weak solution is differentiable almost everywhere with respect to time. Having the existence of $D_t\psi$, we know by the Leibniz formula (1.6)

$$\frac{d}{dt}\int_{\Omega_t} u\,\psi\,dx = \int_{\Omega_t} D_t u\,\psi + u\,D_t\psi + u\,\psi\ \operatorname{div}\underline{v}\,dx\,.$$

Thus a weak solution of (1.11) always fulfills for each $\psi \in F$ both: (1.15) and

$$\int_{\Omega_t} D_t u\,\psi\,dx + \int_{\Omega_t} u\,\psi\operatorname{div}\underline{v}\,dx + \alpha\int_{\Omega_t} \nabla u\cdot\nabla\psi\,dx\ = 0 \qquad \text{almost everywhere in } [0,T]. \tag{1.16}$$

We call the above characterization of the weak solution of (1.11) the second formulation.

We start with proving two basic energy equations for the weak solution.

**Lemma 1.8** ( [3, Lemma 4.2] )**.** *Let $u$ be a weak solution of* (1.11)*. Then*

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} u^2\,dx\ + \alpha\int_{\Omega_t} |\nabla u|^2\ dx\ + \frac{1}{2}\int_{\Omega_t} u^2\operatorname{div}\underline{v}\,dx\ = 0. \tag{1.17}$$

*Proof.* We choose $\psi = u$ in (1.15) and get

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} u^2\,dx\ + \alpha\int_{\Omega_t} |\nabla u|^2\ dx\ &= \int_{\Omega_t} u\,D_t u\,dx\\ &= \frac{1}{2}\int_{\Omega_t} D_t[u^2]\,dx\\ &= \frac{1}{2}\frac{d}{dt}\int_{\Omega_t} u^2\,dx\ - \frac{1}{2}\int_{\Omega_t} u^2\operatorname{div}\underline{v}\,dx\,.\end{aligned}$$

□

**Lemma 1.9** ([3, Lemma 4.3])**.** *Let $u$ be a weak solution of* (1.11) *for which the following quantities exist. Then*

$$\int_{\Omega_t}(D_t u)^2\,dx\ + \frac{\alpha}{2}\frac{d}{dt}\int_{\Omega_t} |\nabla u|^2\ dx\ = \frac{\alpha}{2}\int_{\Omega_t} |\nabla u|^2\operatorname{div}\underline{v}\,dx\ - \alpha\int_{\Omega_t}(\nabla u J_x\underline{v})\cdot\nabla u\,dx\ - \int_{\Omega_t} u\,D_t u\operatorname{div}\underline{v}\,dx \tag{1.18}$$

*Proof.* With Lemma A.1 from the appendix we get

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} |\nabla u|^2\ dx\ &= \int_{\Omega_t} D_t[\,|\nabla u|^2]\,dx\ + \int_{\Omega_t} |\nabla u|^2\operatorname{div}\underline{v}\,dx\\ &= 2\int_{\Omega_t} \nabla[D_t u]\cdot\nabla u\,dx\ + \int_{\Omega_t} |\nabla u|^2\operatorname{div}\underline{v}\,dx\ - 2\int_{\Omega_t}(\nabla u J_x\underline{v})\cdot\nabla u\,dx\,.\end{aligned}$$

We choose $\psi = D_t u$ in (1.16) and use the above formula

$$\begin{aligned}0 &= \int_{\Omega_t}(D_t u)^2\,dx\ + \int_{\Omega_t} u\,D_t u\operatorname{div}\underline{v}\,dx\ + \alpha\int_{\Omega_t} \nabla u\cdot\nabla\,[D_t u]\ dx\\ &= \int_{\Omega_t}(D_t u)^2\,dx\ + \int_{\Omega_t} u\,D_t u\operatorname{div}\underline{v}\,dx\ + \alpha\Big(\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} |\nabla u|^2\ dx\\ &\quad + \int_{\Omega_t}(\nabla u J_x\underline{v})\cdot\nabla u\,dx - \frac{1}{2}\int_{\Omega_t} |\nabla u|^2\operatorname{div}\underline{v}\,dx\Big)\,.\end{aligned}$$

□

### 1.2.3 Well-Posedness

**Theorem 1.10** (Well-posedness, [3, Theorem 4.4]). *There exists a unique weak solution of* (1.11) *with initial value* $u_0 \in H_0^1(\Omega_0)$ *which satisfies the following energy estimates:*

$$\begin{aligned} \sup_{t\in[0,T]} \|u\|_{0,\Omega_t}^2 + \int_0^T \|\nabla u\|_{0,\Omega_t}^2 \, dt \le c\, \|u_0\|_{0,\Omega_0}^2 \,, \\ \int_0^T \|D_t u\|_{0,\Omega_t}^2 \, dt + \sup_{t\in[0,T]} \|\nabla u\|_{0,\Omega_t}^2 \le c\, \|u_0\|_{1,\Omega_0}^2 \,. \end{aligned} \tag{1.19}$$

Since the proof of Theorem 1.10 is lengthy, we split it into partial results:

**Lemma 1.11** (Uniqueness). *If $u$ is a weak solution of* (1.11) *with initial value $u_0 \in H_0^1(\Omega_0)$, then $u$ is the unique weak solution of* (1.11) *with initial value $u_0$.*

*Proof.* The linearity of the problem assures that the difference of two weak solutions $u, w$ with initial value $u_0 \in H_0^1(\Omega_0)$ is again a weak solution. We apply Lemma 1.8 to $u - w$ and get

$$\frac{d}{dt} \|u - w\|_{0,\Omega_t}^2 + 2\alpha \, \|\nabla(u - w)\|_{0,\Omega_t}^2 + \int_{\Omega_t} (u - w)^2 \operatorname{div} \underline{v} = 0.$$

Since the second summand is non-negative, this implies

$$\begin{aligned} \frac{d}{dt} \|u - w\|_{0,\Omega_t}^2 &\le - \int_{\Omega_t} (u - w)^2 \operatorname{div} \underline{v} \, dx \\ &\le \left| \int_{\Omega_t} (u - w)^2 \operatorname{div} \underline{v} \, dx \right| \le \sup_{t\in[0,T]} \left\|\operatorname{div} \underline{v}\right\|_{L^\infty(\Omega_t)} \|u - w\|_{0,\Omega(t)}^2 \,. \end{aligned}$$

Since $\sup_{t\in[0,T]} \left\|\operatorname{div} \underline{v}\right\|_{L^\infty(\Omega_t)} \le c$ by assumption, the Gronwall estimate from Proposition A.6 gives the uniqueness:

$$\sup_{t\in[0,T]} \|u(t,\cdot) - w(t,\cdot)\|_{0,\Omega_t} \le \|u_0 - u_0\|_{0,\Omega_0} \, e^{cT} = 0.$$

□

We prove the existence of a weak solution as in (1.16) by showing existence for a sequence of corresponding finite dimensional problems. The boundedness of this sequence together with the reflexivity of our Hilbert space $H^1(\mathcal{N}_T)$ then yield the solution by the limit of a weakly convergent subsequence.

**Lemma 1.12** (Construction of a Galerkin Solution). *For each $N \in \mathbb{N}$, there is a unique Galerkin solution of* (1.15)*. The Galerkin solutions are described in the proof below.*

*Proof.* Let $\hat{\theta}_j$, $j \in \mathbb{N}$ denote the eigenfunctions of the Laplacian $\Delta$ on $H_0^1(\Omega_0)$. We consider $\{\hat{\theta}_j\}_{j\in\mathbb{N}}$ as an orthonormal basis of $H_0^1(\Omega_0)$ and as a consequence $\{\hat{\theta}_j\}_{j\in\mathbb{N}}$ is an orthogonal basis of $L^2(\Omega_0)$ (see e.g. [5, Section 6.5, Theorem 1]). Set

$$\theta_j(t, \Phi(t,\cdot)) = \hat{\theta}_j(\cdot) \qquad \text{on } \Omega_t.$$

This then gives the countable subset $\{\theta_j(t,\cdot) \mid j \in \mathbb{N}\} \subset H_0^1(\Omega_t)$ whose linear span is dense, since the linear mapping $H_0^1(\Omega_0) \to H_0^1(\Omega_t)$, $f \mapsto f \circ \Phi^{-1}(t,\cdot)$ is bijective and continuous. Moreover there exists a constant $c > 0$ independent of $t$ such that $\left\|f \circ \Phi^{-1}(t,\cdot)\right\|_{1,\Omega_t} \le c\, \|f\|_{1,\Omega_0}$. Note that $\{\theta_j(t,\cdot) \mid j \in \mathbb{N}\}$ is not an orthonormal basis of $H_0^1(\Omega_t)$. For all $j \in \mathbb{N}$, we have

$$\frac{d}{dt}[\theta_j(t, \Phi(t, y))] = \frac{d}{dt}[\hat{\theta}_j(y)] = 0$$

and with (1.4)

$$D_t \theta_j = 0 \,.$$

Our ansatz for a Galerkin solution of (1.16) with respect to the finite dimensional subspace

$$X_N(t) := \operatorname{span}\{\theta_1(t,\cdot),\dots,\theta_N(t,\cdot)\} \subset H_0^1(\Omega_t)$$

is

$$u_N(t,x) = \sum_{j=1}^{N} \mu_j(t)\theta_j(t,x)$$

with initial value $\mu_j(0) = (u_0, \hat{\theta}_j)_{1,\Omega_0}$. Because of the vanishing material derivatives of the basis functions $\theta_j$ we have that

$$D_t u_N(t,\cdot) = \sum_{j=1}^{N} \partial_t \mu_j(t)\theta_j(t,\cdot)$$

is in the same finite dimensional space as $u_N(t,\cdot)$. We now proof the unique existence of a solution for the following linear ordinary differential equation (ODE): Find $u_N(t,\cdot) \in X_N(t)$ such that

$$\frac{d}{dt}\int_{\Omega_t} u_N\,\theta\,dx + \alpha\int_{\Omega_t} \nabla u_N\cdot\nabla\theta\,dx = \int_{\Omega_t} u_N\,D_t\theta\,dx \tag{1.20}$$

for all $\theta(t,\cdot) \in X_N(t)$.

We will only state the arguments of the proof of existence for (1.20):

- (1.20) can be reduced rewritten as an ODE for $\mu : [0,T] \to \mathbb{R}^N$, by omitting the arbitrary coefficients in the basis representation of $\theta$.
- We then use the Picard-Lindelöf theorem [17, Analysis 2, Satz 4.10]: The Lipschitz continuity with respect to the vector argument is given by the boundedness of the norm of the matrices. The continuity with respect to time is assured by the continuous evolution of the domain $\Omega_t$.

Note, that when the dimension $N$ of the system increases, the condition numbers of the appearing matrices increases too. The unique existence of a solution of (1.20) is however not endangered by an increased Lipschitz constant. □

**Lemma 1.13** (Stability Estimates for the Galerkin Solution)**.** *Let $u_N$ be the Galerkin solution of* (1.20)*. Then $u_N$ satisfies following stability estimates independently of $N$:*

$$\sup_{t\in[0,T]} \|u_N\|^2_{0,\Omega_t} + \int_0^T \|\nabla u_N\|^2_{0,\Omega_t}\,dt \le c\,\|u_0\|^2_{0,\Omega_0}$$

$$\int_0^T \|D_t u_N\|^2_{0,\Omega_t}\,dt + \sup_{t\in[0,T]} \|\nabla u_N\|^2_{0,\Omega_t} \le c\,\|u_N(0)\|^2_{1,\Omega_0}\ .$$

*Proof.* Now we prove the finite dimensional versions of the stability estimates (1.19):

The following equation can be obtain as in Lemma 1.8

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} u_N^2\,dx + \alpha\int_{\Omega_t} |\nabla u_N|^2\,dx + \frac{1}{2}\int_{\Omega_t} u_N^2\,\operatorname{div}\underline{v}\,dx = 0\,. \tag{1.21}$$

This implies

$$\frac{d}{dt}\|u_N\|^2_{0,\Omega_t} \le \left\|\operatorname{div}\underline{v}\right\|_{L^\infty(\Omega_t)} \|u_N\|^2_{0,\Omega_t}\ .$$

By a Gronwall argument and $\sup_{t\in[0,T]}\left\|\operatorname{div}\underline{v}\right\|_{L^\infty(\Omega_t)} \le c$, we get

$$\sup_{t\in[0,T]} \|u_N(t)\|^2_{0,\Omega_t} \le \|u_N(0)\|^2_{0,\Omega_0}\,e^{cT} \le c\,\|u_N(0)\|^2_{0,\Omega_0}\ .$$

We use the above result, $\sup_{t\in[0,T]} \left\|\operatorname{div} \underline{v}\right\|_{L^\infty(\Omega_t)} \leq c$ and (1.21) to find

$$\begin{aligned}
\alpha \int_0^T \int_{\Omega_t} |\nabla u_N|^2 \, dx\, dt &= \frac{1}{2}\left( \|u_N(0)\|^2_{0,\Omega_0} - \|u_N(T)\|^2_{0,\Omega_T} - \int_0^T \int_{\Omega_t} u_N^2 \operatorname{div} \underline{v} \, dx\, dt \right) \\
&\leq \frac{1}{2}\left( \|u_N(0)\|^2_{0,\Omega_0} + \int_0^T \left\|\operatorname{div} \underline{v}\right\|_{L^\infty(\Omega_t)} \|u_N\|^2_{0,\Omega_t} \, dt \right) \\
&\leq \frac{1}{2}\left( \|u_N(0)\|^2_{0,\Omega_0} + c\, \|u_N(0)\|^2_{0,\Omega_0} \int_0^T \left\|\operatorname{div} \underline{v}\right\|_{L^\infty(\Omega_t)} \, dt \right) \\
&\leq c\left( \|u_N(0)\|^2_{0,\Omega_0} + c\, \|u_N(0)\|^2_{0,\Omega_0} \right) \\
&\leq c\, \|u_N(0)\|^2_{0,\Omega_0} \; .
\end{aligned}$$

Altogether we obtain

$$\sup_{t\in[0,T]} \int_{\Omega_t} u_N^2 \, dx + \int_0^T \int_{\Omega_t} |\nabla u_N|^2 \, dx\, dt \leq c\, \|u_N(0)\|^2_{0,\Omega_0} \tag{1.22}$$

which is the first estimate.

The second stability estimate follows analogously: Lemma 1.9 implies

$$\int_{\Omega_t} (D_t u_N)^2 \, dx + \frac{\alpha}{2}\frac{d}{dt} \int_{\Omega_t} |\nabla u_N|^2 \, dx \leq c \int_{\Omega_t} |\nabla u_N|^2 \, dx + c \int_{\Omega_t} |u_N|\, |D_t u_N| \, dx\, . \tag{1.23}$$

as an easy computation shows. With Young's inequality we get

$$c\, |u_N|\, |D_t u_N| = \left( \frac{c}{2^{1/2}} |u_N| \right) \left( 2^{1/2} |D_t u_N| \right) \leq \frac{c^2}{4} |u_N|^2 + |D_t u_N|^2$$

and (1.23) implies

$$\frac{\alpha}{2}\frac{d}{dt} \|\nabla u_N\|^2_{0,\Omega_t} \leq c\, \|\nabla u_N\|^2_{0,\Omega_t} + c\, \|u_N\|^2_{0,\Omega_t} \; .$$

Another application of the Gronwall estimate from Proposition A.6 yields with (1.22)

$$\begin{aligned}
\|\nabla u_N\|^2_{0,\Omega_t} &\leq e^{cT}\, \|\nabla u_N(0)\|^2_{0,\Omega_0} + c \int_0^T \|u_N\|^2_{0,\Omega_t} \, dt \\
&\leq c\, \|\nabla u_N(0)\|^2_{0,\Omega_0} + c\, \|u_N(0)\|^2_{0,\Omega_0}
\end{aligned}$$

and thus

$$\sup_{t\in[0,T]} \|\nabla u_N\|^2_{0,\Omega_t} \leq c\, \|u_N(0)\|^2_{1,\Omega_0} \; . \tag{1.24}$$

Again with (1.23) and Young's inequality, we get

$$\begin{aligned}
&\int_0^T \|D_t u_N\|^2_{0,\Omega_t} \, dt + \frac{\alpha}{2}\left( \|\nabla u_N(T)\|^2_{0,\Omega_T} - \|\nabla u_N(0)\|^2_{0,\Omega_0} \right) \\
&\qquad \leq c \int_0^T \|\nabla u_N\|^2_{0,\Omega_t} \, dt + c \int_0^T \|u_N\|^2_{0,\Omega_t} \, dt + \frac{1}{2} \int_0^T \|D_t u_N\|^2_{0,\Omega_t} \, dt
\end{aligned}$$

and thus with (1.24) and (1.22)

$$\begin{aligned}
\int_0^T \|D_t u_N\|^2_{0,\Omega_t} \, dt &\leq c\, \|\nabla u_N(0)\|^2_{0,\Omega_0} + c \int_0^T \|\nabla u_N\|^2_{0,\Omega_t} \, dt + c \int_0^T \|u_N\|^2_{0,\Omega_t} \, dt \\
&\leq c\, \|\nabla u_N(0)\|^2_{0,\Omega_0} + cT\, \|u_N(0)\|^2_{1,\Omega_0} + cT\, \|u_N(0)\|^2_{0,\Omega_0} \\
&\leq c\, \|u_N(0)\|^2_{1,\Omega_0} \; .
\end{aligned}$$

Combining the above estimate with (1.24) gives the second estimate. □

**Lemma 1.14** (Boundedness of the Galerkin Sequence)**.** *The sequence of Galerkin solutions $(u_N)_{N\in\mathbb{N}}$ is bounded in $H^1(\mathcal{N}_T)$.*

*Proof.* With $\partial_t u_N = D_t u_N - \nabla u_N \cdot \underline{v}$ (cf. Definition 1.2) and $\sup_{t\in[0,T]} \left\|\underline{v}\right\|_{L^\infty(\Omega_t)} \le c$ we obtain

$$\int_0^T \|\partial_t u_N\|^2_{0,\Omega_t}\, dt \le \int_0^T \|D_t u_N\|^2_{0,\Omega_t} + c\int_0^T \|\nabla u_N\|^2_{0,\Omega_t} \,.$$

And from stability estimates of the last Lemma we know

$$\int_0^T \|D_t u_N\|^2_{0,\Omega_t}\, dt\,,\ \int_0^T \|u_N\|^2_{0,\Omega_t}\, dt\,,\ \int_0^T \|\nabla u_N\|^2_{0,\Omega_t} \le c\, \|u_N(0)\|^2_{1,\Omega_0} \,.$$

Altogether:

$$\begin{aligned}
\|u_N\|^2_{H^1(\mathcal{N}_T)} &= \int_0^T \|\partial_t u_N\|^2_{0,\Omega_t} + \|u_N\|^2_{0,\Omega_t} + \|\nabla u_N\|^2_{0,\Omega_t}\, dt \\
&\le c\int_0^T \|D_t u_N\|^2_{0,\Omega_t} + \|\nabla u_N\|^2_{0,\Omega_t} + \|u_N\|^2_{0,\Omega_t}\, dt \\
&\le c\, \|u_N(0)\|^2_{1,\Omega_0} \,.
\end{aligned}$$

Recalling that $u_N(t,x) = \sum_{j=1}^N (u_0, \hat\theta_j)_{1,\Omega_0}\hat\theta_j(x)$ and that $\{\hat\theta_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $H^1_0(\Omega_0)$, we get with Parseval's equality

$$\|u_N(0)\|^2_{1,\Omega_0} = \sum_{j=1}^N \left|(u_0, \hat\theta_j)_{1,\Omega_0}\right| \le \sum_{j=1}^\infty \left|(u_0, \hat\theta_j)_{1,\Omega_0}\right| = \|u_0\|^2_{1,\Omega_0} \,.$$

This gives the claim. □

*Proof of Theorem 1.10.* The reflexivity of the Hilbert space $H^1(\mathcal{N}_T)$ and the boundedness of $(u_N)_{N\in\mathbb{N}}$ yield the existence of a weakly convergent subsequence which we again call $(u_N)_{N\in\mathbb{N}}$:

$$u_N \rightharpoonup u \qquad \text{for } N \to \infty \text{ in } H^1(\mathcal{N}_T). \tag{1.25}$$

We still have to prove that the weak limit $u$ is a weak solution of (1.11): For $i \in \mathbb{N}$ and $t_0 \in [0,T]$ let

$$f_i^{t_0} : H^1(\mathcal{N}_T) \to \mathbb{R},\ w \mapsto \int_0^{t_0}\int_{\Omega_t} D_t w\, \theta_i + w\, \theta_i \operatorname{div} \underline{v} + \alpha \nabla w \cdot \nabla\theta_i \, dx\, dt$$

with $\theta_i(t,\cdot) \in X_N(t)$ being the $i$th basis function. By the continuity of $H^1_0(\Omega_0) \to H^1_0(\Omega_t), f \mapsto f \circ \Phi^{-1}(t,\cdot)$, we have $\int_0^T \|\theta_i\|^2_{1,\Omega_t}\, dt < \infty$ and therefore we get the continuity of the linear functional $f_i^{t_0}$:

$$\begin{aligned}
&\int_0^{t_0}\int_{\Omega_t} D_t w\, \theta_i + w\, \theta_i \operatorname{div} \underline{v} + \alpha \nabla w \cdot \nabla\theta_i \, dx\, dt \\
&\qquad \le \int_0^{t_0} \|D_t w\|_{0,\Omega_t}\, \|\theta_i\|_{0,\Omega_t} + c\, \|w\|_{0,\Omega_t}\, \|\theta_i\|_{0,\Omega_t} + c\, \|\nabla w\|_{0,\Omega_t}\, \|\nabla\theta_i\|_{0,\Omega_t}\, dt \\
&\qquad \le c\left(\int_0^T \|\theta_i\|^2_{0,\Omega_t}\, dt\right)^{\frac12}\left(\int_0^T \|D_t w\|^2_{0,\Omega_t} + \|w\|^2_{0,\Omega_t}\, dt\right)^{\frac12} \\
&\qquad\quad + c\left(\int_0^T \|\nabla\theta_i\|^2_{0,\Omega_t}\, dt\right)^{\frac12}\left(\int_0^T \|\nabla w\|^2_{0,\Omega_t}\, dt\right)^{\frac12} \\
&\qquad \le c\left(\int_0^T \|\theta_i\|^2_{1,\Omega_t}\, dt\right)^{\frac12} \|w\|_{H^1(\mathcal{N}_T)} \,,
\end{aligned}$$

where we used the Hölder inequality to estimate $\int_0^{t_0} \dots dt$. Since $u_N$ is a solution of (1.20) we have $f_i^{t_0}(u_N) = 0$ for each $i \le N$, $t_0 \in [0,T]$ and due to the weak convergence

$$f_i^{t_0}(u) = \lim_{N\to\infty} f_i^{t_0}(u_N) = 0 \qquad \forall i \in \mathbb{N}, t_0 \in [0,T]\,.$$

Therefore, we know that for every $i \in \mathbb{N}$ and almost every $t \in [0,T]$

$$\int_{\Omega_t} D_t u\, \theta_i + u\, \theta_i \operatorname{div} \underline{v} + \alpha \nabla u \cdot \nabla \theta_i \, dx = 0\,. \tag{1.26}$$

Observe that the null set $\eta_i \subset [0,T]$ on which (1.26) is not satisfied, depends on the index $i$. Therefore the following holds only for $t \notin \bigcup_{i\in\mathbb{N}} \eta_i$, which is, as a countable union of null sets, again a null set.

Now let $\psi \in F$ be a test function in (1.16) and $t \in [0,T]$ be fixed. Since $\operatorname{span}\{\theta_i(t,\cdot) \mid i \in \mathbb{N}\}$ is dense in $H_0^1(\Omega_t)$ there is a sequence $(\eta_j)_{j\in\mathbb{N}} \subset \operatorname{span}\{\theta_i(t,\cdot) \mid i \in \mathbb{N}\}$ such that

$$\eta_j \to \psi(t,\cdot) \qquad \text{for } j \to \infty \text{ in } H_0^1(\Omega_t).$$

With the continuity of

$$H_0^1(\Omega_t) \to \mathbb{R},\ \eta \mapsto \int_{\Omega_t} D_t u\, \eta + u\, \eta \operatorname{div} \underline{v} + \alpha \nabla u \cdot \nabla \eta \, dx$$

and (1.26), we obtain

$$\int_{\Omega_t} D_t u\, \psi + u\, \psi \operatorname{div} \underline{v} + \alpha \nabla u \cdot \nabla \psi \, dx = 0\,.$$

for almost every $t \in [0,T]$. Hence $u$ is a weak solution of (1.11) with an initial value satisfying

$$(u(0,\cdot), \hat{\theta}_i)_{1,\Omega_0} = (u_0, \hat{\theta}_i)_{1,\Omega_0}, \qquad i \in \mathbb{N}\,.$$

Since $\{\hat{\theta}_i\}_{i\in\mathbb{N}}$ forms an orthonormal basis of $H_0^1(\Omega_0)$, we get $u(0,\cdot) = u_0$.

The stability estimates (1.19) for the weak solution $u$ can be derived analogously as for $u_N$. □

For our later error analysis we need more regularity of the solution. The following assumption corresponds to [3, Theorem 4.5]:

**Assumption 1.15.** *Let $u$ be the solution of* (1.16)*, i.e. the weak solution of* (1.11)*. We assume $u(t,\cdot) \in H^2(\Omega_t)$ and*

$$\int_0^T \|u(t,\cdot)\|_{2,\Omega_t}^2 \, dt \le c\, \|u_0\|_{1,\Omega_0}^2\,.$$

The above assumption is often satisfied, e.g. if the boundary $\partial\Omega_t$ stays smooth for all $t \in [0,T]$: If $u$ is a solution of (1.16), then $u(t,\cdot)$ solves the elliptic PDEs

$$\alpha \int_{\Omega_t} \nabla u \cdot \nabla \psi \, dx = -\int_{\Omega_t} \left(D_t u + u \operatorname{div} \underline{v}\right) \psi \, dx$$

for each $t \in [0,T]$. Note that for every fixed $t$ the test function $\psi$ is also a test function in $H_0^1(\Omega_t)$. From [5, Theorem 6.4.3] we then have

$$\|u(t,\cdot)\|_{2,\Omega_t} \le c\big(\|D_t u(t,\cdot)\|_{0,\Omega_t} + \|u(t,\cdot)\|_{0,\Omega_t}\big) \qquad t \in [0,T]\,.$$

The stability estimates from the above Theorem then imply the assumption.

Nevertheless we state the $H^2$-regularity as an assumption, since we are interested in problems on Lipschitz domains, where an analogous result can not be found as easily.

# Chapter 2

# A Non-autonomous Initial Value Problem

We need a theory to handle abstract evolution problems, since the numerical time integrator that will be developed in Chapter 4 builds on that theory. The basis of that theory are $C_0$ semigroups, which are used to solve the Banach space valued differential equations.

**Definition 2.1** ($C_0$ semigroup)**.** *Let $T : [0, \infty) \to \mathcal{L}(X)$ be a mapping. We call $T$ a strongly continuous ($C_0$) semigroup, if*

1. $T(0) = \mathrm{Id}$ *and for $t, s \in [0, \infty)$, we have $T(t + s) = T(t)T(s)$.*
2. *$T$ is strongly continuous on $X$, i.e.* $\lim_{t\downarrow 0} T(t)x = x$ *for all $x \in X$.*

Let $A$ is the infinitesimal generator (see e.g. [15]) of the $C_0$ semigroup $T$. Then

$$u(t) = T(t)u_0 \tag{2.1}$$

is the unique solution of the abstract differential equation

$$\frac{d}{dt}u(t) = Au(t), \qquad t \in [0, T] .$$

with initial value $u(0) = u_0$. Although we need several results from the semigroup theory, we do not give any introduction or go into detail. All results we use without proof can be found in the appendix.

*Remark* 2.2. We assume that the reader is familiar with the basic tools from functional analysis. Throughout this chapter, let $(X, \|\cdot\|)$ be a Banach space. The norm of a linear operator $T : X \to X$ is then given by

$$\|T\| = \|T\|_{X\leftarrow X} = \sup_{x\in X,\ x\neq 0} \frac{\|Tx\|}{\|x\|} .$$

We denote the space of all linear and bounded operators with $\mathcal{L}(X)$

$$\mathcal{L}(X) := \{T : X \to X \mid T \text{ is linear and bounded}\} .$$

## 2.1 Introduction to Evolution Systems

Let $X$ be a Banach space. For every $t \in [0, T]$ let $A(t) : X \supset D(A(t)) \to X$ be a linear operator. We will study the homogeneous initial value problem

$$\begin{cases} \frac{d}{dt}u(t) = A(t)u(t) & \text{for } s < t \leq T \\ u(s) = u_0 . \end{cases} \tag{2.2}$$

The initial value problem (2.2) is called an evolution problem or non-autonomous Cauchy problem.

We search for solutions of (2.2) of the following type:

**Definition 2.3.** *A function $u : [s, T] \to X$ is a classical solution of (2.2) if $u$ is continuous on $[s, T]$, $u(t) \in D(A(t))$ for $s < t \leq T$, $u$ is continuously differentiable on $s < t \leq T$ .*

### 2.1.1 The Bounded Operator Case

In order to obtain some feeling of the behavior of the solutions of (2.2) we recollect the basic results of the simpler case where

- $A(t)$ is a bounded linear operator on $X$
- $t \mapsto A(t)$ is continuous in the operator norm

without proof. The results we state can be found in [15, Section 5.1].

Under the above assumptions (2.2) has a unique solution for every $u_0 \in X$. This can be shown by using the Banach contraction principle. We then define a solution operator by

$$U(t,s)u_0 = u(t) \qquad \text{for } \; 0 \le s \le t \le T$$

where $u$ is the solution of (2.2). The two parameter character of the solution operator is plausible because the effect of $A(t)$ depends on time. Thus propagating the solution from $s$ to $t$ depends not only on the difference $t-s$ but both the starting time $s$ and the stopping time $t$. The most important properties of $U(t,s)$ are

- $\|U(t,s)\| \le \exp\left(\int_s^t \|A(\tau)\|\, d\tau\right)$
- $(t,s) \mapsto U(t,s)$ is continuous in the operator norm
- $\partial_t U(t,s) = A(t)U(t,s)$
- $\partial_s U(t,s) = -U(t,s)A(s)$.

for $0 \le s \le t \le T$.

The two parameter family of operators $U(t,s)$ replaces the one parameter semigroup of the autonomous case. This motivates the following definition

**Definition 2.4** (Evolution System, [15, Definition 5.5.3])**.** *A two parameter family of bounded linear operators $U(t,s)$, $0 \le s \le t \le T$, on $X$ is called an evolution system if the following holds*

1. $U(s,s) = \mathrm{Id}$ *and* $U(t,r)U(r,s) = U(t,s)$
2. $(t,s) \mapsto U(t,s)$ *is strongly continuous*

*for* $0 \le s \le r \le t \le T$.

*Remark* 2.5. Like in the autonomous case we do not expect the solution operator to be uniformly continuous. This holds only true if the operator $A(t)$ is bounded, which is too restrictive for our purposes. We assume $A(t)$ to be closed operators, which results in the strong continuity of the propagator $U(t,s)$, as we will see below.

## 2.2 An Evolution System for the Parabolic Initial Value Problem

In this section we construct a solution for a non-autonomous evolution problem as (2.2). There is no unified theory ([16]), but one complete proof is given in [15, Section 5.6]. Our proof is merely a more detailed and adapted version of [15, Section 5.6], since we only need some results as a theoretical foundation in later chapters of this thesis.

We study the homogeneous initial value problem

$$\begin{cases} \frac{d}{dt}u(t) + A(t)u(t) = 0 & 0 \le s < t \le T \\ u(s) = x\,. \end{cases} \tag{2.3}$$

In the parabolic case it is customary to write $A(t)u(t)$ on the left-hand side of the equation, due to the use of fractional powers of $A(t)$. Though we will not use fractional powers, we keep this notation for convention and easier comparison.

Let us start with a formal computation which will motivate our approach to solve (2.3). Suppose that for each $t \in [0, T]$, $-A(t)$ is the infinitesimal generator of a $C_0$ semigroup $S_t(s)$, $s \geq 0$ on the Banach space $X$. We seek an evolution system $U(t, s)$ for (2.3) and make the ansatz

$$U(t, s) := S_s(t - s) + W(t, s) = S_s(t - s) + \int_s^t S_\tau(t - \tau)\, R(\tau, s)\, d\tau\,, \tag{2.4}$$

which describes the evolution system $U(t, s)$ as the sum of the initial propagator $S_s$ and the collected and weighted propagators $S_\tau$, $s \leq \tau \leq t$.

*Remark* 2.6. Differentiation of an integral with respect to a variable appearing in the limit and integrand is done by:

$$\frac{d}{dt} \int_s^t f(t, \tau)\, d\tau = f(t, t) + \int_s^t \partial_t f(t, \tau)\, d\tau\,.$$

With the above remark, we can (formally) differentiate (2.4) and get

$$\partial_t U(t, s) = -A(s) S_s(t - s) + R(t, s) - \int_s^t A(\tau)\, S_\tau(t - \tau)\, R(\tau, s)\, d\tau$$

where we used $S_t(0) = \mathrm{Id}$. With

$$R_1(t, s) := (A(s) - A(t)) S_s(t - s) \tag{2.5}$$

we then get

$$\begin{aligned} \partial_t U(t, s) + A(t) U(t, s) = R(t, s) &- (A(s) - A(t)) S_s(t - s) \\ &- \int_s^t A(\tau)\, S_\tau(t - \tau)\, R(\tau, s)\, d\tau + \int_s^t A(t) S_\tau(t - \tau)\, R(\tau, s)\, d\tau \\ = R(t, s) &- R_1(t, s) - \int_s^t R_1(t, \tau)\, R(\tau, s)\, d\tau\,. \end{aligned} \tag{2.6}$$

Since $U(t, s)$ is an evolution system for (2.3), the above equation implies

$$R(t, s) = R_1(t, s) + \int_s^t R_1(t, \tau)\, R(\tau, s)\, d\tau\,. \tag{2.7}$$

$R_1(t, s)$ is given by (2.5), hence we try to solve the integral equation (2.7) for $R(t, s)$ and insert it into the characterization (2.4) to obtain $U(t, s)$. This will be our method of constructing the evolution system $U(t, s)$ below.

We need the following assumptions to carry this approach through:

**Assumption 2.7.** *We require three different assumptions to be satisfied:*

$(A_1)$ *The domain $D(A(t)) = D$ of $A(t)$, $0 \leq t \leq T$ is dense in $X$ and independent of $t$.*

$(A_2)$ *For $t \in [0, T]$, the resolvent $R(\lambda, A(t)) = (\lambda - A(t))^{-1}$ exists for all $\lambda$ with $\operatorname{Re} \lambda \leq 0$ and there is a constant $M$ such that*

$$\|R(\lambda, A(t))\| \leq \frac{M}{|\lambda| + 1} \qquad \text{for } \operatorname{Re} \lambda \leq 0,\ t \in [0, T]. \tag{2.8}$$

$(A_3)$ *There exists a constant $L > 0$ such that*

$$\left\|(A(t) - A(s)) A(\tau)^{-1}\right\| \leq L\, |t - s| \qquad \text{for } s, t, \tau \in [0, T]. \tag{2.9}$$

The main result of this chapter is:

**Theorem 2.8** ([15, Theorem 5.6.1]). *Under the assumptions $(A_1)$-$(A_3)$ there is a unique evolution system $U(t,s)$ on $0 \leq s \leq t \leq T$, satisfying:*

**$(E_1)$** $\|U(t,s)\| \leq C$ *for* $0 \leq s \leq t \leq T$.

**$(E_2)$** *For* $0 \leq s < t \leq T$, $U(t,s) : X \to D$ *and* $t \to U(t,s)$ *is strongly differentiable in* $X$. *The derivative* $\partial_t U(t,s) \in \mathcal{L}(X)$ *and it is strongly continuous on* $0 \leq s < t \leq T$. *Moreover, for* $0 \leq s < t \leq T$

$$\partial_t U(t,s) + A(t)U(t,s) = 0 \tag{2.10}$$

$$\|\partial_t U(t,s)\| = \|A(t)U(t,s)\| \leq \frac{C}{t-s} \tag{2.11}$$

*and*

$$\left\|A(t)U(t,s)A(s)^{-1}\right\| \leq C \qquad \textit{for } 0 \leq s \leq t \leq T. \tag{2.12}$$

**$(E_3)$** *For every* $v \in D$ *and* $t \in (0,T]$, $U(t,s)v$ *is differentiable with respect to* $s$ *on* $0 \leq s \leq t \leq T$ *and*

$$\partial_s U(t,s)v = U(t,s)A(s)v\,. \tag{2.13}$$

The rest of this section deals with the proof of Theorem 2.8. It will be split into three parts: First, we will construct the evolution system $U(t,s)$ by solving the integral equation (2.7). In the second part, we will prove the properties stated in $(E_2)$ and in the third part will deal with the uniqueness of the solution as well as $U(t,s) = U(t,r)U(r,s)$.

The assumptions $(A_1)$-$(A_3)$ imply some direct consequences, which will be frequently used: Since $(A_2)$ holds and $D$ is dense in $X$, we know with Theorem A.4 that for each $t \in [0,T]$, the operator $-A(t)$ is the infinitesimal generator of an analytic semigroup $S_t(s)$, $s \geq 0$, satisfying

$$\|S_t(s)\| \leq C \qquad \text{for } s \geq 0 \tag{2.14}$$

$$\|A(t)S_t(s)\| \leq \frac{C}{s} \qquad \text{for } s > 0. \tag{2.15}$$

independently of $t$. Furthermore, $(A_2)$ gives the existence of an angle $\theta \in (0, \frac{\pi}{2})$ such that

$$\rho(A(t)) = \Sigma_\theta = \{\lambda \mid \left|\arg \lambda\right| \geq \theta\} \cup 0\,. \tag{2.16}$$

It can be shown (as in Section 4.2.2) that (2.8) holds for all $\lambda \in \Sigma_\theta$, possibly with a different constant $M$.

Before we begin with the actual proof, we state some more consequences of our assumptions:

**Lemma 2.9** ([15, Lemma 5.6.2]). *Let $(A_1)$-$(A_3)$ be satisfied, then*

1. *For* $s \in (0,T]$ *and* $t_1, t_2 \in [0,T]$

$$\left\|(A(t_1) - A(t_2))S_\tau(s)\right\| \leq \frac{C}{s}\,|t_1 - t_2|\,. \tag{2.17}$$

2. *For* $s_1, s_2 \in (0,T]$ *and* $t, \tau \in [0,T]$

$$\left\|A(t)(S_\tau(s_2) - S_\tau(s_1))\right\| \leq \frac{C}{s_1 s_2}\,|s_2 - s_1|\,. \tag{2.18}$$

3. *For* $s \in (0,T]$ *and* $t, \tau_1, \tau_2 \in [0,T]$

$$\left\|A(t)(S_{\tau_1}(s) - S_{\tau_2}(s))\right\| \leq \frac{C}{s}\,|\tau_1 - \tau_2|\,. \tag{2.19}$$

*Moreover, $A(t)S_\tau(s) \in \mathcal{L}(X)$ for $s \in (0,T]$, $\tau, t \in [0,T]$ and the $\mathcal{L}(X)$ valued function $A(t)S_\tau(s)$ is uniformly continuous in the uniform operator topology for $s \in [\varepsilon, T]$, $t, \tau \in [0,T]$ for every $\varepsilon > 0$.*

*Proof.* 1. From (2.9) and (2.15) we have

$$\left\|(A(t_1) - A(t_2))S_\tau(s)\right\| \leq \left\|(A(t_1) - A(t_2))A(\tau)^{-1}\right\| \ \|A(\tau)S_\tau(s)\| \leq \frac{C}{s}\ |t_1 - t_2|\ ,$$

which proves (2.17).

2. Without loss of generality let $0 < s_1 \leq s_2$ and $x \in X$. We use Theorem A.2 to get

$$A(\tau)S_\tau(s_2)x - A(\tau)S_\tau(s_1)x = -\int_{s_1}^{s_2} A(\tau)^2 S_\tau(\sigma)x\, d\sigma = -\int_{s_1}^{s_2} \left(A(\tau)S_\tau\left(\frac{\sigma}{2}\right)\right)^2 x\, d\sigma$$

and therefore by (2.15)

$$\|A(\tau)S_\tau(s_2)x - A(\tau)S_\tau(s_1)x\| \leq C\ \|x\| \int_{s_1}^{s_2} \frac{1}{\sigma^2}\, d\sigma = C\ \|x\| \left(\frac{1}{s_1} - \frac{1}{s_2}\right) \leq \|x\|\, \frac{C}{s_2 s_1}\ |s_2 - s_1|\ .$$

The closed graph theorem [19, Theorem 1.6] implies the boundedness of $A(t)A(\tau)^{-1}$ for $t, \tau \in [0,T]$. With the Lipschitz continuity (2.9) of $[0,T] \to \mathcal{L}(X)$, $t \mapsto A(t)A(\tau)^{-1}$ uniform in $\tau$, we get $\left\|A(t)A(\tau)^{-1}\right\| \leq C$ independently of $t, \tau \in [0,T]$ and hence

$$\left\|A(t)(S_\tau(s_2) - S_\tau(s_1))\right\| \leq \left\|A(t)A(\tau)^{-1}\right\| \ \|A(\tau)S_\tau(s_2)x - A(\tau)S_\tau(s_1)\| \leq \frac{C}{s_1 s_2}\ |s_2 - s_1|$$

which gives (2.18).

3. Note that from $(A_2)$ and the resolvent identity

$$AR(\lambda, A) = \mathrm{Id} - \lambda R(\lambda, A)$$

it follows that

$$\|A(t)R(\lambda, A(t))\| = \|\mathrm{Id} - \lambda R(\lambda, A(t))\| \leq 1 + |\lambda|\, \frac{M}{|\lambda| + 1} \leq C$$

for $0 \leq t \leq T$. Also recall the following resolvent identity for two operators $A, B$

$$R(\lambda, A) - R(\lambda, B) = R(\lambda, A)(A - B)R(\lambda, B)\ .$$

Hence

$$\begin{aligned}\left\|A(t)(R(\lambda, A(\tau_1)) - R(\lambda, A(\tau_2)))\right\| &= \left\|A(t)R(\lambda, A(\tau_1))(A(\tau_1) - A(\tau_2))R(\lambda, A(\tau_2))\right\| \\ &\leq \left\|A(t)A(\tau_1)^{-1}\right\| \ \|A(\tau_1)R(\lambda, A(\tau_1))\| \\ &\quad \cdot \left\|(A(\tau_1) - A(\tau_2))A(\tau_2)^{-1}\right\| \ \|A(\tau_2)R(\lambda, A(\tau_2))\| \\ &\leq C\ |\tau_1 - \tau_2|\ . \end{aligned} \tag{2.20}$$

where we used (2.9) and applied the closed graph theorem to bound $\left\|A(t)A(\tau_1)^{-1}\right\|$. From Theorem A.3 we get the following representation

$$A(t)S_{\tau_1}x - A(t)S_{\tau_1}x = \frac{1}{2\pi i}\int_\Gamma e^{-\lambda s}A(t)(R(\lambda, A(\tau_1)) - R(\lambda, A(\tau_2)))x\,d\lambda \tag{2.21}$$

with $\Gamma$ being a smooth path in $\Sigma_\theta$ from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$. We apply (2.20) to (2.21) and find

$$\begin{aligned}\left\|A(t)S_{\tau_1}(s)x - A(t)S_{\tau_2}(s)x\right\| &\leq C\;|\tau_1 - \tau_2|\;\|x\| \int_\Gamma \left|e^{-\lambda s}\right|\,d\lambda \\ &\leq \frac{C}{s}\;|\tau_1 - \tau_2|\;\|x\|\;.\end{aligned}$$

Since $S_\tau(s) : X \to D$ is bounded for $s > 0$ (Theorem A.4 3.) and $A(t) : D \to X$ a closed operator,

$$A(t)S_\tau(s) : X \to X$$

is again a closed operator. The closed graph theorem then gives $A(t)S_\tau(s) \in \mathcal{L}(X)$ for $t, \tau \in [0, T]$, $s \in (0, T]$. Finally we use

$$A(t)S_\tau(s) = A(t)S_t(s) + A(t)(S_\tau(s) - S_t(s))\,,$$

the triangle inequality and (2.19) to show the uniform continuity of $A(t)S_\tau(s)$ with respect to $\mathcal{L}(X)$ for $t, \tau \in [0, T]$, $s \in [\varepsilon, T]$. □

**Corollary 2.10** ([15, Cor 5.6.3])**.** *The operator $R_1(t, s) = (A(s) - A(t))S_s(t - s)$ is uniformly continuous with respect to $\mathcal{L}(X)$ on $0 \leq s \leq t - \varepsilon \leq T$ for every $\varepsilon > 0$ and*

$$\|R_1(t, s)\| \leq C \tag{2.22}$$

*for $0 \leq s < t \leq T$.*

*Proof.* The first part of the claim is a direct consequence of

$$R_1(t, s) = A(s)S_s(t - s) - A(t)S_s(t - s)\,,$$

and the uniform continuity of $A(t)S_\tau(s)$ in $\mathcal{L}(X)$ for $t - s \geq \varepsilon > 0$.

The estimate (2.22) follows with (2.9) and (2.15):

$$\begin{aligned}\|R_1(t, s)\| &\leq \left\|(A(t) - A(s))A(s)^{-1}\right\|\;\|A(s)S_s(t - s)\| \\ &\leq C\;|t - s|\;|t - s|^{-1}\;.\end{aligned}$$

□

### 2.2.1 Construction

We begin by solving the integral equation (2.7) for $R(t, s)$. For $m \geq 1$, define the following recursion

$$R_{m+1}(t, s) = \int_s^t R_1(t, s)R_m(t, s)\,d\tau\,. \tag{2.23}$$

which will be used to construct $R(t, s)$.

**Lemma 2.11.** *Let $C$ be the constant from (2.22) and $m!$ denote the factorial of $m$. For $0 \leq s < t \leq T$,*

$$\|R_m(t, s)\| \leq \frac{C^m}{(m-1)!}(t - s)^{m-1}\,, \tag{2.24}$$

*and*

$$\left\|\sum_{m=1}^{\infty} R_m(t, s)\right\| \leq Ce^{C(t-s)}\,. \tag{2.25}$$

*Proof.* We prove the first claim by induction: (2.22) proves the relation for $m = 1$. Now suppose (2.24) holds for one $m$, then

$$\begin{aligned}\|R_{m+1}(t,s)\| \leq \int_s^t \|R_1(t,\tau)\|\, \|R_m(\tau,t)\|\, d\tau &\leq \frac{C^{m+1}}{(m-1)!}\int_s^t (\tau-s)^{m-1}\, d\tau \\ &= \frac{C^{m+1}}{(m-1)!}\int_0^{t-s} \tau^{m-1}\, d\tau \\ &= \frac{C^{m+1}}{(m-1)!}\frac{1}{m}(t-s)^m\end{aligned}$$

which gives (2.24).

Secondly,

$$\begin{aligned}\left\|\sum_{m=1}^{\infty} R_m(t,s)\right\| \leq \sum_{m=1}^{\infty} \|R_m(t,s)\| \leq \sum_{m=1}^{\infty} \frac{C^m}{(m-1)!}(t-s)^{m-1} &\\ = C\sum_{m=1}^{\infty} \frac{(C(t-s))^{m-1}}{(m-1)!} &\\ = C\sum_{m=0}^{\infty} \frac{(C(t-s))^{m}}{m!} &\\ = Ce^{C(t-s)} &\end{aligned}$$

where we used the power series representation of the exponential function. □

*Remark* 2.12. Note that the integral defining $R_{m+1}(t,s)$ is an improper integral whose existence is an immediate consequence of (2.24).

By construction, we have

$$\begin{aligned}\sum_{m=1}^{\infty} R_m(t,s) &= R_1(t,s) + \sum_{m=1}^{\infty}\int_s^t R_1(t,\tau)R_m(\tau,s)\, d\tau \\ &= R_1(t,s) + \int_s^t R_1(t,\tau)\sum_{m=1}^{\infty} R_m(\tau,s)\, d\tau \qquad (2.26)\end{aligned}$$

where we used (2.24), (2.25) and [17, Analysis 3, Korrolar 3.13] to interchange the summation and the integral. Thus, we see that

$$R(t,s) = \sum_{m=1}^{\infty} R_m(t,s)$$

is the solution of the integral equation (2.7). (2.25) implies that the series $\sum_{m=1}^{\infty} R_m(t,s)$ converges uniformly in $\mathcal{L}(X)$ for $0 \leq s \leq t \leq T$. Thus $R(t,s)$ is uniformly continuous in $\mathcal{L}(X)$.

Recall that we defined $U(t,s)$ by (2.4). The strong continuity of $U(t,s)$ for $0 \leq s \leq t \leq T$ then follows readily from the strong continuity of $S_\tau(s)$, (2.14) and (2.25). Moreover, with (2.25)

$$\begin{aligned}\|U(t,s)\| &\leq \|S_s(t-s)\| + \int_s^t \|S_\tau(t-\tau)\|\ \ \|R(\tau,s)\|\, d\tau \\ &\leq C_1 + C_2\int_s^t e^{C(\tau-s)}\, d\tau \\ &\leq C\,. \qquad (2.27)\end{aligned}$$

Therefore $(E_1)$ is satisfied.

### 2.2.2 Differentiability

Again we start with some preliminaries:

**Lemma 2.13** ([15, Lemma 5.6.4]). *For every $\beta$, $0 < \beta \leq 1$, there is a constant $C_\beta$ such that*

$$\|R_1(t,s) - R_1(\tau,s)\| \leq C_\beta (t-\tau)^\beta (\tau - s)^{-\beta} \tag{2.28}$$

*for every $0 \leq s < \tau < t \leq T$.*

*Proof.* We have

$$\begin{aligned} R_1(t,s) - R_1(\tau,s) &= (A(s) - A(t))S_s(t-s) - (A(s) - A(\tau))S_s(\tau - s) \\ &= (A(\tau) - A(t))S_s(t-s) + (A(s) - A(\tau))(S_s(t-s) - S_s(\tau - s)) \,. \end{aligned}$$

Estimating the first term with (2.17), we obtain

$$\begin{aligned} \left\|(A(\tau) - A(t))S_s(t-s)\right\| &\leq C(t-\tau)(t-s)^{-1} \\ &\leq C(t-\tau)(\tau-s)^{-1} \,. \end{aligned}$$

The second term can be bounded with (2.9) and (2.18)

$$\begin{aligned} \left\|(A(s) - A(\tau))(S_s(t-s) - S_s(\tau-s))\right\| &\leq \left\|(A(s) - A(\tau))A(s)^{-1}\right\| \left\|A(s)(S_s(t-s) - S_s(\tau-s))\right\| \\ &\leq L(\tau - s)\frac{C}{(t-s)(\tau-s)} \, |t - s - (\tau - s)| \\ &\leq C(t-s)^{-1}(t-\tau) \\ &\leq C(t-\tau)(\tau-s)^{-1} \,. \end{aligned}$$

Thus

$$\|R_1(t,s) - R_1(\tau,s)\| \leq C_1 \, (t-\tau)(\tau-s)^{-1} \,.$$

On the other hand we have by (2.22)

$$\|R_1(t,s) - R_1(\tau,s)\| \leq C_2$$

Interpolating the two estimates with

$$x \leq a, \; x \leq b \implies x = x^\beta x^{1-\beta} \leq a^\beta b^{1-\beta}$$

we find

$$\begin{aligned} \|R_1(t,s) - R_1(\tau,s)\| &\leq C_1^\beta \left[(t-\tau)(\tau-s)^{-1}\right]^\beta C_2^{1-\beta} \\ &\leq C_\beta (t-\tau)^\beta (\tau-s)^{-\beta} \,. \end{aligned}$$

□

**Corollary 2.14** ([15, Cor 5.6.5]). *For every $\beta$, $0 < \beta < 1$, there is a constant $C_\beta$ such that*

$$\|R(t,s) - R(\tau,s)\| \leq C_\beta (t-\tau)^\beta (\tau-s)^{-\beta} \tag{2.29}$$

*for $0 \leq s < \tau < t \leq T$.*

*Proof.* We know from the integral equation (2.7) that

$$R(t,s) - R(\tau,s) = R_1(t,s) - R_1(\tau,s) + \int_\tau^t R_1(t,\sigma)R(\sigma,s)\,d\sigma + \int_s^\tau \big(R_1(t,\sigma) - R_1(\tau,\sigma)\big)R(\sigma,s)\,d\sigma \,.$$

Each of the different terms then satisfies the same bound:

- With (2.28)
$$\|R_1(t,s) - R_1(\tau,s)\| \le C_\beta (t-\tau)^\beta (t-s)^{-\beta}\,.$$
- Applying (2.25) and (2.22) to the second term yields
$$\begin{aligned}\left\|\int_\tau^t R_1(t,\sigma)R(\sigma,s)\,d\sigma\right\| &\le \int_\tau^t C\,d\tau\\ &\le C(t-\tau)\\ &\le C_\beta (t-\tau)^\beta(\tau-s)^{-\beta}\,,\end{aligned}$$
where the last inequality follows from the lower boundedness of $(t-\tau)^\beta(\tau-s)^{-\beta}$ on $(s,t)$.
- Again with (2.28) and (2.22), we get
$$\begin{aligned}\left\|\int_s^\tau (R_1(t,\sigma) - R_1(\tau,\sigma))R(\sigma,s)\,d\sigma\right\| &\le C(t-\tau)^\beta \int_s^\tau (\tau-\sigma)^{-\beta}\,d\sigma\\ &= C(t-\tau)^\beta \int_0^{\tau-s} \sigma^{-\beta}\,d\sigma\\ &= C(t-\tau)^\beta \frac{1}{1-\beta}(\tau-s)^{1-\beta}\\ &\le C_\beta (t-\tau)^\beta(\tau-s)^{-\beta}\end{aligned}$$

□

**Lemma 2.15** ([15, Lemma 5.6.6])**.** *For every $x \in X$ we have*
$$\lim_{\varepsilon\to 0} S_t(\varepsilon)x = x \tag{2.30}$$
*uniformly in $0 \le t \le T$.*

*Proof.* For $v \in D$ we have with Theorem A.2
$$\begin{aligned}v - S_t(\varepsilon)v &= \int_0^\varepsilon A(t)S_t(\sigma)v\,d\sigma\\ &= \int_0^\varepsilon S_t(\sigma)A(t)v\,d\sigma\,.\end{aligned}$$
Using $\left\|A(t)A(0)^{-1}\right\| \le C$, which can be seen with $(A_3)$, we get
$$\begin{aligned}\|v - S_t(\varepsilon)v\| &\le \int_0^\varepsilon \|S_t(\sigma)\|\;\left\|A(t)A(0)^{-1}\right\|\;\|A(0)v\|\,d\sigma\\ &\le \varepsilon C\;\|A(0)v\|\;.\end{aligned}$$
Thus (2.30) holds for every $v \in D$. Since $D$ is dense in $X$ and $\|S_t(s)\| \le C$ for $s,t \in [0,T]$ the result for every $x \in X$ follows by [18, Cor 3.5] which is an application of the principle of uniform boundedness. □

We want to prove the differentiability of the evolution system
$$U(t,s) = S_s(t-s) + W(t,s) = S_s(t-s) + \int_s^t S_\tau(t-\tau)\,R(\tau,s)\,d\tau\,.$$
Since $S_s(t-s)$ is differentiable for $t > s$ with
$$\partial_t S_s(t-s) = -A(s)S_s(t-s)$$

being a bounded linear operator, continuous in $\mathcal{L}(X)$, it suffices to show the differentiability of $W(t,s)$.

To this end we set

$$W_\varepsilon(t,s) := \int_s^{t-\varepsilon} S_\tau(t-\tau)R(\tau,s)\,d\tau \tag{2.31}$$

for $0<\varepsilon<t-s$. By the continuity of the integral, we have $W_\varepsilon(t,s)\to W(t,s)$ as $\varepsilon\to 0$. We are going to prove the differentiability of $W_\varepsilon(t,s)$ and then construct the derivative of $W(t,s)$ as the limit of the former as $\varepsilon\to 0$.

$W_\varepsilon(t,s)$ is differentiable in $t$ and with an application of the differentiation rule mentioned in Remark 2.6

$$\partial_t W_\varepsilon(t,s) = S_{t-\varepsilon}(\varepsilon)R(t-\varepsilon,s) - \int_s^{t-\varepsilon} A(\tau)S_\tau(t-\tau)R(\tau,s)\,d\tau\,. \tag{2.32}$$

Since $A(t)S_t(t-\tau) = \partial_\tau S_t(t-\tau)$ we have

$$\int_s^{t-\varepsilon} A(t)S_t(t-\tau)R(t,s)\,d\tau = \int_s^{t-\varepsilon} \partial_\tau S_t(t-\tau)R(t,s)\,d\tau = (S_t(\varepsilon)-S_t(t-s))R(t,s)$$

and thus

$$\begin{aligned}
\partial_t W_\varepsilon(t,s) &= S_{t-\varepsilon}(\varepsilon)R(t-\varepsilon,s) - \int_s^{t-\varepsilon} A(\tau)S_\tau(t-\tau)R(\tau,s)\,d\tau \\
&\quad + \int_s^{t-\varepsilon} A(t)S_t(t-\tau)R(t,s)\,d\tau + (S_t(t-s)-S_t(\varepsilon))R(t,s) \\
&= S_{t-\varepsilon}(\varepsilon)R(t-\varepsilon,s) + \int_s^{t-\varepsilon} (A(t)S_t(t-\tau)-A(\tau)S_\tau(t-\tau))R(\tau,s)\,d\tau \\
&\quad + \int_s^{t-\varepsilon} A(t)S_t(t-\tau)(R(t,s)-R(\tau,s))\,d\tau + (S_t(t-s)-S_t(\varepsilon))R(t,s)\,.
\end{aligned} \tag{2.33}$$

Now we are able to derive a bound for $\|\partial_t W_\varepsilon(t,s)\|$:

- From (2.14) and (2.22) we can bound the first and the last term on the right-hand side of (2.33):

$$\|S_{t-\varepsilon}(\varepsilon)R(t-\varepsilon,s)\| + \left\|(S_t(t-s)-S_t(\varepsilon))R(t,s)\right\| \le C\,.$$

- If we use (2.17) and (2.19), we get

$$\begin{aligned}
\|A(t)S_t(t-\tau)-A(\tau)S_\tau(t-\tau)\| &\le \left\|(A(t)-A(\tau))S_t(t-\tau)\right\| + \left\|A(\tau)(S_t(t-\tau)-S_\tau(t-\tau))\right\| \\
&\le \frac{C}{t-\tau}\,|t-\tau| + \frac{C}{t-\tau}\,|t-\tau| \\
&\le C
\end{aligned}$$

  and therefore with (2.25)

$$\begin{aligned}
\left\|\int_s^{t-\varepsilon}(A(t)S_t(t-\tau)-A(\tau)S_\tau(t-\tau))R(\tau,s)\,d\tau\right\| &\le C\int_s^{t-\varepsilon} e^{C(\tau-s)}\,d\tau \\
&\le C\,.
\end{aligned}$$

- Finally, from (2.15) and choosing a $0<\beta<1$ in (2.29) we have

$$\left\|\int_s^{t-\varepsilon} A(t)S_t(t-\tau)(R(t,s)-R(\tau,s))\,d\tau\right\| \le C\int_s^{t-\varepsilon}(t-\tau)^{\beta-1}(\tau-s)^{-\beta}\,d\tau$$

  The integral on the right-hand side is bounded since $-1<\beta-1<0$ and $-1<-\beta<0$ give the integrability of $(t-\tau)^{\beta-1}(\tau-s)^{-\beta}$ over $[s,t]$.

By combining these estimates we get
$$\|\partial_t W_\varepsilon(t,s)\| \le C \tag{2.34}$$
where $C > 0$ is independent of $\varepsilon$. If we pass to the limit on the right-hand side of (2.33) we see with Lemma 2.15 that $\partial_t W_\varepsilon(t,s)$ converges strongly as $\varepsilon \to 0$. Denoting this limit by $W'(t,s)$, we get

$$\begin{aligned} W'(t,s) = S_t(t-s)R(t,s) + \int_s^t (A(t)S_t(t-\tau) - A(\tau)S_\tau(t-\tau))R(\tau,s)\,d\tau \\ + \int_s^t A(t)S_t(t-\tau)(R(t,s) - R(\tau,s))\,d\tau\,. \end{aligned} \tag{2.35}$$

The uniform continuity of $R(t,s)$ and the strong continuity of $A(t)S_t(s)$ (cf. (2.15)) then give the strong continuity of $W'(t,s)$ for $0 \le s < t \le T$. Moreover, given the uniform bound (2.34), we get

$$\left\|W'(t,s)\right\| \le C \tag{2.36}$$

and with
$$W_\varepsilon(t_2,s) - W_\varepsilon(t_1,s) = \int_{t_1}^{t_2} \partial_\tau W_\varepsilon(\tau,s)\,d\tau$$
as we pass to the limit for $\varepsilon \to 0$
$$W(t_2,s) - W(t_1,s) = \int_{t_1}^{t_2} W'(\tau,s)\,d\tau$$
where $t_2 > t_1 > s + \varepsilon$. Thus $W(t,s)$ is strongly continuously differentiable with respect to $t$ for $0 \le s < t \le T$ and the above equation gives
$$\partial_t W(t,s) = W'(t,s)\,.$$

Now, since we have shown the strongly continuous differentiability of $W(t,s)$, it follows readily that
$$\partial_t U(t,s) = -A(s)S_s(t-s) + \partial_t W(t,s)$$
is strongly continuous and by (2.15) and (2.36), we obtain
$$\|\partial_t U(t,s)\| \le \frac{C}{t-s}\,.$$

We set
$$U_\varepsilon(t,s) := S_s(t-s) + W_\varepsilon(t,s)\,.$$
Since
$$S_s(t-s) : X \to D, \qquad t-s > 0\,,$$
and $S_\tau(t-\tau)R(\tau,s) : X \to D$ is strongly continuous for $\tau \in [s, t-\varepsilon]$ with respect to the graph norm of $A(r)$ for some $r \in [0,T]$ (this can be checked by using the results from Lemma 2.9) implies
$$W_\varepsilon(t,s)x = \int_s^{t-\varepsilon} S_\tau(t-\tau)R(\tau,s)x\,d\tau \in D\,,$$
it follows that $U_\varepsilon(t,s) : X \to D$. Moreover by (2.31) and (2.32)
$$\partial_t U_\varepsilon(t,s) + A(t)U_\varepsilon(t,s) = S_{t-\varepsilon}(\varepsilon)R(t-\varepsilon,s) - R_1(t,s) - \int_s^{t-\varepsilon} R_1(t,\tau)R(\tau,s)\,d\tau\,. \tag{2.37}$$

As $\varepsilon \to 0$, the right-hand side of (2.37) tends strongly to zero. Using $\partial_t W_\varepsilon(t,s) \to \partial_t W(t,s)$ strongly, we see $\partial_t U_\varepsilon(t,s) \to \partial_t U(t,s)$ strongly. Thus it follows from (2.37) that $A(t)U_\varepsilon(t,s)$ converges strongly as $\varepsilon \to 0$. Since $U_\varepsilon(t,s)x \to U(t,s)x$ for $x \in X$, the closedness of $A(t)$ implies
$$U(t,s)x \in D$$

and

$$A(t)U_\varepsilon(t,s)x \to A(t)U(t,s)x\,.$$

As we pass to the strong limit in (2.37), we obtain

$$\partial_t U(t,s) + A(t)U(t,s) = 0$$

for $t > s$. Hence we have proven (2.10) and (2.11) from Theorem 2.8.

We turn now to the proof of (2.12). Let $x \in X$ and consider the function $g(s) = S_t(t-s)U(s,\tau)A(\tau)^{-1}x$ for $0 \le \tau < s < t \le T$. Since (2.10) was proven above and $S_t(t-s)$ is differentiable in $s$, the derivative of $g$ exists and

$$g'(s) = S_t(t-s)(A(t) - A(s))U(s,\tau)A(\tau)^{-1}x\,.$$

Now we integrate $g'$ from $\tau$ to $t$

$$U(t,\tau)A(\tau)^{-1}x - S_t(\tau - s)A(\tau)^{-1}x = \int_\tau^t S_t(t-s)(A(t) - A(s))U(s,\tau)A(\tau)^{-1}x\,ds$$

and apply $A(t)$ to find

$$Z(t,\tau) = A(t)S_t(t-\tau)A(\tau)^{-1}x + \int_\tau^t Y(t,s)Z(s,\tau)x\,ds \tag{2.38}$$

where

$$Y(t,s) := A(t)S_t(t-s)(A(t) - A(s))A(s)^{-1}$$

and

$$Z(t,\tau) := A(t)U(t,\tau)A(\tau)^{-1}\,.$$

We can estimate the first term on the right-hand side of (2.38) by using (2.9) and (2.14)

$$\begin{aligned}\left\|A(t)S_t(t-\tau)A(\tau)^{-1}\right\| &= \left\|S_t(t-\tau)A(t)A(\tau)^{-1}\right\| \\ &\le \|S_t(t-\tau)\|\ \left\|A(t)A(\tau)^{-1}\right\| \\ &\le C_1\end{aligned}$$

and $Y(t,s)$ by (2.9) and (2.15),

$$\begin{aligned}\|Y(t,s)\| &\le \|A(t)S_t(t-s)\|\ \left\|(A(t) - A(s))A(s)^{-1}\right\| \\ &\le \frac{C}{t-s}L(t-s) \\ &\le C_2\,.\end{aligned}$$

Applying these to (2.38) we find

$$\|Z(t,\tau),x\| \le C_1\ \|x\| + C_2 \int_\tau^t \|Z(s,\tau)x\|\ ds\,.$$

The Gronwall estimate from Proposition A.5 then implies

$$\|Z(t,\tau),x\| \le e^{C_2 t}C_1\ \|x\|$$

whence

$$\|Z(t,\tau)\| \le \left\|A(t)U(t,\tau)A(\tau)^{-1}\right\| \le C\,.$$

This completes the proof of $(E_2)$.

### 2.2.3 Uniqueness

The main part of this section is considered with proving $(E_3)$, since the uniqueness of the evolution system $U(t,s)$ will be an immediate consequence of $(E_1)$, $(E_2)$ and $(E_3)$.

**Lemma 2.16.** *Let $A(t)v$ be continuously differentiable on $[0,T]$ for every $v \in D$. Then the evolution system $U(t,s)$, as constructed above, satisfies $(E_3)$.*

*Proof.* First observe that $\partial_t A(t)A(0)^{-1}$ is uniformly bounded on $[0,T]$: For a fixed $t_0 \in [0,T]$, we approximate $\partial_t A(t_0)A(0)^{-1}$ by a sequence of difference quotients $D_n(t_0)$ that are bounded by the closed graph theorem. Since $D_n(t_0)x$ converges for arbitrary $x \in X$, Banach-Steinhaus ([18, Corollary 3.5]) gives $\sup_{n\in\mathbb{N}} \|D_n(t_0)\| \leq C(t_0)$ implying

$$\left\|\partial_t A(t_0)A(0)^{-1}x\right\| \leq \sup_{n\in\mathbb{N}} \|D_n(t_0)x\| \leq C(t_0)\,\|x\| \,.$$

Now fix $x \in X$. The continuity of $[0,T] \to \mathbb{R}$, $t \mapsto \left\|\partial_t A(t)A(0)^{-1}x\right\|$ on the compact set $[0,T]$ gives an upper bound $\left\|\partial_t A(t)A(0)^{-1}x\right\| \leq C_x$. Applying the principle of uniform boundedness, we find the uniform boundedness of $\partial_t A(t)A(0)^{-1}$.

Moreover, with the resolvent identity for two operators $A, B$

$$R(\lambda, A) - R(\lambda, B) = R(\lambda, A)(A - B)R(\lambda, B)\,,$$

we have for every $\lambda \in \Sigma_\theta$, that $R(\lambda, A(s))$ is differentiable with respect to $s$ and

$$\partial_s R(\lambda, A(s)) = R(\lambda, A(s))\partial_s A(s)R(\lambda, A(s))\,. \tag{2.39}$$

The uniform boundedness of $\partial_s A(s)R(\lambda, A(s))$ follows as at the beginning of this proof and with $(A_2)$ and (2.39) we deduce that

$$\|\partial_s R(\lambda, A(s))\| \leq \|R(\lambda, A(s))\|\,\|\partial_s A(s)R(\lambda, A(s))\| \leq \frac{c}{|\lambda| + 1} \tag{2.40}$$

for $\lambda \in \Sigma_\theta$. The assumptions $(A_1)$ and $(A_2)$ give with Theorem A.3 the representation

$$S_s(t-s) = \frac{1}{2\pi i}\int_\Gamma e^{-\lambda(t-s)}R(\lambda, A(s))\,d\lambda$$

with $\Gamma$ being a smooth path in $\Sigma_\theta$ that connects $\infty e^{-i\theta}$ to $\infty e^{i\theta}$. If $t - s > 0$, our supplementary assumption then implies the strong differentiability of $S_s(t-s)$ in $s$ and

$$\begin{aligned}\partial_s S_s(t-s) &= \frac{1}{2\pi i}\int_\Gamma \lambda e^{-\lambda(t-s)}R(\lambda, A(s))\,d\lambda + \frac{1}{2\pi i}\int_\Gamma e^{-\lambda(t-s)}\partial_s R(\lambda, A(s))\,d\lambda \\ &= -\partial_t S_s(t-s) + \frac{1}{2\pi i}\int_\Gamma e^{-\lambda(t-s)}\partial_s R(\lambda, A(s))\,d\lambda\,.\end{aligned}$$

Now we prove $(E_3)$: First we construct an operator valued function $(t,s) \mapsto V(t,s)$ satisfying

$$\begin{cases}\partial_s V(t,s)v = V(t,s)A(s)v & \text{for } 0 \leq s \leq t \leq T,\, v \in D\\ V(t,t) = \mathrm{Id}\end{cases} \tag{2.41}$$

and then we prove it coincides with $U(t,s)$.

The construction of $V(t,s)$ is done the same way we constructed $U(t,s)$ above. Set

$$Q_1(t,s) := (\partial_t + \partial_s)S_s(t-s) = \frac{1}{2\pi i}\int_\Gamma e^{-\lambda(t-s)}\partial_s R(\lambda, A(s))\,d\lambda\,,$$

where the second equality follows from the above considerations. Using (2.40), we are able to estimate $Q_1(t, s)$

$$\|Q_1(t,s)\| = \left\| \frac{1}{2\pi i} \int_\Gamma e^{-\lambda(t-s)} \partial_s R(\lambda, A(s))\, d\lambda \right\| \leq C\,,$$

in the typical way (cf. the proof of [15, Theorem 1.7.7]). We continue with solving the integral equation

$$Q(t,s) = Q_1(t,s) + \int_s^t Q(t,\tau) Q_1(\tau,s)\, d\tau\,. \tag{2.42}$$

by successive approximation, as it was done to obtain $R(t, s)$. Again, we find that the solution of (2.42) satisfies

$$\|Q(t,s)\| \leq C\,.$$

We set

$$V(t,s) = S_s(t-s) + \int_s^t Q(t,\tau) S_s(\tau - s)\, d\tau$$

and find that $\|V(t, s)\| \leq C$, since $\|S_s(t - s)\| \leq C$ and $\|Q(t, s)\| \leq C$. Moreover, for $v \in D$, $V(t, s)v$ is differentiable in $s$ and with

- $Q_1(t, s) = (\partial_t + \partial_s)S_s(t - s) = \partial_s S_s(t - s) - A(s)S_s(t - s)$
- By using the above relation:

$$\begin{aligned}\partial_s \int_s^t Q(t,\tau) S_s(\tau-s)\, d\tau &= -\partial_s \int_t^s Q(t,\tau) S_s(\tau-s)\, d\tau \\ &= Q(t,s)S_s(0) - \int_t^s Q(t,\tau)\partial_s S_s(\tau-s)\, d\tau \\ &= Q(t,s) + \int_s^t Q(t,\tau)Q_1(\tau,s)\, d\tau + \int_s^t Q(t,\tau)A(s)S_s(\tau-s)\, d\tau\end{aligned}$$

- For $v \in D$, we have $A(s)S_s(\tau - s)v = S_s(\tau - s)A(s)v$

we find

$$\begin{aligned}&\partial_s V(t,s)v \\ &\quad = Q_1(t,s)v + A(s)S_s(t-s)v + Q(t,s)v + \int_s^t Q(t,\tau)Q_1(\tau,s)v\, d\tau + \int_s^t Q(t,\tau)A(s)S_s(\tau-s)v\, d\tau \\ &\quad = Q_1(t,s)v + Q(t,s)v + \int_s^t Q(t,\tau)Q_1(\tau,s)v\, d\tau + V(t,s)A(s)v \\ &\quad = V(t,s)A(s)v\,,\end{aligned}$$

where the last equality follows from (2.42). Thus $V(t, s)$ is a solution of (2.41), since $V(t,t) = \mathrm{Id}$ is given by definition.

Now let $x \in X$ and $s < r < t$. Then the function $r \mapsto V(t,r)U(r, s)x$ is differentiable in $r$ and

$$\partial_r V(t,r)U(r,s)x = V(t,r)A(r)U(r,s)x - V(t,r)A(r)U(r,s)x = 0\,.$$

Thence $V(t,r)U(r, s)x$ is independent of $r$ for $s < r < t$. Considering the limits $r \downarrow s$ and $r \uparrow t$, then gives $V(t, s)x = U(t, s)x$ for every $x \in X$. Therefore $U(t, s) = V(t, s)$ satisfies

$$\partial_s U(t,s)v = U(t,s)A(s)v \tag{2.43}$$

for $v \in D$ as desired. □

Now we need to transfer the above result to our case, where $A(t)v$ is only Lipschitz continuous instead of differentiable for $v \in D$. Since we want to use Lemma 2.16, our approach is clear: We approximate $A(t)$ by a sequence of operators $A_n(t)$ for which $A_n(t)A_n(0)^{-1}$ is continuously differentiable. Afterward the corresponding evolution system $U_n(t,s)$, which satisfies $(E_3)$ by construction, will be proven to approximate $U(t,s)$. By transferring $(E_3)$ through this approximation process we obtain the result.

Let $\rho(t) \geq 0$ be a continuously differentiable real-valued function on $\mathbb{R}$ satisfying $\rho(t) = 0$ for $|t| \geq 1$ and $\int_{-\infty}^{\infty} \rho(t)\, dt = 1$. Using $\rho$ as a prototype, we define a sequence of functions $\rho_n$ by

$$\rho_n(t) = n\rho(nt), \qquad n \in \mathbb{N}\,.$$

We use the these functions as mollifiers: Extend $A(t)$ to all of $\mathbb{R}$ by defining $A(t) = A(0)$ for $t < 0$ and $A(t) = A(T)$ for $t < T$. For $v \in D$ we set

$$A_n(t)v = \int_{-\infty}^{\infty} \rho_n(t-\sigma)A(\sigma)v\, d\sigma = \int_{-\infty}^{\infty} \rho_n(t)A(t-\sigma)v\, d\sigma\,.$$

The first characterization shows that $A_n(t)v$ thus defined is continuously differentiable on $[0,T]$. We continue by showing that $A_n(t)$ satisfy the conditions $(A_1)$ - $(A_3)$.

- The domain $D(A_n(t)) = D$ by definition, thus $(A_1)$ is fulfilled.
- Observe that $\int_{-\infty}^{\infty} \rho_n(t)\, dt = 1$ for all $n \in \mathbb{N}$ and $\text{supp}(\rho_n) \subset [-1/n, 1/n]$. Now let $\lambda \in \Sigma_\theta$, then

$$\begin{aligned} x - (\lambda - A_n(t))R(\lambda, A(t))x &= (\lambda - A(t))R(\lambda, A(t))x - (\lambda - A_n(t))R(\lambda, A(t))x \\ &= -(A(t) - A_n(t))R(\lambda, A(t))x \\ &= \int_{-\infty}^{\infty} \rho_n(t-\tau)\big(A(\tau) - A_n(t)\big)R(\lambda, A(t))x\, d\tau\,. \end{aligned}$$

For $|t-\tau| \leq 1/n$, we have by (2.8) and (2.9)

$$\left\|(A(t) - A(\tau))R(\lambda, A(t))\right\| \leq \left\|(A(t) - A(\tau))A(0)^{-1}\right\| \|A(0)R(\lambda, A(t))\| \leq Cn^{-1}\,.$$

Inserting this result in the above equation gives

$$\left\|x - (\lambda - A_n(t))R(\lambda, A(t))x\right\| \leq Cn^{-1} \|x\| \tag{2.44}$$

and in particular, by choosing $\lambda = 0$,

$$\left\|(A(t) - A_n(t))A(t)^{-1}\right\| \leq Cn^{-1}\,. \tag{2.45}$$

Let $v \in D$. Choosing $x = (\lambda - A(t))v$ in (2.44) and applying the reverse triangle inequality, we get

$$\pm \left\|(\lambda - A(t))v\right\| \mp \left\|(\lambda - A_n(t))v\right\| \leq \left\|(\lambda - A(t))v - (\lambda - A_n(t))v\right\| \leq Cn^{-1} \left\|(\lambda - A(t))v\right\|$$

which implies

$$(1 - Cn^{-1}) \|(\lambda - A(t))v\| \leq \|(\lambda - A_n(t))v\| \leq (1 + Cn^{-1}) \|(\lambda - A(t))v\|\,. \tag{2.46}$$

Given $n$ sufficiently large so that $Cn^{-1} < 1$ and $\lambda \in \Sigma_\theta$, we see with (2.46) that the graph norms of $\lambda - A(t)$ and $\lambda - A_n(t)$ are equivalent. With $\lambda - A(t)$ being closed, [19, Lemma 1.5] this implies that $\lambda - A_n(t)$ is also closed. Using $\text{range}(\lambda I - A_n(t)) = X$, (2.46) implies

$$\|(\lambda - A(t))R(\lambda, A_n(t))x\| \leq c\, \|x\|$$

and thus, by commutation of the operators,

$$\|R(\lambda, A_n(t))x\| \leq c\, \|R(\lambda, A(t))x\| \leq \frac{C}{|\lambda| + 1}\,.$$

Hence $(A_2)$ is satisfied.

- Again, we choose $n$ sufficiently large so that $Cn^{-1} < 1$. Then by (2.45)

$$\left\| \mathrm{Id} - A_n(t)A(t)^{-1} \right\| \le 1$$

and thus a Neumann series representation yields

$$A(t)A_n(t)^{-1} = (\mathrm{Id} - (\mathrm{Id} - A_n(t)A(t)^{-1}))^{-1} = \sum_{k=0}^{\infty} \left[ \mathrm{Id} - A_n(t)A(t)^{-1} \right]^k .$$

and

$$\left\| A(t)A_n(t)^{-1} \right\| \le C .$$

From the definition of $A_n(t)$ and (2.9) it follows that

$$\begin{aligned} \left\| (A_n(t) - A_n(s))A(\tau)^{-1} v \right\| &\le \int_{-\infty}^{\infty} \rho_n(\sigma) \left\| (A(t-\sigma) - A(s-\sigma))A(\tau)^{-1} v \right\| d\sigma \\ &\le C \ |t-s| \ \|v\| \end{aligned}$$

and thus

$$\begin{aligned} \left\| (A_n(t) - A_n(s))A_n(\tau)^{-1} v \right\| &\le \left\| (A_n(t) - A_n(s))A(\tau)^{-1} \right\| \left\| A(\tau)A_n(\tau)^{-1} \right\| \\ &\le C \ |t-s| \ . \end{aligned}$$

Hence $(A_3)$ is also satisfied by $A_n(t)$.

Considering the first part of the proof, with $A_n(t)$ satisfying $(A_1)$ - $(A_3)$, there is an operator valued function $(t, s) \mapsto U_n(t, s)$ satisfying $\|U_n(t, s)\| \le C$. Observe that this bound holds independent of $n$, since the deduction (2.27) only uses relations that we have proven above to hold independently of $n$. Moreover, we know

$$\partial_t U_n(t, s) = -A_n(t)U_n(t, s)$$

for $0 \le s < t \le T$. Since $A_n(t)v$ is continuously differentiable in $t$ for $v \in D$ by construction, we get by Lemma 2.16

$$\partial_t U_n(t, s) = U_n(t, s)A_n(s)v . \tag{2.47}$$

(2.47) and the already shown properties of $U(t, s)$ assure the differentiability of the function $r \mapsto U_n(t, r)U(r, s)v$ for every $v \in D$ and

$$\begin{aligned} U(t, s)v - U_n(t, s) &= \int_s^t \partial_r [U_n(t, r)U(r, s)v] \, dr \\ &= \int_s^t U_n(t, r)(A_n(r) - A(r))U(r, s)v \, dr \\ &= \int_s^t U_n(t, r)(A_n(r) - A(r))A(r)^{-1}A(r)U(r, s)A(s)^{-1}A(s)v \, dr . \end{aligned} \tag{2.48}$$

We now apply (2.45) and (2.12) to estimate (2.48) and obtain

$$\begin{aligned} \|U(t, s)v - U_n(t, s)v\| &\le \int_s^t \|U_n(t, r)\| \left\| (A_n(r) - A(r))A(r)^{-1} \right\| \left\| A(r)U(r, s)A(s)^{-1} \right\| \|A(s)v\| \, dr \\ &\le Cn^{-1}(t-s) \ \|A(s)v\| \\ &\le Cn^{-1} \ \|A(0)v\| \ . \end{aligned}$$

Thus $U_n(t,s)v$ converges uniformly in $t, s$ to $U(t,s)v$. Using the density of $D$ in $X$ we can apply [18, Corollary 3.5] and get the uniform convergence $U_n(t,s)x \to U(t,s)x$ in $t$ and $s$ for every $x \in X$. For $v \in D$, we obtain from

$$\begin{aligned}\|U_n(t,s)A_n(s)v - U(t,s)A(s)v\| &\leq \left\|U_n(t,s)(A_n(s) - A(s))A(0)^{-1}A(0)v\right\| + \left\|(U_n(t,s) - U(t,s))A(s)v\right\| \\ &\leq \|U_n(t,s)\| \left\|(A_n(s) - A(s))A(0)^{-1}\right\| \|A(0)v\| \\ &\quad + \left\|(U_n(t,s) - U(t,s))A(s)v\right\| \\ &\leq Cn^{-1} \|A(0)v\| + \left\|(U_n(t,s) - U(t,s))A(s)v\right\|\end{aligned}$$

that $U_n(t,s)A_n(t,s)v \to U(t,s)A(s)v$ uniformly in $t$ and $s$. Now let $r < s < t$ and $v \in D$. It is clear that

$$U_n(t,s)v - U_n(t,r)v = \int_r^s \partial_\sigma U_n(t,\sigma)v \, d\sigma = \int_s^t U_n(t,\sigma)A_n(\sigma)v \, d\sigma$$

and as we take the limit $n \to \infty$ the uniform convergence gives

$$U(t,s)v - U(t,r)v = \int_r^s U(t,\sigma)A(\sigma)v \, d\sigma \, .$$

Thus (2.43) holds in general.

At this point, we need to show the uniqueness of $U(t,s)$ and $U(t,r)U(r,s) = U(t,s)$ for $0 \leq s \leq r \leq t \leq T$, to complete the proof of Theorem 2.8. These are consequences of the next Theorem:

**Theorem 2.17** ([15, Theorem 6.8]). *Let $A(t)$, $0 \leq t \leq T$ satisfy the conditions $(A_1)$-$(A_3)$. For every $0 \leq s < T$ and $x \in X$ the initial value problem*

$$\begin{cases} \frac{d}{dt}u(t) + A(t)u(t) = 0 & 0 \leq s < t \leq T \\ u(s) = x \end{cases} \tag{2.49}$$

*has a unique solution $u$ given by $u(t) = U(t,s)x$ where $U(t,s)$ is the evolution system constructed above.*

*Proof.* (2.10) implies that $u(t) = U(t,s)x$ is a solution of the above problem. Now let $v(t)$ be a second solution. Given that $v(r) \in D$ for every $r > s$, we know the function $r \mapsto U(t,r)v(r)$ to be differentiable and

$$\partial_r U(t,r)v(r) = U(t,r)A(r)v(r) - U(t,r)A(r)v(r) = 0 \, .$$

Thus $U(t,r)v(r)$ is constant for $s < r < t$. Its continuity on $s \leq r \leq t$ allows to consider the limits $r \to t$ and $r \to s$ and we get $U(t,s)x = v(t)$. This was the claim. □

The uniqueness shown in the above Theorem, then yields

$$U(t,s)x = U(t,r)U(r,s)x$$

for $x \in X$ and $0 \leq s \leq t \leq T$. Thus we have shown that $U(t,s)$ satisfies $(E_1)$ - $(E_3)$ and the proof of Theorem 2.8 is closed. □

## 2.3 A Remark on Evolution Problems based on PDEs on Evolving domains

Since we want to apply the theory, we developed in this chapter, to partial differential equations, this seems worth some notes. Recall the PDE (1.11):

$$\begin{cases} D_t u + u \operatorname{div} \underline{v} = \alpha \Delta u & \text{in } \mathcal{N}_T \\ u(t,x) = 0 & x \in \partial\Omega_t \\ u(0,\cdot) = u_0 & \text{in } \Omega_0 \end{cases}$$

Normally the standard semigroup and evolution problem theory, deals with PDEs by understanding the problem as the description of a trajectory of the solution $u$ in some function space $X$. I.e. $u : [0,T] \to X$ for some Banach space $X$ and

$$\begin{cases} \frac{d}{dt}u(t) = A(t)u(t) & 0 \leq s < t \leq T \\ u(s) = x\,. \end{cases}$$

In our case, it is not that easy: We require the solution $u$ of (1.11) to satisfy

$$u(t,\cdot) \in H_0^1(\Omega_t) \subset L^2(\Omega_t), \qquad t \in [0,T].$$

In fact we want the solution for each $t \in [0,T]$ to lie in a different function space. Since function spaces are the substructure on which the theory is build, a direct formulation of (1.11) as an evolution problem in a fixed function space is not possible. There are different approaches to tackle this problem and these are two of them:

1. One can use theories that allow the domains $D(A(t))$ to vary in time (e.g. [16] or [15, Section 5.2 - 5.5]). Nevertheless, as far as we know, they require each domain to be a subspace of the same Banach space, i.e. $D(A(t)) \subset X$ for $t \in [0,T]$. To avoid this problem, one can choose a function space of functions with a domain $\mathbb{R}^2$ or $\bigcup_{t\in[0,T]} \Omega_t$ and then choose $D(A(t))$ such that

   $$f \in D(A(t)) \implies f\big|_{\Omega_t} \in H_0^1(\Omega_t)\,,$$

   as it is done in [2]. The downside of this approach, is the additional complexity of the problem and a probably more difficult numerical analysis.

2. Assume the evolution of the domain $\Omega_t$ is determined by a family of well-behaved functions $\Phi(t,\cdot)$

   $$\Omega_t = \Phi(t,\Omega_0)\,,$$

   then a change of variables is an option: Let

   $$w(t,y) := u(t,\Phi(t,y)), \qquad y \in \Omega_0\,.$$

   $w$ is then a function defined on a fixed domain $[0,T]\times\Omega_0$. We then try to find a problem which is solved by $w$ and fits into the evolution problem framework of this chapter. Hopefully, we are then able to obtain the solution $u$ of our initial problem by a retransformation.

The rest of this section is dedicated to the change of variables approach. Observe that we just want to state some basic considerations and we do not want to give a rigorous analysis.

### 2.3.1 A Change of Variables

Let $\mathcal{N}_T := \bigcup_{t\in[0,T]}\{t\}\times\Omega_t$ be a time-space domain. We consider the problem

$$\begin{cases} D_t u(t,x) = A(t)u(t,x) & \forall (t,x) \in \mathcal{N}_T \\ u(0,x) = u_0(x) & \forall x \in \Omega_0 \end{cases} \tag{2.50}$$

Observe the first difference to a Cauchy problem is that a material derivative replaces the time derivative. We search a function $u$ such that the material derivative $D_t u$ exists and $u(t,\cdot) \in D(A(t))$. We assume $A(t)$ to act similar for each $t \in [0,T]$ and thus consider the following case:

- Let $X(\Omega_t)$ be a Banach space of functions: For $f \in X(\Omega_t)$, we have $f : \Omega_t \to \mathbb{R}$ and $f$ has some properties that specify functions being in $X(\Omega_t)$. The standard example would be $X(\Omega_t) := L^2(\Omega_t)$.

- Let $V(\Omega_t)$ be a function space such that $V(\Omega_t) \subset X(\Omega_t)$. We want $V(\Omega_t)$ to be the domain of $A(t)$:

  $$D(A(t)) = V(\Omega_t)\,.$$

  Since normally $A(t)$ is some kind of differential operator a simple example is $V(\Omega_t) := H_0^1(\Omega_t)$ with $X(\Omega_t) = L^2(\Omega_t)$.

Now we turn to the transformation of variables: We consider the transformed function

$$w(t, y) := u(t, \Phi(t, y))$$

and we seek for a problem such that

$$\begin{cases} \partial_t w(t, y) = \tilde{A}(t) w(t, y) & \forall (t, y) \in [0, T] \times \Omega_0 \\ w(0.y) = u_0(y) & \forall y \in \Omega_0 \end{cases}$$

is equivalent to (2.50). In this case, if we are able to solve the transformed problem for $w$, we get the solution $u$ by a simple retransformation.

We start by changing the variables in (2.50), which gives the equivalent problem

$$\begin{cases} D_t u(t, \Phi(t, y)) = (A(t)u(t, \cdot))(\Phi(t, y)) & \forall (t, y) \in [0, T] \times \Omega_0 \\ u(0, x) = u_0(x) & \forall x \in \Omega_0 \end{cases}$$

The equivalence holds since $\Phi(t, \cdot)$ is bijective for each $t \in [0, T]$.

The left-hand side can be transformed as follows: As we have seen before, the material derivative corresponds to the time derivative on constant domains. Thus the transformation of the left-hand side of (2.50) is easy and can be done by (1.4)

$$D_t u(t, \Phi(t, y)) = \frac{d}{dt}[u(t, \Phi(t, y))] = \frac{d}{dt}[w(t, y)]\,.$$

The rigth-hand side requires more considerations: Assume we have $A(t) : X(\Omega_t) \supset V(\Omega_t) \to X(\Omega_t)$. Ideally we find a linear operator $\tilde{A}(t) : X(\Omega_0) \supset V(\Omega_0) \to X(\Omega_0)$ which acts on $w$ the same way as $A(t)$ acts on $u$.

**Assumption 2.18.** *Let $\mathcal{T}_t : X(\Omega_t) \to X(\Omega_0)$ with $(\mathcal{T}_t f)(y) := f(\Phi(t, y))$ be an isomorphism (linear, bounded and with a bounded inverse) for all $t \in [0, T]$. Moreover, we assume*

$$\mathcal{T}_t V(\Omega_t) = V(\Omega_0), \qquad t \in [0, T]. \tag{2.51}$$

Observe that we already have

$$w(t, \cdot) := \mathcal{T}_t u(t, \cdot)$$

and by (1.4)

$$\mathcal{T}_t D_t u(t, \cdot) = \frac{d}{dt}[\mathcal{T}_t u(t, \cdot)] = \partial_t w(t, \cdot)\,.$$

*Remark* 2.19. The properties of $\mathcal{T}_t$ depend essentially on the properties of the transformations $\Phi(t, \cdot)$. Obviously a higher order Sobolev space requires a family of smoother mappings $\Phi(t, \cdot)$ to guarantee (2.51).

With $\mathcal{T}_t$ at hand we can now completely transform (2.50):

$$\begin{aligned} D_t u(t) = A(t)u(t) &\iff D_t u(t, \cdot) = A(t)u(t, \cdot) \\ &\iff \mathcal{T}_t D_t u(t, \cdot) = \mathcal{T}_t A(t) u(t, \cdot) \\ &\iff \frac{d}{dt}[\mathcal{T}_t u(t, \cdot)] = \mathcal{T}_t A(t) \mathcal{T}_t^{-1} \mathcal{T}_t u(t, \cdot) \\ &\iff \frac{d}{dt} w(t, \cdot) = \mathcal{T}_t A(t) \mathcal{T}_t^{-1} w(t, \cdot) \\ &\iff \frac{d}{dt} w(t) = \tilde{A}(t) w(t)\,, \end{aligned}$$

where

$$\tilde{A}(t) := \mathcal{T}_t A(t) \mathcal{T}_t^{-1} . \tag{2.52}$$

Since $\mathcal{T}_t V(\Omega_t) = V(\Omega_0)$ we have $\tilde{A}(t) : X(\Omega_0) \supset V(\Omega_0) \to X(\Omega_0)$ and the transformed problem is found:

$$\begin{cases} \partial_t w(t,y) = \mathcal{T}_t A(t) \mathcal{T}_t^{-1} w(t,y) & \forall (t,y) \in [0,T] \times \Omega_0 \\ w(0.y) = u_0(y) & \forall y \in \Omega_0 \end{cases}$$

*Remark* 2.20. If the time dependence of $A(t)$ is only due to the changing domains of the functions, $\tilde{A}(t) = \mathcal{T}_t A(t) \mathcal{T}_t^{-1}$ is a time dependent operator. Thus we exchanged the time dependence of the domain against a time dependent operator. Since for the latter case there exists a useful theory, this seems fair. E.g. if we consider $A(t) = \Delta : H_0^1(\Omega_t) \to L^2(\Omega_t)$ the operator $\tilde{A}(t)$ is a second order elliptic operator with time dependent coefficients that depend on $\Phi(t,\cdot)$, $\Phi^{-1}(t,\cdot)$ and their derivatives.

### 2.3.2 Conclusion

We saw that it is basically possible to transform a PDE on an evolving domain such that it can be understood as an evolution problem in a fixed Banach space $X$. Of course the lack of details in the above considerations interdicts to state any consequences, nevertheless it seems realistic to expect - for reasonable operators $A(t)$ - that $\tilde{A}(t)$ are in fact generating an evolution system.

Moreover, if we apply a numerical time integrator

$$w_{n+1} = T(t_n, \tau) w_n, \qquad t_{n+1} = \tau + t_n$$

to the transformed problem, satisfying

$$\|w(t_{n+1}) - w_{n+1}\| \leq C\tau^p$$

we can expect - under additional assumptions on $\mathcal{T}_t$ - that the approximation

$$u_{n+1} = \mathcal{T}_{t_{n+1}}^{-1} w_{n+1} \approx u(t_{n+1})$$

satisfies

$$\|u(t_{n+1}) - u_{n+1}\| = \left\| \mathcal{T}_{t_{n+1}}^{-1} (w(t_{n+1}) - w_{n+1}) \right\| \leq \tilde{C}\tau^p .$$

Finally, we can deduce from (2.52) that relations, as the Lipschitz continuity in $(A_3)$, are satisfied if the transformations $\mathcal{T}_t$ and the mappings $\Phi(t,\cdot)$ satisfy corresponding conditions.

# Chapter 3

# Spatial Discretization of the Diffusion Equation on Evolving Domains

With the full discretization of (1.11) being our final objective, we follow the general idea of the method of lines: find a way to discretize the appearing function spaces and then approximate the resulting ODE with a numerical time integrator of your choice. Since we actually want to test a time integrator we seek a semi-discretization that is easy to use and easy to implement. Again inspired by the method in [3], our approach is the approximation of the evolving domain by a somehow discrete domain.

## 3.1 A Finite Element Approximation

### 3.1.1 Triangulation Basics

We recollect some basic definitions and properties of triangulations of domains:

**Definition 3.1.** *Let $\Omega \subset \mathbb{R}^2$ be some open, bounded domain. A triangulation or a mesh of $\Omega$ is a partition $T = \{K^e \mid e = 1, \dots, E\}$ into a finite number of triangles $K$, with the following properties*

1. $\overline{\Omega} = \bigcup_{K \in T} K$
2. *Each $K$ is closed, with a nonempty interior $\mathring{K}$.*
3. *For distinct $i \neq j$, $\mathring{K}^i \cap \mathring{K}^j = \emptyset$.*

*For each $K^e \in T$ we call $a_1^e, a_2^e, a_3^e \in \Omega$ with $K^e = \operatorname{conv}\{a_1^e, a_2^e, a_3^e\}$ the nodes of $K^e$. We denote the entity of all nodes $\bigcup_{e \in E}\{a_1^e, a_2^e, a_3^e\}$ by the set $\{a_1, \dots, a_N\}$, $N \in \mathbb{N}$. Thus for each $K \in T$ there exist indices $1 \leq m, n, p \leq N$ such that*

$$K = \operatorname{conv}\{a_n, a_m, a_p\}\,.$$

*For an arbitrary $K \in T$, we denote*

$$h_K = \operatorname{diam}(K) = \max_{x,y \in K} |x - y| \qquad \textit{(diameter of circumscribed circle)}$$

$$\rho_K = \textit{diameter of the largest sphere } S_K \textit{ inscribed on } K\,.$$

*The quantity $h_K$ describes the size of $K$ while the ratio $\frac{h_K}{\rho_K}$ is an indication whether the triangle is flat.*

The diameters $h_K$ and the ratio $\frac{h_K}{\rho_K}$ are the most important parameter of of a triangulation $T$ . A triangulation $T$ sufficing

$$h \geq \max_{K \in T} h_K$$

is indicated by a subscript $h$, i.e. $T = T_h$. $h$ is often called the mesh parameter. The smaller $h$, the finer the mesh. Hence we expect convergence of the semi discrete solution for $h \to 0$.

The following definition deals with a sequence of triangulations $\mathcal{T}_h = (T_h)$.

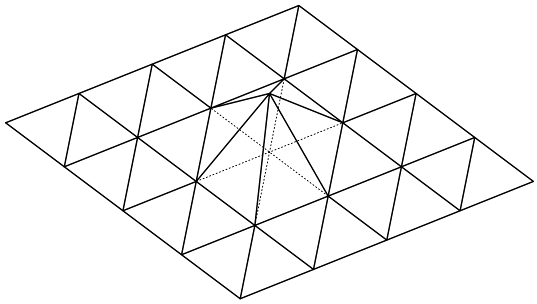

Figure 3.1: A symbolic sketch of a hat functions $\varphi_i$ over a triangulation.

**Definition 3.2.** *A family of triangulations $\mathcal{T}_h = (T_h)$ is said to be regular if*

1. *the mesh parameter $h \to 0$*
2. *there exists a constant $\sigma$ such that $\frac{h_K}{\rho_K} \leq \sigma$ for all $K \in T_h$ and $T_h \in \mathcal{T}_h$.*

We use triangulations of domains to construct discrete function spaces. The discretization of the function space is directly connected to some local discretization of the domain.

**Definition 3.3.** *Let $T_h$ be a triangulation of $\Omega$. We call*

$$\hat{S}_h = \{f \in C^0(\Omega) \mid f \text{ is linear affine on each } K \in T_h\}$$

*the corresponding linear finite element space. A function $f \in S_h$ is said to be piecewise linear.*

Let $\varphi_i \in \hat{S}_h$ such that

$$\varphi_i(a_j) = \delta_{i,j}\,, 1 \leq i, j \leq N\,.$$

Figure 3.1 shows a sketch of the so called hat function $\varphi_i$. It is easy to show that the $\varphi_i$ are uniquely given and form a basis

$$\hat{S}_h = \text{span}\{\varphi_1, \ldots, \varphi_N\}\,.$$

Given this basis of $\hat{S}_h$, the piecewise linear interpolation of a function $\eta : \Omega \to \mathbb{R}$ at the nodes $a_i$, $1 \leq i \leq N$ of a triangulation is very easy to compute:

$$\eta(x) \approx \sum_{i=1}^{N} \eta(a_i)\varphi_i(x)\,.$$

*Remark* 3.4. A representation in barycentric coordinates of $x \in K = \text{conv}\{a_m, a_n, a_p\}$

$$x = \lambda_m a_m + \lambda_m a_n + \lambda_p a_p$$

yields

$$\varphi_i(x) = \lambda_i,\ i = m, n, p$$

with $\lambda_m + \lambda_n + \lambda_p = 1$.

### 3.1.2 The Discrete Evolving Domain

We turn again to our problem (1.11) on the evolving domain $\Omega_t$. A triangulation of $\Omega_t$ for each $t \in [0, T]$ is computationally very expensive, thus we try to transform the initial triangulation of $\Omega_0$ such that it triangulates $\Omega_t$ as good as possible:

**Definition 3.5.** *(Discrete Evolving Domain) Let $T_h$ be a triangulation of $\Omega_0$.*

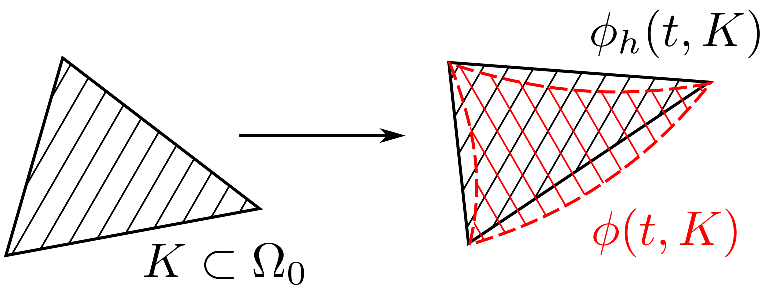


Figure 3.2: This picture shows how the $\Phi(t, \cdot)$ and its interpolation $\Phi_h(t, \cdot)$ map the triangle $K$.

1. *Let* $\mathrm{span}\{\hat{\varphi}_1, \ldots, \hat{\varphi}_N\} \subset H^1(\Omega_0)$ *be the linear finite element space over* $T_h$ *such that* $\hat{\varphi}_i(a_j) = \delta_{ij}$*. We define the family of mappings*

$$\Phi_h(t, y) := \sum_{i=1}^{N} \Phi(t, a_i)\hat{\varphi}_i(y) \qquad \text{for } t \in [0, T] \text{ and } y \in \Omega_0. \tag{3.1}$$

*These mappings define a new, discrete evolving domain:*

2. *Let*

$$\Omega_t^h := \Phi_h(t, \Omega_0)$$

*denote the discrete evolving domain.*

3. *Let* $K \in T_h$ *with* $K = \mathrm{conv}\{a_m, a_m, a_p\}$ *and*

$$a_i(t) := \Phi(t, a_i) \in \Omega_t \,, \qquad 1 \leq i \leq N \,.$$

*We define the corresponding evolving triangle*

$$K(t) := \mathrm{conv}\left\{a_m(t), a_n(t), a_p(t)\right\} \,.$$

*and make it part of the evolving triangulation* $T_h(t) = \{K(t) \,|\, K \in T_h\}$*.*

*Remark* 3.6. The above definition implies:

- The piecewise linear map $\Phi_h(t, \cdot)$ is invertible for each $t \in [0, T]$.
- For all $t \in [0, T]$, $T_h(t)$ is a triangulation of $\Omega_t^h$ and

$$\Omega_t^h = \mathrm{int}\left(\bigcup_{T_h(t)} K(t)\right) ,$$

where int denotes the interior of a set.

- The evolving triangle $K(t) = \mathrm{conv}\left\{a_m(t), a_n(t), a_p(t)\right\}$ can also be found by

$$K(t) == \Phi_h(t, K(0)) \,.$$

The difference between $K(t)$ and $\Phi(t, K(0))$ is presented in Figure 3.2.

The discrete evolving domain $\Omega_t^h$ is determined by the evolution of the nodes $a_i$ of the initial triangulation $T_h = T_h(0)$. Nonetheless we need $\Phi_h(t, \cdot)$ as we want to model material flows in the evolving domain. The family $\Phi_h(t, \cdot)$ can be understood as the piecewise linear interpolation of $\Phi(t, \cdot)$ on the triangulation $T_h$.

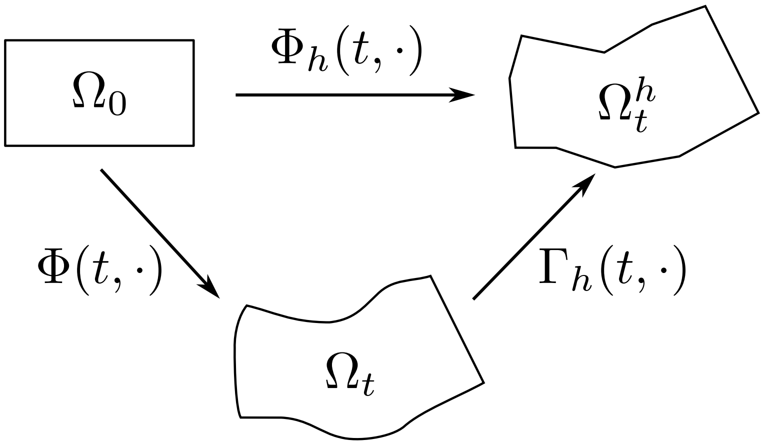


Figure 3.3: Schematic sketch of the relations between $\Omega_t$ and $\Omega_t^h$ and the mappings that are used to transform the domains.

How are $\Omega_t$ and $\Omega_t^h$ related? Though we have $\Omega_0 = \Omega_0^h$, the discrete evolving domain $\Omega_t^h$ is in general neither contained in $\Omega_t$ nor a superset of $\Omega_t$. Nevertheless

$$\Phi(t, a_i) = \Phi_h(t, a_i) \qquad , 1 \leq i \leq N$$

justifies naming $\Omega_t^h$ an interpolation of $\Omega_t$ with respect to the nodes $a_i$, $1 \leq i \leq N$. Figure 3.3 tries to visualize the difference between $\Omega_t$ and $\Omega_t^h$.

*Remark* 3.7. Of course our approach is limited to cases where for each $t \in [0, T]$ each $K(t)$ is a triangle . This excludes combinations of diffeomorphisms $\Phi(t, \cdot)$ and triangulations $T_h$, where $T_h$ is too coarsely meshed and e.g. material transport is happening at a finer scale than $h$.

Observe a notational difficulty: the maximal diameter $h(t)$ of all $K(t) \in T_h(t)$ is, though the index $h$ suggests it, in general not $h$, i.e. $h(t) \neq h$. If $h_K \leq h$ for each $K \in T_h$, we get by the Lipschitz continuity of $\Phi(t, \cdot)$

$$h_{K(t)} \leq h(t) \qquad K(t) \in T_h(t)$$

for some $h(t) > 0$. $h(t)$ is bounded by the maximal dilation of area of $\Phi(t, \cdot)$. The diameter of the inscribed sphere $\rho_{K(t)}$ has a lower bound since we assume that all triangles stay triangles

$$\rho_{K(t)} \geq \rho(t) \, .$$

Considering Assumption 1.1, the smoothness of $\Phi$ with respect to time yields uniform bounds for $h(t)$ and $\rho(t)$:

$$h^* := \max_{t \in [0,T]} h(t)$$
$$\rho_* := \min_{t \in [0,T]} \rho(t) \, .$$

This implies

$$h_{K(t)} \leq h(t) \leq h^*$$
$$\rho_{K(t)} \geq \rho(t) \geq \rho_* \, .$$

Let $t \in [0, T]$. If $\mathcal{T}_h$ is regular, $\mathcal{T}_h(t) := (T_h(t))$ is regular, since

$$h(t) \to 0$$

and

$$\frac{h_{K(t)}}{\rho_{K(t)}} \leq \frac{h(t)}{\rho(t)} \leq \sigma(t) \leq \sigma^*$$

with

$$\sigma^* := \max_{t\in[0,T]} \frac{h(t)}{\rho(t)} .$$

**Assumption 3.8.** *We assume our triangulations $T_h$ result from a regular family of triangulations $\mathcal{T}_h$. Moreover, for a given triangulation $T_h$, we assume that the uniform bounds $h^*$, $\sigma^*$ are of reasonable size, to assure the same order of convergence for all $t \in [0,T]$.*

At this point we have a discrete evolving domain $\Omega_t^h$ and a corresponding evolving triangulation $T_h(t)$. Hence we can now define the corresponding discrete function spaces:

$$S_h(t) = \{f \in C^0(\Omega_t^h) \mid f \text{ is linear affine on each } K \in T_h(t)\}. \tag{3.2}$$

$S_h(t)$ can also be constructed from the initial basis functions $\hat{\varphi}_1, \ldots, \hat{\varphi}_N$ of the linear finite element space over $T_h(0) = T_h$:

$$S_h = \{f \in C^0(\Omega_0^h) \mid f \text{ is linear affine on each } K \in T_h\} = \text{span}\{\hat{\varphi}_1, \ldots, \hat{\varphi}_N\} .$$

Let

$$\varphi_i(t, \Phi_h(t,y)) = \hat{\varphi}_i(y) \tag{3.3}$$

for $1 \le i \le N$ then

$$S_h(t) = \text{span}\{\varphi_1(t,\cdot), \ldots, \varphi_N(t,\cdot)\}.$$

This basis representation is the reason why we defined a discrete evolving domain in the first place. An alternative strategy would have been to move the mesh with $\Phi(t,\cdot)$. But in this case the moved mesh is not a triangulation, since in general $\Phi(t,\cdot)$ is not assumed to be linear or - as needed - piecewise linear with respect to the triangulation $T_h$. Our objective is to obtain an algorithm that is easy to use and a construction as in (3.2) and (3.3) preserves the possibility to use a linear affine mappings to map arbitrary $K(t) \in T_h(t)$ to a reference triangle $\hat{K}$.

Again we want to point out that the discrete evolving domain $\Omega_t^h$ and the original evolving domain $\Omega_t$ only share the initial domain $\Omega_0 = \Omega_0^h$. The evolution of the discrete evolving domain is given by

$$\Phi_h(t,\cdot) = \sum_{i=1}^{N} \Phi(t, a_i)\hat{\varphi}_i(\cdot) ,$$

whereas the evolution of $\Omega_t$ is given by $\Phi(t,\cdot)$. To solve (1.11) on $\Omega_t^h$ with weak solutions as in (1.16), we need the velocity of the evolution. Analogously to the continuous case the discrete velocity $\underline{v}_h$ satisfies

$$\underline{v}_h(t, \Phi_h(t,y)) = \partial_t \Phi_h(t,y) \tag{3.4}$$

for all $y \in \Omega_0^h$.

**Lemma 3.9.** *Let $\underline{v}_h$ be the velocity of the discrete evolving domain $\Omega_t^h$, i.e. satisfy (3.4). Then $\underline{v}_h$ can be considered as an interpolation of $\underline{v}$, since*

$$\underline{v}_h(t,x) = \sum_{i=1}^{N} \underline{v}(t, a_i(t))\varphi_i(t,x) .$$

*Proof.* The proof is a simple application of (3.1) and (3.3)

$$\partial_t \Phi_h(t,x) = \sum_{i=1}^{N} \partial_t \Phi(t,a_i)\hat{\varphi}_i(x) = \sum_{i=1}^{N} \underline{v}(t, a_i(t))\varphi_i(t, \Phi_h(t,x)) .$$

□

### 3.1.3 Preliminary Approximation Results

This chapter is concerned with some preliminaries needed for the convergence proof.

For the convergence analysis we need to evaluate functions that live on the discrete time-space domain

$$\mathcal{N}_T^h := \bigcup_{t\in[0,T]} \{t\} \times \Omega_t^h$$

on the original time-space domain $\mathcal{N}_T = \bigcup_{t\in[0,T]}\{t\} \times \Omega_t$. We have $\Phi(t,\cdot)$ and $\Phi_h(t,\cdot)$ and their inverse mappings $\Phi^{-1}(t,\cdot)$ and $\Phi_h^{-1}(t,\cdot)$ at hand to map values from $\Omega_t$ to $\Omega_t^h$.

**Definition 3.10.** *Let $X$ be a vector space and $\mu_h : \mathcal{N}_T^h \to X$. We define the lift function*

$$\Gamma_h(t,\cdot) : \Omega_t \to \Omega_t^h,\ x \mapsto \left(\Phi_h(t,\cdot) \circ \Phi^{-1}(t,\cdot)\right)(x)$$

*and the lifted version of $\mu_h$ as*

$$\mu_h^l : \mathcal{N}_T \to X,\ \mu_h^l(t,x) = \mu_h(t,\Gamma_h(t,x))\,.$$

See again Figure 3.3.

*Remark* 3.11. The term lift function originates from the evolving surface problem in [3], where functions living on a triangulation of a surface are lifted onto the original surface with the help of bijective mappings. To give an example of the way the lift function works we lift our basis functions $\varphi_i$, $1 \le i \le N$. Recall the characterization (3.3) of $\varphi_i(t,x)$. As always $\Phi_h^{-1}(t,x)$ denotes the inverse with respect to the space variable $x$, where $t$ is kept constant. We then have for $(t,x) \in \mathcal{N}_T$

$$\begin{aligned}\varphi_i^l(t,x) = \varphi_i(t,\Gamma_h(t,x)) &= \left(\hat\varphi_i \circ \Phi_h^{-1}(t,\cdot) \circ \Phi_h(t,\cdot) \circ \Phi^{-1}(t,\cdot)\right)(x)\\ &= \hat\varphi_i(\Phi^{-1}(t,x))\,.\end{aligned}$$

**Lemma 3.12.** *There exists a constant $c > 0$ such that*

$$\sup_{t\in[0,T]}\ \sup_{x\in\Omega_t} \left|\underline{v}_h^l(t,x) - \underline{v}(t,x)\right|^2 \le c\,(h^*)^2\ . \tag{3.5}$$

*Proof.* This estimate is a result of the mean value theorem (e.g. [17, Analysis 1, Satz 5.15]): For a function $f : \mathbb{R}^2 \supset U \to \mathbb{R}^2$ that is continuously differentiable we know that for $a, b \in U$ there exists a

$$\xi_i \in [a,b] := \{ra + (1-r)b \mid r \in [0,1]\}$$

such that

$$f_i(a) - f_i(b) = \nabla f_i(\xi_i) \cdot (a-b), \qquad i = 1,2\,.$$

Combined, we have

$$f(a) - f(b) = \begin{pmatrix}\nabla f_1(\xi_1)\\ \nabla f_2(\xi_2)\end{pmatrix}(a-b)\,.$$

If it is possible to bound all partial derivatives $\partial_j f_i(x)$, $i,j = 1,2$ uniformly on $U$ then

$$|f(a) - f(b)|^2 \le c \max_{i=1,2} |\nabla f_i(\xi_i)|^2\ |a-b|^2 \le c\ |a-b|^2\ . \tag{3.6}$$

With (3.6) at hand, we now turn to the lifted discrete velocity: With the above remark and $\Phi_h(t,a_i) = \Phi(t,a_i)$ for $1 \le i \le N$ (cf. (3.1)), we obtain

$$\underline{v}_h^l(t,x) = \underline{v}_h(t,\Gamma_h(t,x)) = \sum_{i=1}^N \underline{v}(t,a_i(t))\varphi_i^l(t,x) = \sum_{i=1}^N \underline{v}(t,a_i(t))\left(\hat\varphi_i \circ \Phi^{-1}\right)(t,x)\,.$$

Now let $y \in \Omega_0 = \Omega_0^h$. Without loss of generality $y \in K = \operatorname{conv}\{a_1, a_2, a_3\}$ for a $K \in T_h$. Moreover let $\lambda_1, \lambda_2, \lambda_3 \in [0,1]$ be the barycentric coordinates of $y$ with respect to $K$, i.e. $y = \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Remember $\varphi_i(y) = \lambda_i$ as stated in Remark 3.4. Then by Lemma 3.9 and (3.3)

$$\begin{aligned}\underline{v}(t,\Phi(t,y)) - \underline{v}_h^l(t,\Phi(t,y)) &= \underline{v}(t,\Phi(t,y)) - \sum_{i=1}^N \underline{v}(t,a_i(t))\varphi_i(t,\Phi(t,y)) \\ &= (\lambda_1+\lambda_2+\lambda_3)\underline{v}(t,\Phi(t,y)) \\ &\quad - \Big(\lambda_1\underline{v}(t,a_1(t)) + \lambda_2\underline{v}(t,a_2(t)) + \lambda_3\underline{v}(t,a_3(t))\Big) \\ &= \lambda_1\Big(\underline{v}(t,\Phi(t,y)) - \underline{v}(t,\Phi(t,a_1)\Big) + \lambda_2(\dots) + \lambda_3(\dots)\,.\end{aligned}$$

Since $\underline{v}(t,\cdot)$ is uniformly continuously differentiable in $t$, we can apply the initial considerations of this proof and obtain some $\xi_1^i \in [\Phi(t,y), \Phi(t,a_1)]$ such that

$$\lambda_1\left(\underline{v}(t,\Phi(t,y)) - \underline{v}(t,\Phi(t,a_1))\right) = \lambda_1 \begin{pmatrix}\nabla \underline{v}_1(t,\xi_1^1) \\ \nabla \underline{v}_2(t,\xi_1^2)\end{pmatrix} (\Phi(t,y) - a_1(t))\ .$$

Analogously for $\lambda_2(\dots)$ and $\lambda_3(\dots)$. Since $\Phi(t,\cdot)$ is bijective the equation holds for arbitrary $x = \Phi(t,y) \in \Omega_t$. Moreover, by Assumption 1.1 each partial derivative of $\underline{v}$ is uniformly bounded in $\Omega_t$. Thus we can apply (3.6) and obtain

$$\begin{aligned}\left|\underline{v}(t,x) - \underline{v}_h^l(t,x)\right|^2 &\le c \max_{\hat{x}\in\Omega_t}\max_{i,j=1,2}\left|\partial_i \underline{v}_j(t,\hat{x})\right| \left(\sum_{i=1}^3 \lambda_i\, |x - a_i(t)|^2\right) \\ &\le c\sum_{i=1}^3 \lambda_i\, |x-a_i(t)|^2 \\ &\le c\,(h^*)^2\end{aligned}$$

where we used $x \in K(t)$ and thus $|x - a_i(t)| \le h^*$ for $i = 1,2,3$. This bound holds independently of $x$ and $t$ which gives the claim. □

Later on we want to evaluate the semi-discrete solution of (1.11) on the evolving domain $\Omega_t$. Thus the lift function $\Gamma_h$ will play an important role: It lifts functions defined on $\Omega_t^h$ onto $\Omega_t$ and it transforms $\int_{\Omega_t^h}$ to $\int_{\Omega_t} \delta_h$ with $\delta_h$ being the functional determinant of $\Gamma_h(t,\cdot) : \Omega_t \to \Omega_t^h$. Since $\Phi_h(t,\cdot)$ can be interpreted as an interpolation of $\Phi(t,\cdot)$ we have $\Phi_h(t,\cdot) \approx \Phi(t,\cdot)$ and we hope to get

$$\Gamma_h(t,\cdot) = \left(\Phi_h(t,\cdot)\circ\Phi^{-1}(t,\cdot)\right)(\cdot) \approx \mathrm{Id}\,.$$

The next lemmata will prove this conjecture and other approximation results.

*Remark* 3.13. (Reference Element Technique) In the following proofs we are going to use the reference element technique. This is an introduction into the notation and results we will need.

Let $K \in T_h$ be a triangle, without loss of generality we assume

$$K = \operatorname{conv}\{a_1, a_2, a_3\}$$

There exists a linear affine mapping $F_K$ that maps the reference triangle

$$\hat{K} = \operatorname{conv}\{\hat{a}_1, \hat{a}_2, \hat{a}_3\}$$

onto $K$. We choose $\hat{a}_1 = (0,0)$, $\hat{a}_2 = (1,0)$ and $\hat{a}_3 = (0,1)$ and set $\hat{h}$ and $\hat{\rho}$ for the spatial parameters of $\hat{K}$. Observe $\hat{a}_j - \hat{a}_1 = \hat{e}_j$, where $\hat{e}_j$ denotes the $j$th unit vector. Since $F_K$ is linear affine there exists a matrix $T_K \in \mathbb{R}^{2\times 2}$ and a vector $b_K \in \mathbb{R}^2$ such that

$$F_K(\hat{y}) = T_K\hat{y} + b_K,\ F_K(\hat{K}) = K,\ F_K(\hat{a}_i) = a_i \qquad \text{for } i = 1,2,3.$$

For each point $y \in K$ we denote its reference point in $\hat{K}$ by $\hat{y}$, i.e. $F_K(\hat{y}) = y$. A more detailed insight is given in [1, Chapter 9].

From [1, Lemma 9.2.3] we know

$$|T_K| \left|T_K^{-1}\right| \leq \frac{h_K}{\rho_K} \frac{\hat{h}}{\hat{\rho}} .$$

Thus, with our previous considerations and $\frac{\hat{h}}{\hat{\rho}} \leq c$, we have for $K(t) \in T_h(t)$

$$\left|T_{K(t)}\right| \left|T_{K(t)}^{-1}\right| \leq \frac{h_{K(t)}}{\rho_{K(t)}} \frac{\hat{h}}{\hat{\rho}} \leq c\sigma^*.$$

**Lemma 3.14.** *There is a constant* $c > 0$ *such that*

$$\sup_{t\in[0,T]} \operatorname*{ess\,sup}_{x\in\Omega_t} |I_2 - J_x\Gamma_h(t,x)| \leq ch^* ,$$

*where* $I_2$ *denotes the two dimensional identity matrix.*

*Proof.* We start by rewriting $\Gamma_h$ as

$$\begin{aligned}\Gamma_h(t,x) &= \Phi_h(t,\Phi^{-1}(t,x)) - \Phi(t,\Phi^{-1}(t,x)) + x \\ &= \left[\Phi_h(t,\cdot) - \Phi(t,\cdot)\right] \circ \Phi^{-1}(t,x) + x ,\end{aligned}$$

which is the characterization of $\Gamma_h$ we work with. Applying the chain rule, we obtain the Jacobian

$$J_x\Gamma_h(t,x) = [J_x\Phi_h(t,\cdot) - J_x\Phi(t,\cdot)] \circ \Phi^{-1}(t,x)\, J_x\Phi^{-1}(t,x) + I_2 . \tag{3.7}$$

As $\Phi_h(t,\cdot)$ is piecewise linear, its Jacobian $J_x\Phi_h(t,\cdot)$ only exists almost everywhere on $\Omega_0$. A consequence is the ess sup in the statement of this Lemma. However, since this almost everywhere differentiability leaves rest of this proof unaffected we ignore it in the following.

The rest of the proof is rather technical due to the fact the $\Gamma_h$ maps from $\mathbb{R}^2$ to $\mathbb{R}^2$. It will be given in three steps:

- We use the reference element technique in this part of the proof. Consult Remark 3.13 for notation. For now we will omit the time dependencies, they will be dealt with separately in the second part.

  Suppose without loss of generality $x \in K = \mathrm{conv}\{a_1, a_2, a_3\}$ for $K \in T_h$. Recall that the chain rule for differentiation yields for $g : \mathbb{R}^2 \to \mathbb{R}^2$

  $$J_x[g \circ F_K](\hat{x}) = J_xg(F_K(\hat{x}))\, J_xF_K(\hat{x}) = J_xg(F_K(\hat{x}))\, T_K \tag{3.8}$$

  and that $\Phi_h$ coincides with $\Phi$ on the nodes

  $$\Phi_h(a_i) = \Phi(a_i) , \qquad i = 1,2,3.$$

  The (partial) derivatives of linear affine mappings (as $\Phi_h \circ F_K$ on $\hat{K}$) are constant and we can evaluate the difference quotient at arbitrary points lying in the corresponding direction

  $$\begin{aligned}J_x[\Phi_h \circ F_K](\hat{x}) &= \begin{pmatrix} \partial_{x_1}\Phi_h(F_K(\hat{x})) & \partial_{x_2}\Phi_h(F_K(\hat{x})) \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{\Phi_h(F_K(\hat{a}_2)) - \Phi_h(F_K(\hat{a}_1))}{|\hat{a}_2 - \hat{a}_1|} & \dfrac{\Phi_h(F_K(\hat{a}_3)) - \Phi_h(F_K(\hat{a}_1))}{|\hat{a}_3 - \hat{a}_1|} \end{pmatrix} \\ &= \begin{pmatrix} \Phi(a_2) - \Phi(a_1) & \Phi(a_3) - \Phi(a_1) \end{pmatrix} ,\end{aligned}$$

  where we used $\left|\hat{a}_i - \hat{a}_j\right| = |\pm e_i| = 1$.

The first row of $J_x[\Phi_h \circ F_K]$ is an approximation to $\partial_{x_1}[\Phi \circ F_K]$: We use the boundedness of the second spatial derivative of $\Phi$ in a truncated Taylor series, the Lipschitz continuity of $J_x\Phi$ and apply (3.8) to get

$$\begin{aligned}\Phi(a_2) - \Phi(a_1) &= J_x\Phi(a_1)\,(a_2 - a_1) + O(\,|a_2 - a_1|^2) \\ &= J_x\Phi(x)(a_2 - a_1) + O(\,|x - a_1|)(a_2 - a_1) + O(\,|a_2 - a_1|^2) \\ &= J_x[\Phi \circ F_K](\hat{x})\,T_k^{-1}(a_2 - a_1) \\ &\quad + O(\,|x - a_1|)(a_2 - a_1) + O(\,|a_2 - a_1|\ \ |a_2 - a_1|) \\ &= J_x[\Phi \circ F_K](\hat{x})\,\hat{e}_1 + O(\,|x - a_1|)\,T_K\hat{e}_1 + O(\,|a_2 - a_1|\ |T_K\hat{e}_1|)\,.\end{aligned}$$

Recall that $a_2 - a_1 = F_K(\hat{a}_2) - F_K(\hat{a}_1) = T_K\,(\hat{a}_2 - \hat{a}_1) = T_K\hat{e}_1$. Now we use the above equation and $|x - a_1| \leq ch_K$ for $x \in K$ to estimate

$$\begin{aligned}|J_x[\Phi_h \circ F_K](\hat{x})\hat{e}_1 - J_x[\Phi \circ F_K](\hat{x})\hat{e}_1| &= |\Phi(a_2) - \Phi(a_1) - J_x[\Phi \circ F_K](\hat{x})\hat{e}_1| \\ &\leq O(\,|x - a_1|)\ \ |T_K\hat{e}_1| + O(\,|a_2 - a_1|\ |T_K\hat{e}_1|) \\ &\leq\ |T_K|\ \ O(\,|a_2 - a_1|)\ |\hat{e}_1| \\ &\leq c\ |T_K|\,h_K\ \ |\hat{e}_1|\ \ .\end{aligned}$$

An analogous computation for $\hat{e}_2$ yields

$$|J_x[\Phi_h \circ F_K](\hat{x})\hat{e}_2 - J_x[\Phi \circ F_K](\hat{x})\hat{e}_2| \leq c\ |T_K|\,h_K\ \ |\hat{e}_2|\ \ .$$

Since $\hat{e}_1, \hat{e}_2$ form a basis of $\mathbb{R}^2$ we get for arbitrary $\hat{x} \in \hat{K}$

$$|J_x[\Phi_h \circ F_K](\hat{x}) - J_x[\Phi \circ F_K](\hat{x})| \leq c\ |T_K|\,h_K\,.$$

As stated in Remark 3.13, we have $|T_K|\ \left|T_K^{-1}\right| \leq c\frac{h_K}{\rho_K}$, because our triangulations are part of a regular family. Using the above result and $F_K$ being bijective, we get

$$\begin{aligned}\sup_{x\in K}|J_x\Phi_h(x) - J_x\Phi(x)| &= \sup_{\hat{x}\in\hat{K}}\left|J_x[\Phi_h \circ F_K](\hat{x})\,T_K^{-1} - J_x[\Phi \circ F_K](\hat{x})\,T_K^{-1}\right| \\ &\leq c\ |T_K|\,h_K\ \left|T_K^{-1}\right| \\ &\leq c\frac{h_K}{\rho_K}h_K\,.\end{aligned}$$

With $h_K \leq h$ we obtain

$$\sup_{x\in\Omega_0}|J_x\Phi_h(x) - J_x\Phi(x)| \leq c\max_{K\in T_h}\frac{h_K}{\rho_K}h\,. \tag{3.9}$$

- Now we consider the time dependencies: Since we have

$$\frac{h_K(t)}{\rho_K(t)} \leq \sigma^*$$

and $h(t) \leq h^*$, (3.9) implies

$$\sup_{t\in[0,T]}\ \sup_{x\in\Omega_0}\ |J_x\Phi_h(t,x) - J_x\Phi(t,x)| \leq ch^*\ .$$

- To obtain the complete estimate as stated, we need the boundedness of $J_x\Phi^{-1}(t,x)$,i.e.

$$\sup_{t\in[0,T]}\ \sup_{x\in\Omega_t}\left|J_x\Phi^{-1}(t,x)\right| \leq c$$

and the fact that $\Phi^{-1}(t,\cdot):\Omega_t\to\Omega_0$ is bijective. (3.7) and (3.9) then imply

$$\begin{aligned}|I_2 - J_x\Gamma_h(t,x)| &= \left|[J_x\Phi_h(t,\cdot) - J_x\Phi(t,\cdot)]\circ\Phi^{-1}(t,x)\, J_x\Phi^{-1}(t,x)\right| \\ &\le c\left|[J_x\Phi_h(t,\cdot) - J_x\Phi(t,\cdot)]\circ\Phi^{-1}(t,x)\right| \\ &= c\sup_{y\in\Omega_0}|J_x\Phi_h(t,y) - J_x\Phi(t,y)| \\ &\le c\sup_{t\in[0,T]}\sup_{y\in\Omega_0}|J_x\Phi_h(t,y) - J_x\Phi(t,y)| \\ &\le ch^*\end{aligned}$$

The almost everywhere differentiability of $\Phi_h(t,\cdot)$ then forces us to replace the sup by an ess sup, which completes the proof. □

**Lemma 3.15.** *Let*

$$\delta_h(t,x) := |\det J_x\Gamma_h(t,x)|$$

*denote the functional determinant of* $\Gamma_h(t,\cdot)$*, then*

$$\sup_{t\in[0,T]}\operatorname*{ess\,sup}_{x\in\Omega_t}|1-\delta_h(t,x)| \le ch^*\,. \tag{3.10}$$

*Proof.* We need the following basic results for $2\times 2$ matrices:

1. $\det(I_2 + A) = 1 + \operatorname{tr}(A) + \det(A)$ where $\operatorname{tr}(A) := A_{1,1} + A_{2,2}$ denotes the trace of $A$.
2. $\left|A_{i,j}\right| \le a$ implies $\det(A) \le 2a^2$.
3. Let $B\in\mathbb{R}^{2\times 2}$ with $\left|B_{i,j}\right| \le \hat{c}$, then $|\operatorname{tr}(AB)| \le \hat{c}\sum_{i,j=1}^2\left|A_{i,j}\right| = c\,|A|_F$ where $|\cdot|_F$ denotes the Frobenius norm.

We apply the first result to

$$J_x\Gamma_h(t,x) = \left[J_x\Phi_h(t,\cdot) - J_x\Phi(t,\cdot)\right]\circ\Phi^{-1}(t,x)\,J_x\Phi^{-1}(t,x) + I_2$$

and get

$$\begin{aligned}\det J_x\Gamma_h(t,x) =& 1 + \operatorname{tr}\left(\left[J_x\Phi_h(t,\cdot) - J_x\Phi(t,\cdot)\right]\circ\Phi^{-1}(t,x)\,J_x\Phi^{-1}(t,x)\right) \\ &+ \det\left(\left[J_x\Phi_h(t,\cdot) - J_x\Phi(t,\cdot)\right]\circ\Phi^{-1}(t,x)\right)\det\left(J_x\Phi^{-1}(t,x)\right)\,.\end{aligned}$$

By assumption all partial derivatives of $\Phi^{-1}(t,\cdot)$ are bounded, thus $\left|(J_x\Phi^{-1}(t,x))_{i,j}\right| \le c$ for all $t\in[0,T]$ and $x\in\Omega_t$. Using this boundedness in the above equation, gives

$$\begin{aligned}|1-\delta_h(t,x)| &= |1 - |\det J_x\Gamma_h(t,x)|| \\ &\le c\left|[J_x\Phi_h(t,\cdot) - J_x\Phi(t,\cdot)]\circ\Phi^{-1}(t,x)\right|_F \\ &\quad + c\det\left([J_x\Phi_h(t,\cdot) - J_x\Phi(t,\cdot)]\circ\Phi^{-1}(t,x)\right)\,.\end{aligned}$$

Since all matrix norms are equivalent, there exists an $r>0$ such that

$$\left|A_{i,j}\right| \le |A|_F \le r\,|A| \text{ for } i,j = 1,2.$$

By the second initial result, we can use these inequalities to dominate $\det(A)$ by $c\,|A|^2$:

$$\begin{aligned}|A|_F &\le c\,|A| \\ \det(A) &\le c\,|A|^2\end{aligned}$$

Considering this and using

$$\sup_{t\in[0,T]} \sup_{x\in\Omega_0} |J_x\Phi_h(t,x) - J_x\Phi(x)| \le ch^*$$

from the proof of the previous Lemma, yields

$$\begin{aligned}|1-\delta_h(t,x)| &\le ch^* + c(h^*)^2 \\ &\le ch^* .\end{aligned}$$

□

## 3.2 Semi-Discrete Approximation

Again this section is mainly motivated by [3]. The proof of Theorem 3.20 is an analogous and more detailed version of the proof of [3, Theorem 6.2].

### 3.2.1 Spatial Discretization

The ansatz for our moving domain finite element method (MDFEM) is the variational formulation (1.16). We want to emphasize again that a weak solution as in (1.16) implies classical differentiability with respect to $t$.

The discretization is now done in a Petrov-Galerkin like manner: Since a weak solution $u$ of (1.11) and admissible test functions $\psi \in F$ are only required to satisfy $u(t,\cdot), \psi(t,\cdot) \in H_0^1(\Omega_t)$ almost without any other restriction, we want to define the semi-discrete weak solution accordingly. Until now we only have the discrete function space $S_h(t) = \operatorname{span}\{\varphi_1(t,\cdot), \dots, \varphi_N(t,\cdot)\}$. However since

$$S_h(t) \not\subseteq H_0^1(\Omega_t^h)$$

a solution from $S_h(t)$ does in general not suffice our BC. Let

$$V_h(t) := \{f \in S_h(t) \mid f(x) = 0 \text{ for } x \in \partial\Omega_t^h\} . \tag{3.11}$$

Observe, if $a_1, \dots, a_d \notin \partial\Omega_0$ are the inner nodes of the triangulation, then

$$V_h(t) = \operatorname{span}\{\varphi_1(t,\cdot), \dots, \varphi_d(t,\cdot)\} . \tag{3.12}$$

$S_h(t)$ is a discrete subspace of $H^1$ and $V_h(t)$ is a discrete subspace of $H_0^1(\Omega_t)$.

**Definition 3.16** (Semi-Discrete Solution, [3, Definition 5.6])**.** *We call $u_h : \mathcal{N}_T^h \to \mathbb{R}$ the discrete weak solution of (1.11) if $u_h(t,\cdot) \in V_h(t)$ and it satisfies*

$$\frac{d}{dt}\int_{\Omega_t^h} u_h\,\psi_h\,dx + \alpha\int_{\Omega_t^h} \nabla u_h\cdot\nabla\psi_h\,dx = \int_{\Omega_t^h} u_h\,D_t\psi_h\,dx, \qquad t\in[0,T] \tag{3.13}$$

*for all $\psi_h(t,\cdot) \in V_h(t)$, where we assume that all appearing quantities exist.*

As in the continuous case the above definition is equivalent to

$$\int_{\Omega_t^h} D_t u_h \psi_h\,dx + \int_{\Omega_t^h} u_h\,\psi_h \operatorname{div}\underline{v}_h\,dx + \alpha\int_{\Omega_t^h}\nabla u_h\cdot\nabla\psi_h\,dx = 0 \tag{3.14}$$

which can be seen by the use of the Leibniz formula. Observe that a classical application of a Galerkin-like method to our discretization is not possible due to the different domains $\Omega_t$ and $\Omega_t^h$ of the discrete and the continuous problem.

We want to express this linear problem in matrices and vectors. First of all note that for a given $h$ we have that $d = \dim V_h(t)$ is independent of $t$ by construction. We then define for each $t \in [0, T]$ the mapping

$$\Pi_h(t) : \mathbb{R}^d \to V_h(t),\ \mathbf{y} = (y_i)_{i=1}^d \mapsto \sum_{i=1}^d y_i \varphi_i(t, \cdot)\,. \tag{3.15}$$

We use the coefficient vectors $\mathbf{u}, \mathbf{y} : [0, T] \to \mathbb{R}^d$ and set

$$\begin{aligned} u_h(t, x) &= \Big(\Pi_h(t)\mathbf{u}(t)\Big)(x) = \sum_{i=1}^d u_i(t)\varphi_i(t, x) \\ \psi_h(t, x) &= \Big(\Pi_h(t)\mathbf{y}(t)\Big)(x) \end{aligned}$$

for $(t, x) \in \mathcal{N}_T^h$. Note that the existence and uniqueness of such a basis representation is assured by $u_h(t, \cdot), \psi_h(t, \cdot) \in V_h(t)$.

*Remark* 3.17. Let $f_h(t, \cdot) \in V_h(t)$ with $\Pi_h(t)\mathbf{f}(t) = f_h(t, \cdot)$. The material derivative of $f_h$ satisfies

$$\begin{aligned} D_t f_h(t, \Phi_h(t, y)) &= \frac{d}{dt}[f_h(t, \Phi_h(t, \cdot))](y) \\ &= \frac{d}{dt} \sum_{i=1}^d f_i(t)\varphi_i(t, \Phi_h(t, y)) \\ &= \frac{d}{dt} \sum_{i=1}^d f_i(t)\hat{\varphi}_i(y) \\ &= \sum_{i=1}^d \partial_t f_i(t)\hat{\varphi}_i(y) \\ &= \Big(\Pi_h(t)\partial_t \mathbf{f}(t)\Big)(\Phi_h(t, y)) \end{aligned}$$

by (1.4). Thus

$$D_t f_h(t, \cdot) = \Pi_h(t)\partial_t \mathbf{f}(t)\,.$$

We insert the basis representation of $u_h$ and $\psi_h$ and obtain the following equivalent formulation: Find $\mathbf{u}(t)$ such that

$$\frac{d}{dt} \int_{\Omega_t^h} \sum_{i,j=1}^d u_j(t)\varphi_j\, y_i(t)\varphi_i\, dx + \alpha \int_{\Omega_t^h} \sum_{i,j=1}^d (u_j(t)\nabla\varphi_j) \cdot (y_i(t)\nabla\varphi_i)\, dx = \int_{\Omega_t^h} \sum_{i,j=1}^d u_j(t)\varphi_j\, \partial_t y_i(t)\varphi_i\, dx$$

for all $\mathbf{y}(t)$. Now let $\mathbf{M}_h(t)$ denote the mass matrix

$$\mathbf{M}_h(t)_{i,j} := \int_{\Omega_t^h} \varphi_i(t, x)\, \varphi_j(t, x)\, dx$$

and $\mathbf{S}_h(t)$ the stiffness matrix

$$\mathbf{S}_h(t)_{i,j} := \alpha \int_{\Omega_t^h} \nabla\varphi_i(t, x) \cdot \nabla\varphi_j(t, x)\, dx\,.$$

The above problem can then be written as

$$\frac{d}{dt}\Big[\mathbf{y}(t)^\top \mathbf{M}_h(t)\mathbf{u}(t)\Big] + \mathbf{y}(t)^\top \mathbf{S}_h(t)\mathbf{u}(t) = \partial_t \mathbf{y}(t)^\top \mathbf{M}_h(t)\mathbf{u}(t) \tag{3.16}$$

where $\mathbf{y}^\top$ denotes the transposed vector of $\mathbf{y}$. If we apply

$$\frac{d}{dt}\Big[\mathbf{y}(t)^\top \mathbf{M}_h(t)\mathbf{u}(t)\Big] = \partial_t \mathbf{y}(t)^\top \mathbf{M}_h(t)\mathbf{u}(t) + \mathbf{y}(t)^\top \frac{d}{dt}\Big[\mathbf{M}_h(t)\mathbf{u}(t)\Big]\,.$$

to (3.16), we obtain

$$\mathbf{y}(t)^\intercal \left( \frac{d}{dt} \left[ \mathbf{M}_h(t)\mathbf{u}(t) \right] + \mathbf{S}_h(t)\mathbf{u}(t) \right) = 0 \,.$$

We can omit $\mathbf{y}(t)$ in the above equation, since $\psi_h$ is arbitrary and obtain that the solution $u_h(t,\cdot) = \Pi_h(t)\mathbf{u}(t)$ of (3.13) is equivalently given by

$$\frac{d}{dt}\Big[\mathbf{M}_h(t)\mathbf{u}(t)\Big] + \mathbf{S}_h(t)\mathbf{u}(t) = 0 \,. \tag{3.17}$$

Observe that the mass matrix $\mathbf{M}_h(t)$ is uniformly positive definite on $[0,T]$ and the stiffness matrix $\mathbf{S}_h(t)$ is positive semi-definite. The continuity of $\Phi_h(\cdot, y)$ and the Lipschitz continuity of $\Phi_h(t,\cdot)$ then yield existence and uniqueness of the semi-discrete finite element solution (3.13).

We want to point out that there is already a numerical method for PDEs on evolving domains: The so called arbitrary Lagrangian Eulerian (ALE) formulation results in a ODE which is quite similar to (3.17). Although, the approach using ALE is slightly different there exist a lot results about this topic. A basic introduction into the ALE approach is given in [6].

*Remark* 3.18. Our aim was to find an easy-to-use method for discretizing (1.11). With (3.17) we can be satisfied:

- $\mathbf{M}_h(t)$ and $\mathbf{S}_h(t)$ are as easy to assemble as standard FEM matrices, since we can use linear affine mappings from each triangle to a reference element.
- We only need one triangulation of the initial domain $\Omega_0$.
- Our method does not require any evaluation of the velocity or functional determinant of $\Phi(t,\cdot)$, since it only depends on the knowledge of the position of the nodes.

### 3.2.2 Convergence of the Semi-Discrete Solution

**Lemma 3.19** (Stability of the Lifted Solution, [3, Lemma 6.1])**.** *Let $u_h$ be the solution of (3.13) with initial value $u_h(0,\cdot) = u_0$ and $u_h^l$ its lifted version. The following stability estimates hold:*

$$\sup_{t\in[0,T]} \left\|u_h^l\right\|_{0,\Omega_t}^2 + \int_0^T \left\|\nabla u_h^l\right\|_{0,\Omega_t}^2 \,dt \le c\,\left\|u_h^l(0)\right\|_{0,\Omega_0}^2 \,,$$

$$\int_0^T \left\|D_t u_h^l\right\|_{0,\Omega_t}^2 \,dt \;+\; \sup_{t\in[0,T]} \left\|\nabla u_h^l\right\|_{0,\Omega_t}^2 \le c\,\left\|u_h^l(0)\right\|_{1,\Omega_0}^2 \,.$$

*Proof.* The estimates for $u_h$ follow from the energy estimates in Lemma 1.8 and Lemma 1.9 which hold and are proven for $u_h$ as for the solution of (1.16). We then lift $u_h$ to $\Omega_t$ (cf. Definition 3.10) and use the boundedness of

$$H_0^1(\Omega_t^h) \to H_0^1(\Omega_t),\ \eta_h \mapsto \eta_h^l$$

and its inverse. To handle the material derivative we refer to (3.20) from the proof of Theorem 3.20. □

The next theorem is the main result of this section. The proof is done analogously to [3, Theorem 6.2], but gives a slightly different result: The preliminary approximation results we use only allow for order $1/2$ convergence. Observe that our findings are in no sense optimal and can be improved.

**Theorem 3.20** (Convergence of the Semi-Discretization, [3, Theorem 6.2])**.** *Let $u$ be a sufficiently smooth solution of (1.16) and let $u_h$ be the discrete solution of (3.13). For the lifted version $u_h^l$ of $u_h$ we then have the following error estimate:*

$$\sup_{t\in[0,T]} \left\|u - u_h^l\right\|_{0,\Omega_t}^2 + \int_0^T \left\|\nabla(u - u_h^l)\right\|_{0,\Omega_t}^2 \,dt$$
$$\le ch\,\|u_0\|_{2,\Omega_0}^2 + ch^2\,\|u_0\|_{2,\Omega_0}^2 + ch^4\,\|u_0\|_{2,\Omega_0}^2 + ch^2 \int_0^T \|D_t u\|_{2,\Omega_t}^2 \,dt$$

*Remark* 3.21. The spatial parameter $h$ in the above error estimate is in fact $h^*$ which was introduced in Section 3.1.2, but since we assume $h^* \leq ch$, we can use the mesh parameter of the initial triangulation $T_h(0)$.

The proof of Theorem 2.8 is given in three parts:

**Error Relation between the Lifted Discrete and the Continuous Solution**

The error bounds rely on a suitable form of the error equation. In order to compare discrete and continuous solution, both should be defined on the same domain which we take to be the continuous one. The continuous problem reads:

$$\frac{d}{dt}\int_{\Omega_t} u\,\psi\,dx + \alpha\int_{\Omega_t}\nabla u\cdot\nabla\psi\,dx = \int_{\Omega_t} u\,D_t\psi\,dx \qquad \forall\psi(t,\cdot)\in H_0^1(\Omega_t)\,. \tag{3.18}$$

The semi-discrete problem reads:

$$\frac{d}{dt}\int_{\Omega_t^h} u_h\,\psi_h\,dx + \alpha\int_{\Omega_t^h}\nabla u_h\cdot\nabla\psi_h\,dx = \int_{\Omega_t^h} u_h\,D_t\psi_h\,dx \qquad \forall\psi_h(t,\cdot)\in V_h(t)\,. \tag{3.19}$$

We want to lift the discrete equation on the evolving domain $\Omega_t$. These preliminary results are necessary: Let $\eta_h : \mathcal{N}_T^h \to \mathbb{R}$ be a function defined on the discrete evolving domain $\Omega_t^h$ such that all of the following derivatives of $\eta_h$ exist. The lifted function is then defined as

$$\eta_h^l(t,x) = \eta_h(t,\Gamma_h(t,x)) \qquad \text{with } \Gamma_h(t,x) = \Phi_h(t,\Phi^{-1}(t,x))\,.$$

The material derivative of a lifted function is in general not the lift of its material derivative: Let $R_h(t,x) := J_x\Gamma_h(t,x)$. An application of the chain rule for differentiation yields

$$\nabla\eta_h^l = (\nabla\eta_h)^l R_h \iff (\nabla\eta_h)^l = \nabla\eta_h^l R_h^{-1}$$

and thus

$$\begin{aligned}
D_t\eta_h^l(t,x) &= \partial_t\eta_h^l(t,x) + \nabla\eta_h^l(t,x)\cdot\underline{v}(t,x)\\
&= (\partial_t\eta_h)^l(t,x) + (\nabla\eta_h)^l(t,x)\,R_h(t,x)\cdot\underline{v}(t,x) + (\nabla\eta_h)^l(t,x)\cdot(\underline{v}_h^l(t,x) - \underline{v}_h^l(t,x))\\
&= (\partial_t\eta_h)^l(t,x) + (\nabla\eta_h\cdot\underline{v}_h)^l(t,x) + (\nabla\eta_h)^l(t,x)\cdot\left[R_h^\top(t,x)\,\underline{v}(t,x) - \underline{v}_h^l(t,x)\right]\\
&= (D_t\eta_h)^l(t,x) + (\nabla\eta_h)^l(t,x)\cdot\left[\left(R_h^\top(t,x) - I_2\right)\underline{v}(t,x) - \left(\underline{v}(t,x) - \underline{v}_h^l(t,x)\right)\right].
\end{aligned}$$

With Lemma 3.12, Lemma 3.14 and the boundedness of $\underline{v}(t,x)$, the above identity yields

$$D_t\eta_h^l(t,x) - (D_t\eta_h)^l(t,x) = O(\,h\,\left|(\nabla\eta_h)^l(t,x)\right|\,) \tag{3.20}$$

and as a simple consequence we obtain

$$\begin{aligned}
u_h^l(D_t\eta_h)^l = u_h^l\left(D_t\eta_h^l + O(h\,\left|(\nabla\eta_h)^l\right|)\right) &= u_h^l D_t\eta_h^l + u_h^l\,O(h\,\left|(\nabla\eta_h)^l\right|)\\
&\geq u_h^l D_t\eta_h^l - \left|u_h^l\,O(h\,\left|(\nabla\eta_h)^l\right|)\right|\\
&\geq u_h^l D_t\eta_h^l - ch\,\left|u_h^l\right|\,\left|(\nabla\eta_h)^l\right|\,.
\end{aligned}$$

Since $\Gamma_h(t,\cdot)$ is invertible and a concatenation of diffeomorphism, we know that $\left|R_h^{-1}\right|$ is bounded and therefore

$$u_h^l(D_t\eta_h)^l \geq u_h^l D_t\eta_h^l - \left|u_h^l\right|c\,h\,\left|\nabla\eta_h^l\right|\,. \tag{3.21}$$

Now we turn to the differential equation that is satisfied by the discrete solution on the evolving domain $\Omega_t$. Recall that we defined the abbreviation

$$\delta_h(t,x) := \left|\det J_x\Gamma_h(t,x)\right|$$

for the functional determinant of $\Gamma_h(t,\cdot)$. Let $\psi_h(t,\cdot) \in V_h(t)$ and $u_h$ a solution of the discrete problem (3.19), then

$$\frac{d}{dt}\int_{\Omega_t} u_h^l\,\psi_h^l\,\delta_h\,dx + \alpha\int_{\Omega_t}\left(\nabla u_h^l R_h^{-1}\right)\cdot\left(\nabla\psi_h^l R_h^{-1}\right)\delta_h\,dx = \int_{\Omega_t} u_h^l\,(D_t\psi_h)^l\,\delta_h\,dx$$

is satisfied and with (3.21)

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} u_h^l\,\psi_h^l\,\delta_h\,dx &+ \alpha\int_{\Omega_t}\left(\nabla u_h^l R_h^{-1}\right)\cdot\left(\nabla\psi_h^l R_h^{-1}\right)\delta_h\,dx\\ &\geq \int_{\Omega_t} u_h^l\,D_t\psi_h^l\,\delta_h\,dx - c\,h\int_{\Omega_t}\left|u_h^l\right|\,\left|\nabla\psi_h^l\right|\,\delta_h\,dx\,. \end{aligned}\tag{3.22}$$

In the rest of this proof we are going to use the following abbreviations and notation for a more compact representation:

- We omit the lift superscript $l$. If discrete functions (marked by a subscript $h$) get evaluated or integrated on $\mathcal{N}_T = \bigcup_{t\in[0,T]}\{t\}\times\Omega_t$ we always have their lifted versions at hand.
- It is convenient to define the function space

  $$V_h^l(t) := \{f_h^l = f\circ\Gamma_h(t,\cdot)\mid f\in V_h(t)\}$$

  which consists of the lifted functions in $V_h(t)$. It is easy to verify that $V_h^l(t)$ is contained in $H_0^1(\Omega_t)$.
- Set $\mathcal{R}_h := R_h^{-1}R_h^{-\top}$. The integral over the spatial derivatives then reads:

  $$\alpha\int_{\Omega_t}\left(\nabla u_h^l R_h^{-1}\right)\cdot\left(\nabla\psi_h^l R_h^{-1}\right)\delta_h\,dx = \alpha\int_{\Omega_t}(\nabla u_h\mathcal{R}_h)\cdot\nabla\psi_h\,\delta_h\,dx\,.$$

We take the difference of (3.18) at $\psi_h = \psi_h^l$ and (3.22). The error relation between continuous and lifted discrete solution is then given by

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t}(u-u_h\,\delta_h)\,\psi_h\,dx &+ \alpha\int_{\Omega_t}\left(\nabla u - \nabla u_h\mathcal{R}_h\,\delta_h\right)\cdot\nabla\psi_h\,dx\\ &\leq \int_{\Omega_t}(u-u_h\,\delta_h)D_t\psi_h\,dx + ch\int_{\Omega_t}|u_h|\,\,|\nabla\psi_h|\,\delta_h\,dx \qquad \forall\psi_h(t,\cdot)\in V_h^l(t),\end{aligned}$$

or written in a more convenient form

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t}(u-u_h)\,\psi_h\,dx &+ \alpha\int_{\Omega_t}\nabla(u-u_h)\cdot\nabla\psi_h\,dx\\ \leq\ &\frac{d}{dt}\int_{\Omega_t}u_h\psi_h(\delta_h-1)\,dx + \alpha\int_{\Omega_t}\left(\nabla u_h(\mathcal{R}_h\delta_h - I)\right)\cdot\nabla\psi_h\,dx\\ &+\int_{\Omega_t}(u-u_h)D_t\psi_h\,dx + \int_{\Omega_t}u_h\,D_t\psi_h(1-\delta_h)\,dx + ch\int_{\Omega_t}|u_h|\,\,|\nabla\psi_h|\,\delta_h\,dx\,.\end{aligned}$$

**Some Kind of Energy Estimate**

We choose $\psi_h = \eta_h - u_h = u - u_h + \eta_h - u$ in the above estimate for some arbitrary $\eta_h(t,\cdot)\in V_h^l(t)$. Since $\psi_h(t,\cdot)\in V_h^l(t)$ for every $\eta_h$, it is an admissible test function. Therefore the following estimate is satisfied for every $\eta_h(t,\cdot) \in V_h^l(t)$

$$
\begin{aligned}
\frac{d}{dt}\int_{\Omega_t} (u-u_h)^2\,dx \;+\alpha \int_{\Omega_t} |\nabla(u-u_h)|^2\,dx \;-\int_{\Omega_t}(u-u_h)(D_t u - D_t u_h)\,dx \\
\le\; \frac{d}{dt}\int_{\Omega_t}(u-u_h)(u-\eta_h)\,dx \;+\alpha\int_{\Omega_t}\nabla(u-u_h)\cdot\nabla(u-\eta_h)\,dx \\
-\int_{\Omega_t}(u-u_h)(D_t u - D_t\eta_h)\,dx + \alpha\int_{\Omega_t}\Big((\mathcal{R}_h\delta_h - I)\nabla u_h\Big)\cdot\nabla(\eta_h-u_h)\,dx \\
+\int_{\Omega_t} u_h(D_t\eta_h - D_t u_h)(1-\delta_h)\,dx \;+\frac{d}{dt}\int_{\Omega_t} u_h(\eta_h-u_h)(\delta_h-1)\,dx \\
+\,ch\int_{\Omega_t}|u_h|\;|\nabla(\eta_h-u_h)|\;\delta_h\,dx\,.
\end{aligned}
\tag{3.23}
$$

Let us state some basic results we are about to apply:

- An application of the Leibniz formula gives

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega_t}(u-u_h)^2\,dx \;-\int_{\Omega_t}(u-u_h)(D_t u - D_t u_h)\,dx \\
&\quad= 2\int_{\Omega_t}(u-u_h)(D_t u - D_t u_h)\,dx + \int_{\Omega_t}(u-u_h)^2\,\mathrm{div}\,\underline{v}\,dx \;-\int_{\Omega_t}(u-u_h)(D_t u - D_t u_h)\,dx \\
&\quad= \frac{1}{2}\frac{d}{dt}\int_{\Omega_t}(u-u_h)^2\,dx \;+\frac{1}{2}\int_{\Omega_t}(u-u_h)^2\,\mathrm{div}\,\underline{v}\,dx \\
&\quad\ge \frac{1}{2}\frac{d}{dt}\int_{\Omega_t}(u-u_h)^2\,dx - c\int_{\Omega_t}(u-u_h)^2\,dx
\end{aligned}
$$

  where we used $\sup_{t\in[0,T]}\left\|\mathrm{div}\,\underline{v}(t,\cdot)\right\|_{L^\infty(\Omega_t)} \le c$.

- A similar computation shows

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega_t}(u-u_h)(u-\eta_h)\,dx \;-\int_{\Omega_t}(u-u_h)(D_t u - D_t\eta_h)\,dx \\
&\quad= \int_{\Omega_t}(D_t u - D_t u_h)(u-\eta_h)\,dx \;+\int_{\Omega_t}(u-u_h)(u-\eta_h)\,\mathrm{div}\,\underline{v}\,dx \\
&\quad\le c\int_{\Omega_t}\big(|u-u_h| + |D_t u - D_t u_h|\big)\;|u-\eta_h|\;dx\,.
\end{aligned}
$$

- The statement of Lemma 3.14 holds analogously for $R_h^{-1} = J_x\Gamma_h^{-1}$ and thus

$$
|I_2-\mathcal{R}_h| = \left|I_2 - R_h^{-1}R_h^{-\top}\right| = \left|R_h^{-1}\right|\left(\left|I_2-R_h^{-1}\right| + |I_2-R_h|\right) \le ch\,.
$$

  With the above estimate and Lemma 3.15 we then obtain

$$
\begin{aligned}
&\alpha\int_{\Omega_t}\Big(\nabla u_h(\mathcal{R}_h\delta_h - I)\Big)\cdot\nabla(\eta_h-u_h)\,dx \\
&\quad\le \alpha\int_{\Omega_t}|\nabla u_h(\mathcal{R}_h\delta_h-\delta_h+\delta_h-I)|\;|\nabla(\eta_h-u_h)|\;dx \\
&\quad\le \alpha\int_{\Omega_t}\Big(\delta_h\;|\mathcal{R}_h-I| + |\delta_h-1|\Big)\,|\nabla u_h|\;|\nabla(\eta_h-u_h)|\;dx \\
&\quad\le ch\int_{\Omega_t}|\nabla u_h|\Big(\;|\nabla(u-u_h)| + |\nabla(u-\eta_h)|\Big)\,dx\,.
\end{aligned}
$$

  Recall that $\delta_h$ and $\left|R_h^{-1}\right|$ are bounded by assumption.

- Again by Lemma 3.15

$$\int_{\Omega_t} u_h(D_t\eta_h - D_t u_h)\,(1-\delta_h)\,dx \le \int_{\Omega_t} |u_h|\,|D_t\eta_h - D_t u_h|\,|1-\delta_h|\,dx$$
$$\le ch \int_{\Omega_t} |u_h| \Big( |D_t u - D_t u_h| + |D_t u - D_t\eta_h| \Big)\,dx\;.$$

- This last inequality is straightforward

$$ch \int_{\Omega_t} |u_h|\,|\nabla(\eta_h - u_h)|\,\delta_h\,dx \le ch \int_{\Omega_t} |u_h| \Big( |\nabla(u-u_h)| + |\nabla(u-\eta_h)| \Big)\,dx\,.$$

We apply all these results to (3.23). Note that even if we apply exactly the estimates above, we rearrange summands for an easier representation. Thus for any $\eta_h(t,\cdot) \in V_h^l(t)$ we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} (u-u_h)^2\,dx + \alpha \int_{\Omega_t} |\nabla(u-u_h)|^2\,dx \\
&\le c\int_{\Omega_t} (u-u_h)^2\,dx + c\int_{\Omega_t} \big(|u-u_h| + |D_t u - D_t u_h|\big)\;|u-\eta_h|\,dx \\
&+ \alpha \int_{\Omega_t} |\nabla(u-u_h)|\,|\nabla(u-\eta_h)|\,dx \\
&+ ch \int_{\Omega_t} \Big( |u_h| + |\nabla u_h| \Big)\,|\nabla(u-u_h)|\,dx \\
&+ ch \int_{\Omega_t} \Big( |u_h| + |\nabla u_h| \Big)\,|\nabla(u-\eta_h)|\,dx \\
&+ ch \int_{\Omega_t} |u_h|\,|D_t u - D_t u_h|\,dx + ch \int_{\Omega_t} |u_h|\,|D_t u - D_t\eta_h|\,dx \\
&+ \frac{d}{dt}\int_{\Omega_t} u_h(u-u_h)(\delta_h - 1)\,dx - \frac{d}{dt}\int_{\Omega_t} u_h(u-\eta_h)(\delta_h-1)\,dx\,.
\end{aligned}$$

Again we state the single inequalities we want to apply:

- We apply Young's inequality and the Cauchy-Schwarz inequality to get

$$\begin{aligned}
c\int_{\Omega_t} \big(|u-u_h| + |D_t u - D_t u_h|\big)\,|u-\eta_h|\,dx \\
&= c\int_{\Omega_t} |u-u_h|\,|u-\eta_h|\,dx + c\int_{\Omega_t} |D_t u - D_t u_h|\,|u-\eta_h|\,dx \\
&\le c\int_{\Omega_t} |u-u_h|^2\,dx + c\int_{\Omega_t} |u-\eta_h|^2\,dx + c\,\|D_t u - D_t u_h\|_{0,\Omega_t}\,\|u-\eta_h\|_{0,\Omega_t} \\
&= c\,\|u-u_h\|^2_{0,\Omega_t} + c\,\|u-\eta_h\|^2_{0,\Omega_t} + c\,\|D_t u - D_t u_h\|_{0,\Omega_t}\,\|u-\eta_h\|_{0,\Omega_t}\;.
\end{aligned}$$

- Again with Young's inequality

$$\begin{aligned}
ch\int_{\Omega_t} \Big(|u_h| + |\nabla u_h|\Big)\,|\nabla(u-u_h)|\,dx \\
&= \int_{\Omega_t} ch\,|u_h|\;|\nabla(u-u_h)|\,dx + \int_{\Omega_t} ch\,|\nabla u_h|\;|\nabla(u-u_h)|\,dx \\
&\le ch^2\,\|u_h\|^2_{0,\Omega_t} + ch^2\,\|\nabla u_h\|^2_{0,\Omega_t} + \frac{\alpha}{4}\,\|\nabla(u-u_h)\|^2_{0,\Omega_t} \\
&\le ch^2\,\|u_h\|^2_{1,\Omega_t} + \frac{\alpha}{4}\,\|\nabla(u-u_h)\|^2_{0,\Omega_t}\;.
\end{aligned}$$

  Observe that we applied Young's inequality such that we have $\alpha/4$ as factor in front of $\|\nabla(u-u_h)\|^2_{0,\Omega_t}$.

- Analogously

$$ch \int_{\Omega_t} \Big( |u_h| + |\nabla u_h| \Big) \, |\nabla(u-\eta_h)| \, dx \le ch^2 \, \|u_h\|^2_{1,\Omega_t} + \|\nabla(u-\eta_h)\|^2_{0,\Omega_t}$$

- The Cauchy-Schwarz inequality yields

$$ch \int_{\Omega_t} |u_h| \, |D_t u - D_t u_h| \, dx \le ch \, \|u_h\|_{0,\Omega_t} \, \|D_t u - D_t u_h\|_{0,\Omega_t}$$

and

$$ch \int_{\Omega_t} |u_h| \, |D_t u - D_t \eta_h| \, dx \le ch \, \|u_h\|_{0,\Omega_t} \, \|D_t u - D_t \eta_h\|_{0,\Omega_t} \, .$$

- With Young's inequality and a factor modification

$$\alpha \int_{\Omega_t} |\nabla(u-u_h)| \, |\nabla(u-\eta_h)| \, dx \le \frac{\alpha}{4} \, \|\nabla(u-u_h)\|^2_{0,\Omega_t} + c \, \|\nabla(u-\eta_h)\|^2_{0,\Omega_t} \, .$$

Apply all these estimates at once to get:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} (u-u_h)^2 \, dx + \frac{\alpha}{2}\int_{\Omega_t} |\nabla(u-u_h)|^2 \, dx \\
&\quad \le c \, \|u-u_h\|^2_{0,\Omega_t} + c \, \|u-\eta_h\|^2_{0,\Omega_t} + c \, \|\nabla(u-\eta_h)\|^2_{0,\Omega_t} + c \, \|D_t u - D_t u_h\|_{0,\Omega_t} \, \|u-\eta_h\|_{0,\Omega_t} \\
&\qquad + ch \, \|u_h\|_{0,\Omega_t} \, \|D_t u - D_t u_h\|_{0,\Omega_t} + ch \, \|u_h\|_{0,\Omega_t} \, \|D_t u - D_t \eta_h\|_{0,\Omega_t} + ch^2 \, \|u_h\|^2_{1,\Omega_t} \\
&\qquad + \frac{d}{dt}\int_{\Omega_t} u_h(u-u_h)(\delta_h - 1) \, dx - \frac{d}{dt}\int_{\Omega_t} u_h(u-\eta_h)(\delta_h - 1) \, dx \, . 
\end{aligned} \tag{3.24}$$

Now we integrate with respect to time: Denote $u_{h0} = u_h(0,\cdot) \in V_h^l(0)$. The last summands in (3.24) can then be bounded by Lemma 3.15 and an application of the Cauchy-Schwarz inequality

$$\begin{aligned}
&\int_0^t \frac{d}{d\tau}\int_{\Omega_\tau} u_h(u-u_h)(\delta_h - 1) \, dx \, d\tau - \int_0^t \frac{d}{d\tau}\int_{\Omega_\tau} u_h(u-\eta_h)(\delta_h - 1) \, dx \, d\tau \\
&\quad = \int_{\Omega_t} u_h(u-u_h)(\delta_h - 1) \, dx - \int_{\Omega_0} u_h(u-u_h)(\delta_h - 1) \, dx \\
&\qquad - \int_{\Omega_t} u_h(u-\eta_h)(\delta_h - 1) \, dx + \int_{\Omega_0} u_h(u-\eta_h)(\delta_h - 1) \, dx \\
&\quad \le ch \, \|u_h\|_{0,\Omega_t} \, \|u-u_h\|_{0,\Omega_t} + ch \, \|u_{h0}\|_{0,\Omega_0} \, \|u_0 - u_{h0}\|_{0,\Omega_0} \\
&\qquad + ch \, \|u_h\|_{0,\Omega_t} \, \|u-\eta_h\|_{0,\Omega_t} + ch \, \|u_{h0}\|_{0,\Omega_0} \, \|u_0 - \eta_{h0}\|_{0,\Omega_0}
\end{aligned}$$

and we obtain from (3.24)

$$\begin{aligned}
&\|u-u_h\|^2_{0,\Omega_t} + \frac{\alpha}{2}\int_0^t \|\nabla(u-u_h)\|^2_{0,\Omega_\tau} \, d\tau \\
&\quad \le \|u_0 - u_{h0}\|^2_{0,\Omega_0} + c\int_0^t \|u-u_h\|^2_{0,\Omega_\tau} \, d\tau + c\int_0^t \|u-\eta_h\|^2_{0,\Omega_\tau} \, d\tau \\
&\qquad + c\int_0^t \|\nabla(u-\eta_h)\|^2_{0,\Omega_\tau} \, d\tau + ch\int_0^t \|u_h\|_{0,\Omega_\tau} \, \|D_t u - D_t u_h\|_{0,\Omega_\tau} \, d\tau \\
&\qquad + ch\int_0^t \|u_h\|_{0,\Omega_\tau} \, \|D_t u - D_t \eta_h\|_{0,\Omega_\tau} \, d\tau + ch^2\int_0^t \|u_h\|^2_{1,\Omega_\tau} \, d\tau \\
&\qquad + ch \, \|u_h\|_{0,\Omega_t} \, \|u-u_h\|_{0,\Omega_t} + ch \, \|u_{h0}\|_{0,\Omega_0} \, \|u_0 - u_{h0}\|_{0,\Omega_0} \\
&\qquad + ch \, \|u_h\|_{0,\Omega_t} \, \|u-\eta_h\|_{0,\Omega_t} + ch \, \|u_{h0}\|_{0,\Omega_0} \, \|u_0 - \eta_{h0}\|_{0,\Omega_0} \, .
\end{aligned} \tag{3.25}$$

### Inserting the Interpolation

Remark 3.22, which follows after this proof, considers the following interpolations: We choose $\eta_h = \mathcal{I}_h u$ as the interpolation of $u$ at the nodes $a_i(t)$, $i = 1, \ldots, d$ on $\Omega_t$ and set the initial value of $u_h$ as $u_{h0} = I_h u_0$. The $H^2$-regularity (see Assumption 1.15) of the problem and Remark 3.22 give

$$\|u - \eta_h\|_{1,\Omega_t} = \|u(t,\cdot) - (\mathcal{I}_h u)(t,\cdot)\|_{1,\Omega_t} \le ch\, \|u\|_{2,\Omega_t} \ .$$

Additionally, from the inverse triangle inequality we know

$$\|u_{h0}\|_{0,\Omega_0} - \|u_0\|_{0,\Omega_0} \le \|u_{h0} - u_0\|_{0,\Omega_0} \le ch\, \|u_0\|_{1,\Omega_0}$$

which implies

$$\|u_{h0}\|_{0,\Omega_0} \le c\, \|u_0\|_{1,\Omega_0} \ . \tag{3.26}$$

We will apply the following estimates to (3.25):

- We use the results discussed in Remark 3.22:

$$\|u_0 - u_{h0}\|^2_{0,\Omega_0} = \|u_0 - I_h u_0\|^2_{0,\Omega_0} \le ch^4\, \|u_0\|^2_{2,\Omega_0} \ .$$

- The $H^2$-regularity assures

$$\begin{aligned} c\int_0^t \|u - \eta_h\|^2_{0,\Omega_\tau}\, d\tau + c\int_0^t \|\nabla(u - \eta_h)\|^2_{0,\Omega_\tau}\, d\tau &= c\int_0^t \|u(\tau,\cdot) - (\mathcal{I}_h u)(\tau,\cdot)\|^2_{1,\Omega_\tau}\, d\tau \\ &\le ch^2 \int_0^t \|u(\tau,\cdot)\|^2_{2,\Omega_\tau}\, d\tau \\ &\le ch^2\, \|u_0\|^2_{1,\Omega_0} \ . \end{aligned}$$

- With Young's inequality, (3.26) and the stability estimate for the lifted solution from Lemma 3.19, we get

$$\begin{aligned} ch\int_0^t \|u_h\|_{0,\Omega_\tau}\, \|D_t u - D_t u_h\|_{0,\Omega_\tau}\, d\tau &= \int_0^t \left(ch^{1/2}\, \|u_h\|_{0,\Omega_\tau}\right)\left(h^{1/2}\, \|D_t u - D_t u_h\|_{0,\Omega_\tau}\right) d\tau \\ &\le ch\int_0^t \|u_h\|^2_{0,\Omega_\tau}\, d\tau + ch\int_0^t \|D_t u - D_t u_h\|^2_{0,\Omega_\tau}\, d\tau \\ &\le ch\, \|u_{h0}\|^2_{0,\Omega_0} + ch\int_0^t \|D_t u - D_t u_h\|^2_{0,\Omega_\tau}\, d\tau \\ &\le ch\, \|u_0\|^2_{1,\Omega_0} + ch\int_0^t \|D_t u - D_t u_h\|^2_{0,\Omega_\tau}\, d\tau \ . \end{aligned}$$

- Similarly

$$ch\int_0^t \|u_h\|_{0,\Omega_\tau}\, \|D_t u - D_t \eta_h\|_{0,\Omega_\tau}\, d\tau \le ch\, \|u_0\|^2_{1,\Omega_0} + ch\int_0^t \|D_t u - D_t \eta_h\|^2_{0,\Omega_\tau}\, d\tau \ .$$

- The stability results from Lemma 3.19 and (3.26) yield

$$ch^2\int_0^t \|u_h\|^2_{1,\Omega_\tau}\, d\tau \le ch^2\, \|u_{h0}\|^2_{0,\Omega_0} \le ch^2\, \|u_0\|^2_{1,\Omega_0} \ .$$

- We use the triangle inequality and the two stability results for $u$ and the lifted semi-discrete solution $u_h$ to bound :

$$\begin{aligned}
ch\ \|u_h\|_{0,\Omega_t}\ \|u-u_h\|_{0,\Omega_t} + ch\ \|u_{h0}\|_{0,\Omega_0}\ \|u_0-u_{h0}\|_{0,\Omega_0} \\
&= ch\ \|u_h(t,\cdot)\|_{0,\Omega_t}\ \|u(t,\cdot)-u_h(t,\cdot)\|_{0,\Omega_t} + ch\ \|u_{h0}\|_{0,\Omega_0}\ \|u_0-u_{h0}\|_{0,\Omega_0} \\
&\le ch\ \|u_h(t,\cdot)\|_{0,\Omega_t}\left(\ \|u(t,\cdot)\|_{0,\Omega_t} + \|u_h(t,\cdot)\|_{0,\Omega_t}\right) \\
&\qquad + ch\ \|u_{h0}\|_{0,\Omega_0}\left(\ \|u_0\|_{0,\Omega_0} + \|u_{h0}\|_{0,\Omega_0}\right) \\
&\le ch\ \|u_{h0}\|_{0,\Omega_0}\left(\ \|u_0\|_{0,\Omega_0} + \|u_{h0}\|_{0,\Omega_0}\right) + ch\ \|u_0\|^2_{1,\Omega_0} \\
&\le ch\ \|u_0\|^2_{1,\Omega_0}
\end{aligned}$$

- An analogous computation, which uses the interpolation estimates and the $H^2$-regularity of $u$, gives

$$ch\ \|u_h\|_{0,\Omega_t}\ \|u-\eta_h\|_{0,\Omega_t} + ch\ \|u_{h0}\|_{0,\Omega_0}\ \|u_0-\eta_{h0}\|_{0,\Omega_0} \le ch^2\ \|u_0\|^2_{1,\Omega_0}$$

Gathering these results, we get from (3.25)

$$\begin{aligned}
\|u-u_h\|^2_{0,\Omega_t} + \frac{\alpha}{2}\int_0^t \|\nabla(u-u_h)\|^2_{0,\Omega_\tau}\,d\tau \\
&\le c\int_0^t \|u-u_h\|^2_{0,\Omega_\tau}\,d\tau \\
&\quad + ch\ \|u_0\|^2_{1,\Omega_0} + ch^2\ \|u_0\|^2_{1,\Omega_0} + ch^4\ \|u_0\|^2_{2,\Omega_0} \\
&\quad + ch\int_0^t \|D_t u - D_t u_h\|^2_{0,\Omega_\tau}\,d\tau + ch\int_0^t \|D_t u - D_t\,[\mathcal{I}_h u]\|^2_{0,\Omega_\tau}\,d\tau\,.
\end{aligned} \tag{3.27}$$

For the last two terms on the right-hand side of (3.27):

- With the stability estimates and a $\|u_{h0}\|_{1,\Omega_0} \le c\ \|u_0\|_{2,\Omega_0}$ (cf. (3.26))

$$\begin{aligned}
\int_0^t \|D_t u - D_t u_h\|^2_{0,\Omega_\tau}\,d\tau \le \int_0^t \|D_t u\|^2_{0,\Omega_\tau}\,d\tau + \int_0^t \|D_t u_h\|^2_{0,\Omega_\tau}\,d\tau &\le c\ \|u_0\|^2_{1,\Omega_0} + c\ \|u_{h0}\|^2_{1,\Omega_0} \\
&\le c\ \|u_0\|^2_{2,\Omega_0}\ .
\end{aligned}$$

- And with (3.31) we get

$$\begin{aligned}
\int_0^t \left\|D_t u - D_t[\mathcal{I}_h u]\right\|^2_{0,\Omega_\tau}\,d\tau = \int_0^t \|D_t u - \mathcal{I}_h D_t u\|^2_{0,\Omega_\tau}\,d\tau \le ch^4\int_0^t \|D_t u\|^2_{2,\Omega_\tau}\,d\tau \\
\le ch^4\int_0^T \|D_t u\|^2_{2,\Omega_\tau}\,d\tau\,.
\end{aligned}$$

Thus (3.27) gives

$$\begin{aligned}
\|u-u_h\|^2_{0,\Omega_t} + \frac{\alpha}{2}\int_0^t \|\nabla(u-u_h)\|^2_{0,\Omega_\tau}\,d\tau \le c\int_0^t \|u-u_h\|^2_{0,\Omega_\tau}\,d\tau + ch^4\int_0^T \|D_t u\|^2_{2,\Omega_\tau}\,d\tau \\
+ ch\ \|u_0\|^2_{2,\Omega_0} + ch^2\ \|u_0\|^2_{1,\Omega_0} + ch^4\ \|u_0\|^2_{2,\Omega_0}\ .
\end{aligned} \tag{3.28}$$

We collect the time independent terms:

$$R := ch\ \|u_0\|^2_{2,\Omega_0} + ch^2\ \|u_0\|^2_{1,\Omega_0} + ch^4\ \|u_0\|^2_{2,\Omega_0} + ch^4\int_0^T \|D_t u\|^2_{2,\Omega_\tau}\,d\tau\,.$$

An application of the Gronwall lemma mentioned in Proposition A.5 to

$$\|u(t,\cdot) - u_h(t,\cdot)\|^2_{0,\Omega_t} \leq c \int_0^t \|u(\tau,\cdot) - u_h(\tau,\cdot)\|^2_{0,\Omega_\tau} \; d\tau \; + R$$

implies

$$\|u(t,\cdot) - u_h(t,\cdot)\|^2_{0,\Omega_t} \leq R + \int_0^t e^{c(t-s)} Rc \, ds \, .$$

And we get the first part of our error estimate:

$$\sup_{t\in[0,T]} \|u - u_h\|^2_{0,\Omega_t} = \sup_{t\in[0,T]} \|u(t,\cdot) - u_h(t,\cdot)\|^2_{0,\Omega_t} \leq R + \int_0^t e^{c(t-s)} Rc \, ds \; \leq cR \, . \tag{3.29}$$

We use (3.28) and apply (3.29), to get the second part:

$$\int_0^T \|\nabla(u - u_h)\|^2_{0,\Omega_\tau} \; d\tau \leq c \int_0^T \|u(\tau,\cdot) - u_h(\tau,\cdot)\|^2_{0,\Omega_\tau} \; d\tau \; + R \leq cR + R \leq cR \, .$$

Combining these two parts, we have

$$\sup_{t\in[0,T]} \|u - u_h\|^2_{0,\Omega_t} + \int_0^T \|\nabla(u - u_h)\|^2_{0,\Omega_\tau} \; d\tau$$
$$\leq ch \; \|u_0\|^2_{2,\Omega_0} + ch^2 \; \|u_0\|^2_{1,\Omega_0} + ch^4 \; \|u_0\|^2_{2,\Omega_0} + ch^4 \int_0^T \|D_t u\|^2_{2,\Omega_\tau} \; d\tau$$

and the Theorem 2.8 is proved. □

*Remark* 3.22. In the above proof we need an interpolant of the solution of the continuous problem $u$. This remark is concerned with construction of this interpolant and an error bound:

Let $\eta : \mathcal{N}_T \to R$ with $\eta(t,x) = 0$ for $x \in \partial\Omega_t$. We want to interpolate $\eta$ at the moving nodes $a_i(t) \in \Omega_t$, $i = 1,\ldots,d$ and such that the interpolant is an admissible test function, i.e. it lies in $V_h^l(t)$ for all $t \in [0,T]$. Let

$$\tilde{\eta}(t,y) := \eta(t,\Phi(t,y)) \qquad , y \in \Omega_0 \, .$$

denote the transformation of $\eta$ onto the initial domain $\Omega_0$. We set the interpolant of $\tilde{\eta}$ with respect to some triangulation $T_h$ of $\Omega_0$ as

$$\Big(I_h\tilde{\eta}(t,\cdot)\Big)(y) := \sum_{i=1}^{d} \tilde{\eta}(t,a_i(0))\hat{\varphi}_i(y) = \sum_{i=1}^{d} \eta(t,a_i(t))\hat{\varphi}_i(y) \, .$$

This is the standard piecewise interpolation with respect to a given triangulation. The time dependency of $\eta$ corresponds to time dependent coefficients in the basis representation. Formally we then define the wished for interpolation as

$$\Big(\mathcal{I}_h\eta\Big)(t,x) := \Big(I_h\tilde{\eta}(t,\cdot)\Big) \circ \Phi^{-1}(t,x) \, . \tag{3.30}$$

Or given in a more direct way: We call

$$\Big(\mathcal{I}_h\eta\Big)(t,x) := \sum_{i=1}^{d} \eta(t,a_i(t))\hat{\varphi}_i(\Phi^{-1}(t,x)) \, .$$

the interpolant of $\eta$ on the continuously evolving domain $\Omega_t$. By construction of the $\hat{\varphi}_i$, the interpolant can be written as

$$\Big(\mathcal{I}_h\eta\Big)(t,x) = \sum_{i=1}^{d} \eta(t,a_i(t))\varphi_i(t,\Phi_h(t,\Phi^{-1}(t,x))) = \sum_{i=1}^{d} \eta(t,a_i(t))\varphi_i(t,\Gamma_h(t,x)) \, .$$

Hence $\left(\mathcal{I}_h\eta\right)(t,\cdot)$ is for each $t \in [0,T]$ the lift of a function from $V_h(t)$ and therefore an admissible test function.

Moreover the interpolation of the material derivative of $\eta$ is the material derivative of the interpolant of $\eta$, since

$$\begin{aligned} D_t\left[\mathcal{I}_h\eta\right](t,\Phi(t,y)) = \frac{d}{dt}\left[\,(\mathcal{I}_h\eta)\,(t,\Phi(t,y))\right] &= \frac{d}{dt}\left[(I_h\tilde{\eta}(t,\cdot))(y)\right] \\ &= \frac{d}{dt}\sum_{i=1}^{d}\eta(t,a_i(t))\hat{\varphi}_i(y) \\ &= \sum_{i=1}^{d}\frac{d}{dt}[\eta(t,\Phi(t,a_i))]\hat{\varphi}_i(y) \\ &= \sum_{i=1}^{d}D_t\eta(t,a_i(t))\hat{\varphi}_i(y) \\ &= \left(I_hD_t\eta(t,\cdot)\right)(y) \end{aligned}$$

implies

$$D_t\left[\mathcal{I}_h\eta\right] = \mathcal{I}_hD_t\eta\,. \tag{3.31}$$

Error bounds can then be obtained quite simply: Let $f:\Omega_0\to\mathbb{R}$. We know (e.g. from [1, Theorem 9.3.8] ) that the piecewise linear affine interpolant of $f$ with respect to $T_h$, which is again denoted by $I_hf$, satisfies

$$\|f-I_hf\|_{0,\Omega_0}\le ch\;|f|_{1,\Omega_0}$$

if $f\in H^1(\Omega_0)$ and

$$\|f-I_hf\|_{1,\Omega_0}\le ch\;|f|_{2,\Omega_0}$$

if $f\in H^2(\Omega_t)$. The regularity of $\Phi(t,\cdot)$ assures the existence of constants $C,c>0$ such that

$$c\;\left\|f\circ\Phi^{-1}(t,\cdot)\right\|_{k,\Omega_t}\le\;\|f\|_{k,\Omega_0}\le C\;\left\|f\circ\Phi^{-1}(t,\cdot)\right\|_{k,\Omega_t}\qquad k=0,1,2\,.$$

Now let $\eta:\mathcal{N}_T\to R$, as above, with $\tilde{\eta}(t,\Phi^{-1}(t,x))=\eta(t,x)$. With the above estimates and (3.30), we have

$$\|\eta(t,\cdot)-(\mathcal{I}_h\eta)(t,\cdot)\|_{0,\Omega_t}\le c\;\|\tilde{\eta}(t,\cdot)-I_h\tilde{\eta}(t,\cdot)\|_{0,\Omega_0}\le ch\;|\tilde{\eta}(t,\cdot)|_{1,\Omega_0}\le ch\;|\eta(t,\cdot)|_{1,\Omega_t}\;.$$

The results for higher order Sobolev spaces follow analogously.

# Chapter 4

# An Exponential Integrator for Parabolic Evolution Problems

Eventually we are able to present the primary motivation for this thesis: A new numerical time integrator. The diffusion equation on evolving domains (1.11) and its semi-discrete version (3.17) serve as a non-trivial model example for a non-autonomous evolution problem.

We use (3.17) to test an exponential time integrator, which was proposed by Marlis Hochbruck and Alexander Ostermann and presented in [10] for the first time.

There are several numerical methods (see e.g. [14] for Runge-Kutta methods) which can be applied to linear initial value problems with time dependent operators. But as far as we know, there is no exponential integrator which explicitly takes advantage of the structure of the evolution problem:

$$\begin{cases} \frac{d}{dt}u(t) + A(t)u(t) = 0 \quad 0 \le t \le T \\ u(0) = u_0 \, . \end{cases} \tag{4.1}$$

The construction of the exponential integrator is closely related to the construction of the solution of (4.1) in Chapter 2. In order to approximate a solution of (4.1) in a reasonable way, we need the problem to be well-posed:

**Assumption 4.1** (Well-posedness)**.** *These are the same assumptions as in Chapter 2:*

**$(A_1)$** *The domain $D(A(t)) = D$ of $A(t)$, $0 \le t \le T$ is dense in $X$ and independent of $t$.*

**$(A_2)$** *For $t \in [0, T]$, the resolvent $R(\lambda, A(t)) = (\lambda - A(t))^{-1}$ exists for all $\lambda$ with $\operatorname{Re}\lambda \le 0$ and there is a constant $M$ such that*

$$\|R(\lambda, A(t))\| \le \frac{M}{|\lambda| + 1} \qquad \textit{for } \operatorname{Re}\lambda \le 0,\ t \in [0, T].$$

**$(A_3)$** *There exists a constant $L > 0$ such that*

$$\left\|(A(t) - A(s))A(\tau)^{-1}\right\| \le L\,|t - s| \qquad \textit{for } s, t, \tau \in [0, T].$$

*Remark* 4.2. In contrast to the notation of Chapter 2, now and in the following, $-A(t)$ is the generator of the semigroup $e^{-sA(t)}$ for $s \ge 0$. Since numerical time integrators are in practice often applied to semi-discretized PDEs and the semigroup generated by a matrix $\mathbf{A}$ coincides with the exponential function of $\mathbf{A}$, this change of notation is evident.

## 4.1 Construction of the Integrator

This section is based on [12]. A full numerical analysis including the proof of convergence is not available at this point and will be reported elsewhere.

Let $\tau \in [0, T]$ be small. We start by deriving an approximation for $u(\tau)$: Theorem 2.8 states that the solution $u$ of (4.1) at time $\tau$ is given by

$$u(\tau) = U(\tau, 0)u(0) = U(\tau, 0)u_0 \,,$$

where $(t, s) \mapsto U(t, s)$ is the evolution system that we constructed to solve (4.1). It serves as the propagator of the solution. We want to approximate this evolution system and use the ansatz (2.4) to find

$$u(\tau) = U(\tau, 0)u(0) = e^{-\tau A_0}u_0 + \int_0^\tau e^{-(\tau-s)A(s)}R(s, 0)u_0 \, ds \,, \tag{4.2}$$

where the operator $R(\tau, 0)$ is by (2.7) the solution of the integral equation

$$R(\tau, 0) = R_1(\tau, 0) + \int_0^\tau R_1(\tau, s)\, R(s, 0)\, ds$$

with

$$R_1(\tau, s) := (A(s) - A(\tau))e^{-(\tau-s)A(s)} \,.$$

*Remark* 4.3. The final numerical scheme will only require evaluations of $t \mapsto A(t)$ at certain nodes, thus we use the following abbreviations

$$A_0 := A(0), \qquad A_{1/2} := A(\tfrac{\tau}{2}), \qquad A_1 := A(\tau) \,.$$

We apply the following two approximations

- From Section 2.2.1, we know

  $$R(\tau, 0) = \sum_{m=1}^{\infty} R_m(\tau, 0), \qquad \|R_m(\tau, 0)\| \le \frac{C^m}{(m-1)!}\tau^{m-1}$$

  and hence the truncation of the series

  $$R(s, 0) = \sum_{m=1}^{\infty} R_m(s, 0) \approx R_1(s, 0) = (A(0) - A(s))e^{-sA_0}$$

  gives at least a first order approximation.

- In (4.2) we fix the semigroup in the integrand

  $$\int_0^\tau e^{-(\tau-s)A(s)}R(s, 0)u_0 \, ds \approx \int_0^\tau e^{-(\tau-s)A_1}R(s, 0)u_0 \, ds \,.$$

and define $\tilde{u}(t) \approx u(t)$ as

$$\tilde{u}(\tau) = e^{-\tau A_0}u_0 + \int_0^\tau e^{-(\tau-s)A_1}R_1(s, 0)u_0 \, ds \,. \tag{4.3}$$

*Remark* 4.4. Under supplementary assumptions, as the continuous differentiability of $t \mapsto A(t)A(0)^{-1}$, it can be shown that $\tilde{u}(\tau)$ is a fourth order approximation of $u(\tau)$.

Continuing with (4.3), we replace the integrand $R_1(s,0)$ by its quadratic interpolation polynomial at the nodes $\theta = 0, 1/2, 1$. Since $R_1(0,0) = 0$, we obtain

$$R_1(\theta\tau, 0) \approx p(\theta\tau) = \theta\left(4R_1\left(\tfrac{\tau}{2}, 0\right) - R_1(\tau, 0)\right) + 2\theta^2\left(R_1(\tau, 0) - 2R_1\left(\tfrac{\tau}{2}, 0\right)\right). \tag{4.4}$$

Inserting the interpolation polynomial $p$ into $\tilde{u}(\tau)$, we find

$$\begin{aligned}
\tilde{u}(\tau) &= e^{-\tau A_0}u_0 + \tau\int_0^1 e^{-\tau(1-\theta)A_1}R_1(\tau\theta, 0)u_0\,d\theta \\
&\approx e^{-\tau A_0}u_0 + \tau\int_0^1 e^{-\tau(1-\theta)A_1}p(\theta\tau)u_0\,d\theta \\
&= e^{-\tau A_0}u_0 + \tau\int_0^1 e^{-\tau(1-\theta)A_1}\theta\,d\theta\,\left(4R_1\left(\tfrac{\tau}{2}, 0\right) - R_1(\tau, 0)\right) \\
&\quad + 4\tau\int_0^1 e^{-\tau(1-\theta)A_1}\frac{\theta^2}{2}\,d\theta\,\left(R_1(\tau, 0) - 2R_1\left(\tfrac{\tau}{2}, 0\right)\right) \\
&=: \hat{u}(\tau)
\end{aligned}$$

*Remark* 4.5. Exponential integrators are a special class of explicit numerical methods for the time integration of stiff differential equations. Normally, exponential methods are derived by linearizing differential equations and then using the variation of constants formula to find a proper numerical scheme. It is naturally to this approach that there arise functions of the type

$$\varphi_k(z) := \int_0^1 e^{(1-\theta)z}\frac{\theta^{k-1}}{(k-1)!}\,d\theta \qquad k \geq 1. \tag{4.5}$$

By using operator calculus we to evaluate $\varphi_k(-\tau A)$ for linear operators $A$. This is done by understanding

$$e^{rA}$$

as the $C_0$ semigroup generated by $A$. Thus we are only allowed to insert infinitesimal generators of $C_0$ or analytic semigroups into the $\varphi_k$:

$$\varphi_k(-\tau A) := \int_0^1 e^{-\tau(1-\theta)A}\frac{\theta^{k-1}}{(k-1)!}\,d\theta\,.$$

A very readable introduction into the theory of exponential integrators is [13].

**Lemma 4.6** ([13, Lemma 2.4])**.** *For a infinitesimal generator A of a strongly continuous semigroup, the operators $\varphi_k(-\tau A)$, as defined in the above remark, are bounded with respect to the underlying Banach space X.*

*Proof.* The bound follows from $\left\|e^{-\tau A}\right\| \leq C$ and the estimate

$$\begin{aligned}
\|\varphi_k(-\tau A)\| &\leq \int_0^1 \left\|e^{\tau(1-\theta)A}\right\|\frac{\theta^{k-1}}{(k-1)!}\,d\theta \\
&\leq C\int_0^1 \frac{\theta^{k-1}}{(k-1)!}\,d\theta \\
&\leq C\frac{1}{k!}\,.
\end{aligned}$$

□

Applying the definitions of the above remark to $\hat{u}(\tau)$, we obtain

$$\hat{u}(\tau) = e^{-\tau A_0}u_0 + \tau\varphi_2(-\tau A_1)\left(4R_1\left(\tfrac{\tau}{2},0\right) - R_1(\tau,0)\right)u_0 + 4\tau\varphi_3(-\tau A_1)\,\left(R_1(\tau,0) - 2R_1\left(\tfrac{\tau}{2},0\right)\right)u_0$$

and by reordering and expanding the $R_1(\cdot,0)$ terms

$$\begin{aligned}
\hat{u}(\tau) &= e^{-\tau A_0}u_0 + \tau\varphi_2(-\tau A_1)\left(4R_1\left(\tfrac{\tau}{2},0\right) - R_1(\tau,0)\right)u_0 + 4\tau\varphi_3(-\tau A_1)\,\left(R_1(\tau,0) - 2R_1\left(\tfrac{\tau}{2},0\right)\right)u_0 \\
&= e^{-\tau A_0}u_0 + 4\tau\Big(\varphi_2(-\tau A_1) - 2\varphi_3(-\tau A_1)\Big)R_1\left(\tfrac{\tau}{2},0\right)u_0 + \tau\Big(4\varphi_3(-\tau A_1) - \varphi_2(-\tau A_1)\Big)R_1\left(\tfrac{\tau}{2},0\right)u_0 \\
&= e^{-\tau A_0}u_0 + 4\tau\Big(\varphi_2(-\tau A_1) - 2\varphi_3(-\tau A_1)\Big)(A_0 - A_{1/2})e^{-\frac{\tau}{2}A_0}u_0 \\
&\quad + \tau\Big(4\varphi_3(-\tau A_1) - \varphi_2(-\tau A_1)\Big)(A_0 - A_1)e^{-\tau A_0}u_0\,.
\end{aligned}$$

We now define a numerical scheme, that approximates $u(t_{n+1})$ starting at $u(t_n)$.

**Definition 4.7** (Numerical Scheme)**.** *Let $u_n \approx u(t_n)$ be the numerical approximation to the exact solution at time*

$$t_n = n\tau, \qquad n = 0, 1, 2, \ldots$$

*then the above constructed method defines a one-term recursion, given by*

$$u_{n+1} = T_n u_n \tag{4.6}$$

*with the discrete evolution operator*

$$\begin{aligned}
T_n := e^{-\tau A_n} &+ 4\tau\Big(\varphi_2(-\tau A_{n+1}) - 2\varphi_3(-\tau A_{n+1})\Big)(A_n - A_{n+\frac{1}{2}})e^{-\frac{\tau}{2}A_n} \\
&+ \tau\Big(4\varphi_3(-\tau A_{n+1}) - \varphi_2(-\tau A_{n+1})\Big)(A_n - A_{n+1})e^{-\tau A_n},
\end{aligned} \tag{4.7}$$

*We refer to* (4.7) *as ExpInt.*

The construction of $T_n$ suggests that it adheres to a local error bound of order four and a global convergence of order three.

## 4.2 Application of ExpInt to our Problem

Recall the ODE (3.17) we derived:

$$\frac{d}{dt}\Big[\mathbf{M}_h(t)\mathbf{u}(t)\Big] = -\mathbf{S}_h(t)\mathbf{u}(t)$$

with the initial value $\mathbf{u}(0) = \mathbf{u}_0$. We want to apply the integrator (4.7) to this problem and thus have to transform it into a Cauchy problem formulation. Let

$$\mathbf{y}(t) = \mathbf{M}_h(t)\mathbf{u}(t)\,. \tag{4.8}$$

We then apply (4.8) to (3.17) to obtain the equivalent problem

$$\frac{d}{dt}\mathbf{y}(t) + \mathbf{S}_h(t)\mathbf{M}_h(t)^{-1}\mathbf{y}(t) = 0 \tag{4.9}$$

with $\mathbf{y}(0) = \mathbf{M}_h(0)\mathbf{u}(0)$.

Now having the formulation (4.9) at hand we could apply ExpInt right away. But we can not expect the integrator to approximate the solution of (4.9) without verifying $(A_1)$ - $(A_3)$ and the additional smoothness assumptions that are made during the numerical analysis of the time integrator. Since the latter are not yet fully stated, we concentrate on showing that (4.9) satisfies $(A_1)$ - $(A_3)$.

### 4.2.1 First Assumption: Domain of the Operator

First we consider:

($A_1$) The domain $D(A(t)) = D$ of $A(t)$, $0 \leq t \leq T$ is dense in $X$ and independent of $t$.

We apply the integrator (4.7) to the ODE (4.9), thus the operator $A(t)$ is in fact a matrix:

$$A(t) = \mathbf{A}_h(t) = \mathbf{S}_h(t)\mathbf{M}_h(t)^{-1} \in \mathbb{R}^{d\times d} . \tag{4.10}$$

*Remark* 4.8. $\mathbf{A}_h(t)$ can be understood as the discretization of the operator $\tilde{A}(t)$ we constructed in Section 2.3: If we assume there are $\tilde{A}(t)$ such that the theory of chapter 2 can be applied, we can use a FEM as in [7, Chapter 3] to obtain a discrete version of $\tilde{A}(t)$. We expect the discretized $\tilde{A}(t)$ to behave in the same way as $\mathbf{A}_h(t)$. Of course, this association can only be used to better understand the properties of $\mathbf{A}_h(t)$, since e.g. in our example also the mass matrix is time dependent.

At a first glance it seems like all the difficulties we encountered when considering the abstract differential equation in a Banach space setting vanished, since our problem is of finite dimension. But with $\mathbf{A}_h(t)$ being itself the result of a spatial discretization, these difficulties are just hidden.

At this point it is not yet clear how to choose an underlying Banach space, respectively a norm on $\mathbb{R}^d$, which we can use to prove estimates as in ($A_2$) and ($A_3$). This norm will naturally arise in the proof of ($A_2$). Nevertheless, we can make some basic considerations concerning ($A_1$):

1. Obviously $\mathbf{A}_h(t)$ depends on $h$, the parameter of the spatial discretization. As $h \to 0$ the dimension of the linear system $d$ goes to infinity, thus making any constants depending on $d$ useless. These issues arise during a full-discretization of a PDE and can lead to an overall order reduction (see e.g. [20]). Hence for a uniform - independently of the spatial discretization $h$ - convergence of the time integrator it is necessary to find constants fulfilling our assumptions independently of $d$ and $h$.

2. As far as ($A_1$) is concerned we want to emphasize that the identification mapping

$$\Pi_h(t) : \mathbb{R}^d \to V_h(t),\ \mathbf{y} = (y_i)_{i=1}^d \mapsto \sum_{i=1}^{d} y_i\varphi_i(t,\cdot) .$$

   as defined in (3.15) assures that vectors always correspond functions in $H_0^1(\Omega_t^h)$. As a consequence these vectors will always lie in the domain of the formal differential operator $\lim_{h\to 0} \mathbf{A}_h(t)$.

   Nonetheless, if $f \in L^2(\Omega_t^h)$ and $\mathbf{f}_h$ is the sequence of vectors such that $\Pi_h(t)\mathbf{f}_h$ is the discretization of $f$ with respect to $V_h(t)$, we can expect

$$\lim_{h\to 0} \|\mathbf{A}_h(t)\mathbf{f}_h\| \leq c$$

   only if $f \in H_0^1(\Omega_t^h)$, whereas in general

$$\lim_{h\to 0} \|\mathbf{A}_h(t)\mathbf{f}_h\| = \infty .$$

### 4.2.2 Second Assumption: Resolvent Estimate

This section is concerned with the proof of:

($A_2$) For $t \in [0, T]$, the resolvent $R(\lambda, A(t)) = (\lambda - A(t))^{-1}$ exists for all $\lambda$ with $\mathrm{Re}\,\lambda \leq 0$ and there is a constant $M$ such that

$$\|R(\lambda, A(t))\| \leq \frac{M}{|\lambda| + 1} \qquad \text{for } \mathrm{Re}\,\lambda \leq 0,\ t \in [0, T].$$

Note that the right norm $\|\cdot\|$ to use for the above estimate is yet to determine and ($A_2$) and ($A_3$) have to hold with respect to the same norm.

The proof is given in three steps. In the first two steps we omit the time dependence, since the third step tackles these problems separately.

**First Step: Initial Estimate**

For fixed $t \in [0, T]$, let $\Omega = \Omega_t^h$. We set $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. Our starting point is the following stationary problem: Let $\lambda \in \mathbb{C}$, $\alpha > 0$ and $f \in V'$. Find $u \in V$ such that

$$\lambda(u, v)_H + \alpha \int_\Omega \nabla u \cdot \nabla v \, dx = \langle f, v \rangle \qquad \text{for all } v \in V. \tag{4.11}$$

For further treatment of (4.11), we define the bilinear form

$$a(w, v) = \alpha \int_\Omega \nabla w \cdot \nabla v \, dx \tag{4.12}$$

on $V \times V$. It is bounded and $V$-elliptic, i.e. there exist constants $M_a, m_a > 0$ such that

$$|a(w, v)| \leq M_a \; \|v\|_V \; \|w\|_V \qquad \text{for all } w, v \in V \text{ (boundedness)}$$
$$a(w, w) \geq m_a \; \|w\|_V^2 \qquad \text{for all } w \in V \text{ (}V\text{-ellipticity)} .$$

*Remark* 4.9. Recall that we want to transfer our results for the stationary problem to our PDE on an evolving domain. Therefore we have to bear in mind that constants could possibly become time dependent in that process. For example, the constant $m_a > 0$ depends on the domain $\Omega$. Thus it is necessary to take a closer look: By [8, (7.44)] each $w \in H_0^1(\Omega)$ satisfies

$$\|w\|_{L^2(\Omega)} \leq \left( \frac{|\Omega|}{\pi} \right)^{\frac{1}{2}} \; \|\nabla w\|_{L^2(\Omega)}$$

where $|\Omega|$ denotes the area of $\Omega$. Hence

$$\|w\|_{L^2(\Omega)}^2 + \|\nabla w\|_{L^2(\Omega)}^2 \leq \left( 1 + \frac{|\Omega|}{\pi} \right) \|\nabla w\|_{L^2(\Omega)}^2$$

and therefore with $\|w\|_V^2 = \|w\|_{L^2(\Omega)}^2 + \|\nabla w\|_{L^2(\Omega)}^2$

$$\begin{aligned} a(w, w) = \alpha \int_\Omega \nabla w \cdot \nabla w \, dx &= \alpha \; \|\nabla w\|_{L^2(\Omega)}^2 \\ &\geq \frac{\alpha}{1 + \frac{|\Omega|}{\pi}} \; \|w\|_V^2 \; . \end{aligned}$$

Therefore

$$m_a = \frac{\alpha}{1 + \frac{|\Omega|}{\pi}} \; . \tag{4.13}$$

The constant $M_a$ is independent of $\Omega$, because it can be obtained by the Cauchy-Schwarz inequality and a trivial estimate.

As seen in the passage preceding [11, Lemma 14.5] the complex Lax-Milgram lemma [11, Lemma 14.5] is applicable to $a$. Moreover we can find an angle $\phi$ such that

$$\left| \arg\left(\frac{a(v, v)}{\|v\|_V^2}\right) \right| \leq \phi \qquad \text{where } \cos(\phi) = \frac{m_a}{M_a}$$

due to the boundedness and $V$-ellipticity of $a$.

For our purpose it suffices to solve the corresponding finite dimensional problem. Suppose $V_h \subset V$ with $\dim V_h = d$. We are interested in a solution in $V_h$: Let $\lambda \in \mathbb{C}$ and $f \in (V_h)'$. Find $u_h \in V_h$ such that

$$\lambda(u_h, v_h)_H + \alpha \int_\Omega \nabla u_h \cdot \nabla v_h \, dx = \langle f, v_h \rangle \qquad \text{for all } v_h \in V_h. \tag{4.14}$$

**Theorem 4.10** ([11, Lemma 14.6]). *Let $\phi < \theta < \frac{\pi}{2}$ and $f \in H$. Moreover let $\lambda \neq 0$ and*

$$\lambda \in \Sigma_\theta := \{z \in \mathbb{C} \mid \left|\arg(z)\right| \leq \pi - \theta\} .$$

*Then there exists a unique solution $u_h \in V_h$ of (4.14) and a constant $c_\theta = |\sin(\theta - \phi)|^{-1}$ such that*

$$\|u_h\|_H \leq \frac{c_\theta}{|\lambda|} \|f\|_H \ . \tag{4.15}$$

*Proof.* Since $V_h \subset H$ is a Hilbert space with respect to $\|\cdot\|_V$ the theorem is a simple application of [11, Lemma 14.6]. □

*Remark* 4.11. To derive the above result, a complexification method has to be applied to the bilinear form $a(\cdot,\cdot)$. We ignore this fact, since it leaves the rest unchanged. Nonetheless the scalar products $(\cdot,\cdot)$ appearing in the rest of this section are linear with respect to to the second argument and conjugate symmetric.

In the following we will often switch between the finite dimensional problem in the function space $V_h$ and the corresponding problem in $\mathbb{R}^d$. Recall that we chose $\Omega = \Omega_t^h$, in accordance we now set $\varphi_i(\cdot) = \varphi_i(t,\cdot)$ (cf. (3.3)) and $V_h = V_h(t)$ (cf. (3.11)). For fixed $h$ and $d$ such that $V_h = \operatorname{span}\{\varphi_1(\cdot),\ldots,\varphi_d(\cdot)\}$ the transition can then be done by the mapping

$$\Pi_h : \mathbb{R}^d \to V_h,\ \mathbf{y} = (y_i)_{i=1}^d \mapsto \sum_{i=1}^d y_i\varphi_i(\cdot). \tag{4.16}$$

At this point we overload the norm $\|\cdot\|_H$: If $w_h, v_h$ are a functions in $V_h$ with $\dim V_h = d$, we have

$$(w_h, v_h)_H = \int_\Omega w_h\, v_h\, dx$$

whereas for vectors $\mathbf{y},\mathbf{x} \in \mathbb{R}^d$, we set

$$(\mathbf{y},\mathbf{x})_H = \mathbf{y}^*\mathbf{M}_h\mathbf{x}$$

where $\mathbf{y}^*$ denotes the conjugate transposed vector to $\mathbf{y}$. An important property of the above definition is

$$(\mathbf{y},\mathbf{x})_H = (\Pi_h\mathbf{y}, \Pi_h\mathbf{x})_H \tag{4.17}$$

and thus

$$\|\mathbf{y}\|_H = \|\Pi_h\mathbf{y}\|_H \ .$$

**Lemma 4.12.** *Let $\mathbf{f} \in \mathbb{R}^d$, then*

$$\left\|(\lambda + \mathbf{M}_h^{-1}\mathbf{S}_h)^{-1}\mathbf{f}\right\|_H \leq \frac{c_\theta}{|\lambda|} \|\mathbf{f}\|_H \tag{4.18}$$

*for $\lambda \in \Sigma_\theta$ and $\lambda \neq 0$.*

*Proof.* (4.14) is equivalent to

$$\lambda(u_h,\varphi_i)_H + \alpha\int_\Omega \nabla u_h\cdot\nabla\varphi_i\, dx = \langle f,\varphi_i\rangle\,, \qquad 1 \leq i \leq d\,.$$

Assume $f \in V_h \subset H$, then $\langle f,\cdot\rangle = (f,\cdot)_H$. Set $\mathbf{f},\mathbf{u} \in \mathbb{R}^d$ such that $\Pi_h\mathbf{f} = f$ and $\Pi_h\mathbf{u} = u_h$. With the definition of $\mathbf{M}_h$, $\mathbf{S}_h$ we can write the above problem equivalently as

$$(\lambda\mathbf{M}_h + \mathbf{S}_h)\mathbf{u} = \mathbf{M}_h\mathbf{f}\,.$$

By Theorem 4.10 (4.14) has a unique solution. Therefore the existence of the inverse is given and

$$(\lambda\mathbf{M}_h + \mathbf{S}_h)^{-1}\mathbf{M}_h\mathbf{f} = (\lambda + \mathbf{M}_h^{-1}\mathbf{S}_h)^{-1}\mathbf{f} = \mathbf{u}\,. \tag{4.19}$$

With (4.17) and (4.15) we get

$$\left\|(\lambda + \mathbf{M}_h^{-1}\mathbf{S}_h)^{-1}\mathbf{f}\right\|_H = \|\mathbf{u}\|_H = \|u_h\|_H \leq \frac{c_\theta}{|\lambda|}\|f\|_H = \frac{c_\theta}{|\lambda|}\|\mathbf{f}\|_H$$

for $\lambda \in \Sigma_\theta$ and $\lambda \neq 0$. □

A closer look at our operator $\mathbf{A}_h = \mathbf{S}_h\mathbf{M}_h^{-1}$ reveals that (4.18) is not yet the estimate we are looking for:

Note that we used the transformation (4.8) and thus we have to be careful to use the correct norms. Defining

$$\|\mathbf{y}\|_G^2 = \mathbf{y}^*\mathbf{M}_h^{-1}\mathbf{y} \tag{4.20}$$

seems to be appropriate due to the identity

$$\|\mathbf{M}_h\mathbf{u}\|_G = \|\mathbf{u}\|_H . \tag{4.21}$$

Then (4.18) implies

$$\begin{aligned}
\left\|(\lambda + \mathbf{S}_h\mathbf{M}_h^{-1})^{-1}\right\|_G &= \sup_{\mathbf{z}\neq 0} \frac{\left\|(\lambda + \mathbf{S}_h\mathbf{M}_h^{-1})^{-1}\mathbf{z}\right\|_G}{\|\mathbf{z}\|_G} \\
&= \sup_{\mathbf{f}\neq 0} \frac{\left\|(\lambda + \mathbf{S}_h\mathbf{M}_h^{-1})^{-1}\mathbf{M}_h\mathbf{f}\right\|_G}{\|\mathbf{M}_h\mathbf{f}\|_G} \\
&= \sup_{\mathbf{f}\neq 0} \frac{\left\|\mathbf{M}_h(\lambda + \mathbf{M}_h^{-1}\mathbf{S}_h)^{-1}\mathbf{f}\right\|_G}{\|\mathbf{M}_h\mathbf{f}\|_G} \\
&= \sup_{\mathbf{f}\neq 0} \frac{\left\|(\lambda + \mathbf{M}_h^{-1}\mathbf{S}_h)^{-1}\mathbf{f}\right\|_H}{\|\mathbf{f}\|_H} \\
&\leq \frac{c_\theta}{|\lambda|}.
\end{aligned}$$

Since $\mathbf{M}_h$ is invertible for all $h$, the transformation $\mathbf{z} = \mathbf{M}_h\mathbf{f}$ is admissible.

### Results for the Mass and Stiffness Matrices

We will now state some results about the mass and stiffness matrix. Although we omitted the time dependencies, these results hold for each $t \in [0, T]$ in the corresponding way.

Let us recall the definition of the mass and stiffness matrix

$$\begin{aligned}
(\mathbf{M}_h)_{i,j} &= \int_\Omega \varphi_i(x)\,\varphi_j(x)\,dx \\
(\mathbf{S}_h)_{i,j} &= \alpha \int_\Omega \nabla\varphi_i(x)\cdot\nabla\varphi_j(x)\,dx .
\end{aligned}$$

Both $\mathbf{M}_h$ and $\mathbf{S}_h$ are symmetric

$$\mathbf{M}_h = \mathbf{M}_h^\intercal = \mathbf{M}_h^*, \qquad \mathbf{S}_h = \mathbf{S}_h^\intercal = \mathbf{S}_h^*$$

by definition. Moreover, $\mathbf{M}_h$ is positive definite

$$\mathbf{y}^*\mathbf{M}_h\mathbf{y} = \int_\Omega \left(\sum_{i=1}^d y_i\varphi_i(x)\right)^2 dx = \int_\Omega (\Pi_h\mathbf{y})^2\ dx = \|\Pi_h\mathbf{y}\|_H^2 > 0 ,$$

because $\Pi_h\mathbf{y} \neq 0$ for $\mathbf{y} \neq 0$. Note, that there is no uniform lower bound with respect to $h$ since $\operatorname{supp}(\varphi_i) \to 0$ for $h \to 0$. Since the bilinear form $a$ (cf. (4.12)) is $V$-elliptic, we have

$$a(u_h, u_h) \geq m_a\ \|u_h\|_V^2 \geq m_a\ \|u_h\|_H^2$$

for all $u_h \in V_h$. This can be rewritten as

$$\mathbf{y}^*\mathbf{S}_h\mathbf{y} = a(\Pi_h\mathbf{y}, \Pi_h\mathbf{y}) \geq m_a\ \|\Pi_h\mathbf{y}\|_H = m_a\mathbf{y}^*\mathbf{M}_h\mathbf{y} . \tag{4.22}$$

Thus $\mathbf{S}_h$ is also positive definite.

Given $\mathbf{M}_h$ and $\mathbf{S}_h$ positive definite and symmetric, both matrices have an inverse. Moreover, we are even able to define their square roots: There exist matrices $\mathbf{M}_h^{1/2}$ and $\mathbf{S}_h^{1/2}$ such that

$$\mathbf{M}_h^{1/2}\mathbf{M}_h^{1/2} = \mathbf{M}_h, \qquad \mathbf{S}_h^{1/2}\mathbf{S}_h^{1/2} = \mathbf{S}_h \,.$$

Both square roots are themselves positive definite and symmetric. Furthermore we are able to define square roots for the inverse matrices $\mathbf{M}_h^{-1}$ and $\mathbf{S}_h^{-1}$ denoted by $\mathbf{M}_h^{-1/2}$, $\mathbf{S}_h^{-1/2}$. It is easy to verify a matrix, its inverse and the corresponding square roots commute.

Since we decided for a norm on the finite element space, we can now state some bounds.

**Lemma 4.13.** *Let* $|\Omega|$ *denote the area of* $\Omega$*, then*

$$\|\mathbf{M}_h\|_H \le \|\mathbf{M}_h\|_2\,, \qquad \|\mathbf{M}_h\|_G \le \|\mathbf{M}_h\|_2$$

*and*

$$\|\mathbf{M}_h\|_2 \le |\Omega| \;.$$

*Proof.* We give the proof in three steps:

- For $\mathbf{y} \in \mathbb{R}^d$ we have

$$\|\mathbf{M}_h\mathbf{y}\|_H^2 = \mathbf{y}^*\mathbf{M}_h^*\mathbf{M}_h\mathbf{M}_h\mathbf{y} = \mathbf{y}^*\mathbf{M}_h^3\mathbf{y} = \left\|\mathbf{M}_h\mathbf{M}_h^{1/2}\mathbf{y}\right\|_2^2 \le \|\mathbf{M}_h\|_2^2 \left\|\mathbf{M}_h^{1/2}\mathbf{y}\right\|_2^2 = \|\mathbf{M}_h\|_2^2\, \|\mathbf{y}\|_H^2$$
$$\|\mathbf{M}_h\mathbf{y}\|_G^2 = \mathbf{y}^*\mathbf{M}_h\mathbf{y} = \left\|\mathbf{M}_h\mathbf{M}_h^{-1/2}\mathbf{y}\right\|^2 \le \|\mathbf{M}_h\|_2^2\, \|\mathbf{y}\|_G^2$$

  Thus $\|\mathbf{M}_h\|_H\,, \|\mathbf{M}_h\|_G \le \|\mathbf{M}_h\|_2$.

- We show the continuity of $\Pi_h$ with respect to to $\|\cdot\|_H$: Let $\mathbf{y} \in \mathbb{R}^d$, then

$$\begin{aligned}\|\Pi_h\mathbf{y}\|_H^2 = \int_\Omega \left(\sum_{i=1}^d y_i\varphi_i(x)\right)^2 dx &= \int_\Omega \left(\mathbf{y}\cdot(\varphi_i(x))_{i=1}^d\right)^2 dx \\ &\le \int_\Omega \left(\|\mathbf{y}\|_2\, \left\|(\varphi_i(x))_{i=1}^d\right\|_2\right)^2 dx \\ &= \|\mathbf{y}\|_2^2 \int_\Omega \sum_{i=1}^d \varphi_i(x)^2\, dx \\ &\le \|\mathbf{y}\|_2^2\, |\Omega| \,.\end{aligned}$$

  For the last inequality we used $\varphi_i(x) \le 1$, which implies $\varphi_i(x)^2 \le \varphi_i(x)$. Since $\sum_{i=1}^d \varphi_i(x) \le 1$ for all $x \in \Omega$,

$$\int_\Omega \sum_{i=1}^d \varphi_i(x)^2\, dx \le \int_\Omega \sum_{i=1}^d \varphi_i(x)\, dx \le |\Omega| \;.$$

- Let $\mathbf{y} \in \mathbb{R}^d$, then

$$\|\mathbf{M}_h\mathbf{y}\|_2^2 = \sum_{i=1}^d (\Pi_h\mathbf{y}, \varphi_i)_H^2 \le \sum_{i=1}^d \|\Pi_h\mathbf{y}\|_H^2\, \|\varphi_i\|_H^2 \le \|\Pi_h\mathbf{y}\|_H^2 \sum_{i=1}^d \|\varphi_i\|_H^2 \le |\Omega|^2\, \|\mathbf{y}\|_2^2 \,.$$

  where used the second estimate to bound $\|\Pi_h(t)\mathbf{y}\|_\Omega^2$ and the same trick as above to bound $\sum_{i=1}^d \|\varphi_i\|_H^2$.

Overall we have

$$\|\mathbf{M}_h\|_H\,, \|\mathbf{M}_h\|_G \le \|\mathbf{M}_h\|_2 \le |\Omega| \tag{4.23}$$

which holds independently of $h$ and $d$. □

Let us continue with the boundedness of the inverse stiffness matrix:

**Lemma 4.14.** *The following bound holds:*

$$\left\|(\mathbf{S}_h\mathbf{M}_h^{-1})^{-1}\right\|_G \le \frac{1}{m_a}. \tag{4.24}$$

*Proof.* We consider $\mathbf{T}_h := \mathbf{S}_h\mathbf{M}_h^{-1}$ as a positive, self-adjoint operator on a Hilbert space with inner product $(\cdot,\cdot)_G$:

$$(\mathbf{y},\mathbf{T}_h\mathbf{y})_G = \mathbf{y}^*\mathbf{M}_h^{-1}\mathbf{S}_h\mathbf{M}_h^{-1}\mathbf{y} = (\mathbf{S}_h\mathbf{M}_h^{-1}\mathbf{y})^*\mathbf{M}_h^{-1}\mathbf{y} = (\mathbf{T}_h\mathbf{y},\mathbf{y})_G,$$
$$(\mathbf{y},\mathbf{T}_h\mathbf{y})_G = \mathbf{y}^*\mathbf{M}_h^{-1}\mathbf{S}_h\mathbf{M}_h^{-1}\mathbf{y} = (\mathbf{M}_h^{-1}\mathbf{y})^*\mathbf{S}_h\mathbf{M}_h^{-1}\mathbf{y} > 0 \qquad \text{for } y \neq 0.$$

Note that this guarantees the existence of the square root $\mathbf{T}_h^{1/2}$ of $\mathbf{T}_h$ with respect to $(\cdot,\cdot)_G$ always exists: $(\mathbf{x},\mathbf{T}_h\mathbf{y})_G = (\mathbf{T}_h^{1/2}\mathbf{x},\mathbf{T}_h^{1/2}\mathbf{y})_G$ with $\mathbf{T}_h^{1/2} = \mathbf{M}_h^{1/2}\mathbf{S}_h^{1/2}\mathbf{M}_h^{-1}$.

Now we can deduce

$$\sup_{\mathbf{y}\neq 0}\frac{(\mathbf{y},\mathbf{T}_h\mathbf{y})_G}{(\mathbf{y},\mathbf{y})_G} = \sup_{\mathbf{y}\neq 0}\frac{(\mathbf{T}_h^{1/2}\mathbf{y},\mathbf{T}_h^{1/2}\mathbf{y})_G}{(\mathbf{y},\mathbf{y})_G} = \sup_{\mathbf{y}\neq 0}\frac{\left\|\mathbf{T}_h^{1/2}\mathbf{y}\right\|_G^2}{\|\mathbf{y}\|_G^2} = \left\|\mathbf{T}_h^{1/2}\right\|_G^2.$$

With

$$\sup_{\mathbf{y}\neq 0}\frac{(\mathbf{y},\mathbf{T}_h\mathbf{y})_G}{(\mathbf{y},\mathbf{y})_G} \le \sup_{\mathbf{y}\neq 0}\frac{\|\mathbf{T}_h\mathbf{y}\|_G\,\|\mathbf{y}\|_G}{\|\mathbf{y}\|_G^2} \le \|\mathbf{T}_h\|_G$$

and

$$\|\mathbf{T}_h\|_G = \sup_{\mathbf{y}\neq 0}\frac{\|\mathbf{T}_h\mathbf{y}\|_G}{\|\mathbf{y}\|_G} \le \left\|\mathbf{T}_h^{1/2}\right\|_G \sup_{\mathbf{y}\neq 0}\frac{\left\|\mathbf{T}_h^{1/2}\mathbf{y}\right\|_G}{\|\mathbf{y}\|_G} = \left\|\mathbf{T}_h^{1/2}\right\|_G^2 = \sup_{\mathbf{y}\neq 0}\frac{(\mathbf{y},\mathbf{T}_h\mathbf{y})_G}{(\mathbf{y},\mathbf{y})_G}$$

we have

$$\|\mathbf{T}_h\|_G = \sup_{\mathbf{y}\neq 0}\frac{(\mathbf{y},\mathbf{T}_h\mathbf{y})_G}{(\mathbf{y},\mathbf{y})_G}. \tag{4.25}$$

Applying this result to $\mathbf{T}_h^{-1}$, we find

$$\left\|\mathbf{T}_h^{-1}\right\|_G = \sup_{\mathbf{x}\neq 0}\frac{(\mathbf{x},\mathbf{T}_h^{-1}\mathbf{x})_G}{(\mathbf{x},\mathbf{x})_G} = \sup_{\mathbf{x}\neq 0}\frac{(\mathbf{T}_h^{-1/2}\mathbf{x},\mathbf{T}_h^{-1/2}\mathbf{x})_G}{(\mathbf{x},\mathbf{x})_G} = \sup_{\mathbf{y}\neq 0}\frac{(\mathbf{y},\mathbf{y})_G}{(\mathbf{T}_h^{1/2}\mathbf{y},\mathbf{T}_h^{1/2}\mathbf{y})_G} = \sup_{\mathbf{y}\neq 0}\frac{(\mathbf{y},\mathbf{y})_G}{(\mathbf{y},\mathbf{T}_h\mathbf{y})_G}$$

and thus

$$\left\|\mathbf{T}_h^{-1}\right\|_G^{-1} = \inf_{\mathbf{y}\neq 0}\frac{(\mathbf{y},\mathbf{T}_h\mathbf{y})_G}{(\mathbf{y},\mathbf{y})_G}. \tag{4.26}$$

With (4.22) and (4.26), we obtain

$$\left\|\mathbf{T}_h^{-1}\right\|_G^{-1} = \inf_{\mathbf{x}\neq 0}\frac{(\mathbf{x},\mathbf{T}_h\mathbf{x})_G}{(\mathbf{x},\mathbf{x})_G} = \inf_{\mathbf{x}\neq 0}\frac{\mathbf{x}^*\mathbf{M}_h^{-1}\mathbf{S}_h\mathbf{M}_h^{-1}\mathbf{x}}{\mathbf{x}^*\mathbf{M}_h^{-1}\mathbf{x}} = \inf_{\mathbf{y}\neq 0}\frac{\mathbf{y}^*\mathbf{S}_h\mathbf{y}}{\mathbf{y}^*\mathbf{M}_h\mathbf{y}} \ge m_a$$

which proves the bound (4.24). □

**Second Step: Resolvent Estimate on the Half Plane**

In this step we want to obtain an estimate for the resolvent $R(\lambda, \mathbf{A}_h) = (\lambda - \mathbf{A}_h)^{-1}$ of $\mathbf{A}_h$. At this point our bound has a singularity at $\lambda = 0$, this step will deal with proving a bound that holds uniformly on the left half plane of $\mathbb{C}$. Recall that

$$\mathbf{A}_h = \mathbf{S}_h \mathbf{M}_h^{-1}\,.$$

Since $\left|\arg(\lambda)\right| \le \pi - \theta$ implies $\operatorname{Re}\lambda \ge 0$ for $\theta < \frac{\pi}{2}$, we have

$$\left\|(\lambda + \mathbf{A}_h)^{-1}\right\|_G \le \frac{c_\theta}{|\lambda|},\ \lambda \in \Sigma_\theta, \lambda \ne 0 \implies \left\|(\lambda - \mathbf{A}_h)^{-1}\right\|_G \le \frac{c_\theta}{|\lambda|},\ \operatorname{Re}\lambda \le 0, \lambda \ne 0\,.$$

We start with the proof of a new resolvent bound: Let $\operatorname{Re}\lambda \le 0$.

- For $|\lambda| \left\|\mathbf{A}_h^{-1}\right\|_G \le \frac{1}{2}$, we can apply the Neumann series

  $$(\lambda - \mathbf{A}_h)^{-1} = -\mathbf{A}_h^{-1}(1 - \lambda\mathbf{A}_h^{-1})^{-1} = -\mathbf{A}_h^{-1}\sum_{k=0}^{\infty}(\lambda\mathbf{A}_h^{-1})^k$$

  and with $\sum_{k=0}^{\infty}\frac{1}{2^k} = 1$ therefore

  $$\left\|(\lambda - \mathbf{A}_h)^{-1}\right\|_G \le \left\|\mathbf{A}_h^{-1}\right\|_G \le \frac{\left\|\mathbf{A}_h^{-1}\right\|_G}{|\lambda| + 1}\,.$$

- On the other hand, for all other $\lambda$ we find

  $$|\lambda|\,(1 + 2\left\|\mathbf{A}_h^{-1}\right\|_G) > (\,|\lambda| + 1)\,,$$

  since $|\lambda| \left\|\mathbf{A}_h^{-1}\right\|_G > \frac{1}{2}$. Thus

  $$\begin{aligned}\left\|(\lambda - \mathbf{A}_h)^{-1}\right\|_G &\le \frac{c_\theta}{|\lambda|}\\ &< \frac{c_\theta\,(1 + 2\left\|\mathbf{A}_h^{-1}\right\|_G)}{|\lambda| + 1}\,.\end{aligned}$$

Altogether we obtain for $\operatorname{Re}\lambda \le 0$

$$\left\|(\lambda - \mathbf{A}_h)^{-1}\right\|_G \le \frac{c_\theta\,(1 + 2\left\|\mathbf{A}_h^{-1}\right\|_G)}{|\lambda| + 1}\,. \tag{4.27}$$

A bound for $\left\|\mathbf{A}_h^{-1}\right\|_G$ was given in Lemma 4.14. Observe that the bound in (4.27) has no singularity at $\lambda = 0$.

**Third Step: Dealing with Time Dependencies**

At this point it seems like we already have what we need, but we omitted time dependencies in the above steps. Since the domain $\Omega_t^h$ of our discrete partial differential equation evolves in time, even the norms are time dependent: For all $\lambda$ with $\operatorname{Re}\lambda \le 0$, we have

$$\left\|(\lambda - \mathbf{S}_h(t)\mathbf{M}_h(t)^{-1})^{-1}\right\|_{G(t)} \le \frac{c_\theta(t)\,(1 + 2\left\|\mathbf{M}_h(t)\mathbf{S}_h(t)^{-1}\right\|_{G(t)})}{|\lambda| + 1}\,. \tag{4.28}$$

To satisfy $(A_1)$ we have to find a time independent norm and control the size of the constants $c_\theta(t)$ and $\left\|\mathbf{S}_h(t)\mathbf{M}_h(t)^{-1}\right\|_{G(t)}$. We start with choosing a new norm:

We search a norm $\|\cdot\|_*$, independent of $t$, such that it is equivalent to $\|\cdot\|_{G(t)}$ on $\mathbb{R}^d$ in the sense:

$$\|\cdot\|_* \sim \|\cdot\|_{G(t)} :\Longleftrightarrow c\,\|\cdot\|_{G(t)} \le \|\cdot\|_* \le C\,\|\cdot\|_{G(t)} \tag{4.29}$$

for all $t \in [0, T]$ with constants $C > c > 0$ independent of the time $t$, the spatial discretization $h$ and the dimension of the system $d$.

A norm satisfying (4.29) needs to be similar to $\|\cdot\|_{G(t)}$ since any standard euclidean or maximum norm can only be bounded by constants dependent on $d$. Another difficulty is the general handling of $\|\cdot\|_{G(t)}$ since it is defined by the inverse mass matrices $\mathbf{M}_h(t)^{-1}$, $t \in [0, T]$. $\|\cdot\|_{H(t)}$ on $\mathbb{R}^d$ with $\|\mathbf{y}\|^2_{H(t)} = \mathbf{y}^*\mathbf{M}_h(t)\mathbf{y}$ on the other hand can be evaluated more easily. We use the following technical lemma to transfer a $\|\cdot\|_{H(t)}$ equivalency to a $\|\cdot\|_{G(t)}$-norm equivalency:

**Lemma 4.15.** *For symmetric, positive definite matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *assume that the generalized Rayleigh quotient satisfies*

$$c \le \frac{\mathbf{x}^*\mathbf{A}\mathbf{x}}{\mathbf{x}^*\mathbf{B}\mathbf{x}} \le C, \qquad \mathbf{x} \ne 0. \tag{4.30}$$

*Then, for the inverse matrices we have*

$$C^{-1} \le \frac{\mathbf{x}^*\mathbf{A}^{-1}\mathbf{x}}{\mathbf{x}^*\mathbf{B}^{-1}\mathbf{x}} \le c^{-1}, \qquad \mathbf{x} \ne 0. \tag{4.31}$$

*Proof.* Substituting $\mathbf{y} = \mathbf{B}^{1/2}\mathbf{x}$ in (4.30) shows

$$c \le \frac{\mathbf{y}^*\mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2}\mathbf{y}}{\mathbf{y}^*\mathbf{y}} \le C, \qquad \mathbf{y} \ne 0.$$

By (4.25) and (4.26) (for the 2-norm), this yields

$$\left\|\mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2}\right\|_2 \le C, \qquad \left\|\mathbf{B}^{1/2}\mathbf{A}^{-1}\mathbf{B}^{1/2}\right\|_2 \le c^{-1}.$$

Thus we have

$$\sup_{\mathbf{x}\ne 0} \frac{\mathbf{x}^*\mathbf{A}^{-1}\mathbf{x}}{\mathbf{x}^*\mathbf{B}^{-1}\mathbf{x}} = \sup_{\mathbf{y}\ne 0} \frac{\mathbf{y}^*\mathbf{B}^{1/2}\mathbf{A}^{-1}\mathbf{B}^{1/2}\mathbf{y}}{\mathbf{y}^*\mathbf{y}} = \left\|\mathbf{B}^{1/2}\mathbf{A}^{-1}\mathbf{B}^{1/2}\right\|_2 \le c^{-1}$$

and analogously

$$\inf_{\mathbf{x}\ne 0} \frac{\mathbf{x}^*\mathbf{A}^{-1}\mathbf{x}}{\mathbf{x}^*\mathbf{B}^{-1}\mathbf{x}} = \inf_{\mathbf{y}\ne 0} \frac{\mathbf{y}^*\mathbf{B}^{1/2}\mathbf{A}^{-1}\mathbf{B}^{1/2}\mathbf{y}}{\mathbf{y}^*\mathbf{y}} = \left\|\mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2}\right\|_2^{-1} \ge C^{-1}.$$

This completes the proof. □

With the result of the above Lemma we now turn to the proof of

$$\|\cdot\|_{H(0)} \sim \|\cdot\|_{H(t)}\ . \tag{4.32}$$

Note that the choice $t = 0$ is for convenience, any other fixed $t_0 \in [0, T]$ is also acceptable. The equivalency of the norms then follows from the construction of the basis functions $\varphi_i(t, \cdot)$, $i = 1, \dots, d$ and the regularity of $\Phi_h(t, \cdot)$:

$$\begin{aligned}
\mathbf{y}^*\mathbf{M}_h(t)\mathbf{y} = \int_{\Omega_t^h} \left(\sum_{i=1}^d y_i\varphi_i(t,x)\right)^2 dx &= \int_{\Omega_0^h} \left(\sum_{i=1}^d y_i\varphi_i(t,\Phi_h(t,x))\right)^2 |\det J_x\Phi_h(t,x)|\ dx \\
&\le \|\det J_x\Phi_h(t,\cdot)\|_{L^\infty(\Omega_0^h)} \int_{\Omega_0^h} \left(\sum_{i=1}^d y_i\varphi_i(0,x)\right)^2 dx \\
&= \|\det J_x\Phi_h(t,\cdot)\|_{L^\infty(\Omega_0^h)}\ \|\mathbf{y}\|^2_{H(0)} \\
&\le \sup_{t\in[0,T]} \|\det J_x\Phi_h(t,\cdot)\|_{L^\infty(\Omega_0^h)}\ \mathbf{y}^*\mathbf{M}_h(0)\mathbf{y}.
\end{aligned}$$

Since $\Phi_h(t,\cdot)$ is a piecewise interpolation of $\Phi(t,\cdot)$ (cf. (3.1)), $\Phi_h(t,\cdot)$ is a (piecewise) diffeomorphism and $\Phi_h(\cdot,y)$ is continuously differentiable for each $y \in \Omega_0$. Thus, with Assumption 1.1, we find a bound for $\sup_{t\in[0,T]} \|\det J_x \Phi_h(t,\cdot)\|_{L^\infty(\Omega_0^h)}$ independent of $h$. The other inequality in (4.32)

$$\|\cdot\|_* \leq C \, \|\cdot\|_{H(t)}$$

can be obtained analogously by bounding $\sup_{t\in[0,T]} \left\|\det J_x \Phi_h^{-1}(t,\cdot)\right\|_{L^\infty(\Omega_0)}$.

Altogether we obtain the existence of some constants $C, c > 0$ independent of $h$ and $t$ such that

$$c \leq \frac{\mathbf{y}^*\mathbf{M}_h(0)\mathbf{y}}{\mathbf{y}^*\mathbf{M}_h(t)\mathbf{y}} \leq C$$

for all $\mathbf{y} \in \mathbb{R}^d$, $\mathbf{y} \neq 0$ and $t \in [0,T]$. The above Lemma now gives

$$C^{-1} \leq \frac{\|\mathbf{y}\|_{G(0)}^2}{\|\mathbf{y}\|_{G(t)}^2} = \frac{\mathbf{y}^*\mathbf{M}_h(0)^{-1}\mathbf{y}}{\mathbf{y}^*\mathbf{M}_h(t)^{-1}\mathbf{y}} \leq c^{-1}$$

for all $\mathbf{y} \in \mathbb{R}^d$. For an arbitrary operator linear $\mathbf{A} : \mathbb{R}^d \to \mathbb{R}^d$ we thus have

$$\|\mathbf{A}\|_{G(0)} = \sup_{\mathbf{y}\neq 0} \frac{\|\mathbf{A}\mathbf{y}\|_{G(0)}}{\|\mathbf{y}\|_{G(0)}} \leq c^{-1}C^{-1} \sup_{\mathbf{y}\neq 0} \frac{\|\mathbf{A}\mathbf{y}\|_{G(t)}}{\|\mathbf{y}\|_{G(t)}} \leq c^{-1}C^{-1} \, \|\mathbf{A}\|_{G(t)} \; .$$

Therefore by (4.28)

$$\left\|(\lambda - \mathbf{S}_h(t)\mathbf{M}_h(t)^{-1})^{-1}\right\|_{G(0)} \leq C \frac{c_\theta(t)\,(1 + 2\left\|\mathbf{M}_h(t)\mathbf{S}_h(t)^{-1}\right\|_{G(t)})}{|\lambda| + 1} \; . \tag{4.33}$$

with a constant $C$ independent of $h$ and $t$, whereas $c_\theta(t)$ and $\left\|\mathbf{M}_h(t)\mathbf{S}_h(t)^{-1}\right\|_{G(t)}$ can be bounded as follows:

- We can bound the $V$-ellipticity constant $m_a(t)$ of $a(\cdot,\cdot)$ defined in (4.12) by

$$m_a(t) = \frac{\alpha}{1 + \frac{|\Omega_t|}{\pi}} \geq \frac{\alpha}{1 + \frac{\omega_*}{\pi}} =: m_{a*}$$

  where $\omega_* = \max_{t\in[0,T]} |\Omega_t|$. With Lemma 4.14 we then have

$$\left\|\mathbf{M}_h(t)\mathbf{S}_h(t)^{-1}\right\|_{G(t)} \leq \frac{1}{m_a(t)} \leq \frac{1}{m_{a*}} \; . \tag{4.34}$$

- To bound $c_\theta(t)$ we have to take a closer look at several interdependencies: Recall that the angle $\phi(t) > 0$ was determined by the bilinear form $a(\cdot,\cdot)$ and was given by:

$$\cos(\phi(t)) = \frac{m_a(t)}{M_a} \geq \frac{m_{a*}}{M_a} > 0 \, .$$

  The cosine function is monotonously decreasing on $[0, \frac{\pi}{2}]$, hence there exists some $\phi^* > 0$ such that $0 < \phi(t) \leq \phi^* < \frac{\pi}{2}$. For each $t \in [0,T]$ we have to choose some $\theta(t)$ sufficing $\phi(t) < \theta(t) < \frac{\pi}{2}$. The exact sector $\Sigma_{\theta(t)}$ is no longer of interest to us since we only need the estimate (4.15) to hold for $\lambda \neq 0$ in the left half plane $\{\mathrm{Re}\,\lambda \leq 0\}$. The constant $c_\theta(t)$ is then given by

$$c_\theta(t) = \frac{1}{|\sin(\theta(t) - \phi(t))|} = \frac{1}{\sin(\theta(t) - \phi(t))}$$

  since $\theta(t) > \phi(t)$ by assumption. The function $x \mapsto \sin(x)^{-1}$ is monotonously decreasing on $x = \theta(t) - \phi(t) \in (0, \frac{\pi}{2})$, thus we have to find a lower bound for $\theta(t) - \phi(t)$:

Set $\theta(t) \approx \frac{\pi}{2}$ constant in $t$ such that

$$0 < \phi^* < \theta(t) = \theta < \frac{\pi}{2}$$

then

$$\theta(t) - \phi(t) \approx \frac{\pi}{2} - \phi(t) \geq \frac{\pi}{2} - \phi^* > 0$$

and

$$c_\theta(t) = \sin(\theta(t) - \phi(t))^{-1} \leq \sin(\pi/2 - \phi^*)^{-1} =: c_\theta^* \,. \tag{4.35}$$

We define the following abbreviation:

$$(\cdot,\cdot)_* := (\cdot,\cdot)_{G(0)}, \qquad \|\cdot\|_* := \|\cdot\|_{G(0)} \,. \tag{4.36}$$

By taking together (4.34), (4.35) and (4.33) we obtain:

Let $\lambda$ be in the left half plane $\{\operatorname{Re}\lambda \leq 0\}$. Then there exists a constant $M$ independent of $h$ and $t$ such that

$$\begin{aligned}\|R(\lambda, \mathbf{A}_h)\|_* = \left\|(\lambda - \mathbf{S}_h(t)\mathbf{M}_h(t)^{-1})^{-1}\right\|_* &\leq C\frac{c_\theta(t)\,(1 + 2\left\|\mathbf{M}_h(t)\mathbf{S}_h(t)^{-1}\right\|_{G(t)})}{|\lambda| + 1} \\ &\leq \frac{C\, c_\theta^*\,(1 + 2/m_{a*})}{|\lambda| + 1} \\ &\leq \frac{M}{|\lambda| + 1}\,.\end{aligned}$$

This completes the poof of assumption $(A_2)$.

### 4.2.3 Third Assumption: Lipschitz Continuity

We now discuss:

$(A_3)$ There exists a constant $L > 0$ such that

$$\left\|(A(t) - A(s))A(\tau)^{-1}\right\| \leq L\,|t - s| \qquad \text{for } s, t, \tau \in [0, T].$$

Again, at this point we are not able to give the proof of $(A_3)$ but we want to state some basic considerations.

*Remark* 4.16. Observe that it suffices to show, that there exists a constant $L' > 0$ such that

$$\left\|A(t)A(s)^{-1} - \mathrm{Id}\right\| \leq L'\,|t - s|\,.$$

Both Lipschitz estimates are equivalent since $\left\|A(s)A(\tau)^{-1}\right\|$ can be uniformly bounded in $\tau$ and $s$.

The mappings $\Phi(t,\cdot)$ are Lipschitz continuous by assumption. Since the time dependence of the matrices is the consequence of the discrete evolving domain $\Omega_t^h = \Phi_h(t, \Omega_0)$, it seems like the Lipschitz continuity can be transferred directly from $\Phi(t,\cdot)$ to the operators $\mathbf{A}_h(t)$. Taking a closer look at the matrices, the componentwise Lipschitz continuity is in fact easy to see as

$$\begin{aligned}\mathbf{M}_h(t)_{i,j} &= \int_{\Omega_t^h} \varphi_i(t,x)\,\varphi_j(t,x)\,dx \\ &= \int_{\Omega_s^h} \varphi_i(t, \Phi_h(t, \Phi_h^{-1}(s,x)))\,\varphi_j(t, \Phi_h(t, \Phi_h^{-1}(s,x)))\,\left|\det(J_x\Phi_h(t, \Phi_h^{-1}(s,x)))\right|\,dx \\ &= \int_{\Omega_s^h} \varphi_i(s,x)\,\varphi_j(s,x)\,\left|\det(J_x\Phi_h(t, \Phi_h^{-1}(s,x)))\right|\,dx \\ &= \mathbf{M}_h(s)_{i,j} + \int_{\Omega_s^h} \varphi_i(s,x)\,\varphi_j(s,x)\big(\left|\det(J_x\Phi_h(t, \Phi_h^{-1}(s,x)))\right| - 1\big)\,dx \\ &\leq \Big(\left\|\left|\det(J_x\Phi_h(t, \Phi_h^{-1}(s,x)))\right| - 1\right\|_{L^\infty(\Omega_s^h)} + 1\Big)\mathbf{M}_h(s)_{i,j}\,,\end{aligned}$$

and the uniform Lipschitz continuity of $\Phi(\cdot, y)$ implies

$$\sup_{x\in\Omega_s} \left|\Phi_h(t, \Phi_h^{-1}(s, x)) - x\right| \leq C\ |t - s|\ .$$

Analogously but certainly more complicated, we can derive a componentwise Lipschitz estimate for the stiffness matrix $\mathbf{S}_h(t)$.

Recall that the Lipschitz estimate has to hold with respect to the same norm as the resolvent estimate, therefore a componentwise relation, as shown above, is of little use. We have to consider

$$\begin{aligned}\left\|A(t)A(s)^{-1} - \mathrm{Id}\right\| = \left\|\mathbf{A}_h(t)\mathbf{A}_h(s)^{-1} - \mathbf{I}_d\right\|_* &= \left\|\mathbf{S}_h(t)\mathbf{M}_h(t)^{-1}\mathbf{M}_h(s)\mathbf{S}_h(s)^{-1} - \mathbf{I}_d\right\|_{G(0)} \\ &= \sup_{\mathbf{y}\neq 0} \frac{\left\|(\mathbf{S}_h(t)\mathbf{M}_h(t)^{-1}\mathbf{M}_h(s)\mathbf{S}_h(s)^{-1} - \mathbf{I}_d)\mathbf{y}\right\|_{G(0)}}{\|\mathbf{y}\|_{G(0)}}\ .\end{aligned}$$

As we have seen in the proof of $(A_2)$, we probably need functional analytic tools to derive a bound, that holds independently of $h$ and $d$.

# Chapter 5

# Numerical Tests

To show that ExpInt works as it is supposed to, we implemented a numerical simulation to test its order of convergence, before the work on the convergence proof of ExpInt began. Since the theoretical evidence of convergence can be very elaborate and sophisticated, this prevents us from putting to much work into a method that is of little practical benefit.

During this section we will discuss some features of our implementation, the exact example we chose to run our numerical tests and of course the results of our numerical tests.

## 5.1 Implementation

`Matlab 2012a`, a numerical computing environment, perfectly suites our purpose: It provides us with a closed and easy to use environment, including tools for visualization and triangulation of domains, as well as a reasonable performance. Performance issues are always important and the programis not yet optimized with respect to computationally costs. This was beyond the scope of this thesis.

As the discretization process, our program is basically divided into two parts: The assembly of the mass and stiffness matrices for arbitrary times $t$ and the application or adaption of the numerical time integrators to the ODEs.

### 5.1.1 Assembly of the Matrices

Recall the ODE of which we want to approximate the solution

$$\frac{d}{dt}\Big[\mathbf{M}_h(t)\mathbf{u}(t)\Big] + \mathbf{S}_h(t)\mathbf{u}(t) = 0, \qquad \mathbf{u}(0) = \mathbf{u}_0 \tag{5.1}$$

and its equivalent formulation as a Cauchy problem

$$\frac{d}{dt}\mathbf{y}(t) + \mathbf{S}_h(t)\mathbf{M}_h(t)^{-1}\mathbf{y}(t) = 0, \qquad \mathbf{y}(0) = \mathbf{M}_h(0)\mathbf{u}_0\,, \tag{5.2}$$

with the mass matrix

$$\mathbf{M}_h(t)_{i,j} = \int_{\Omega_t^h} \varphi_i(t,x)\,\varphi_j(t,x)\,dx$$

and the stiffness matrix

$$\mathbf{S}_h(t)_{i,j} = \alpha \int_{\Omega_t^h} \nabla\varphi_i(t,x)\cdot\nabla\varphi_j(t,x)\,dx\,.$$

for $i, j = 1, \ldots, d$, as defined in Section 3.2.1.

A typical finite-element program sets up stiffness matrix, mass matrix and the right-hand side within a loop over all triangles. Assume we want to compute the mass matrix $\mathbf{M}_h(t)$ at a time $t_0$. This is done as follows:

1. Given the initial triangulation $T_h(0)$ of $\Omega_0$, we move all nodes to there position at time $t_0$

$$a_i(t_0) = \Phi(t_0, a_i), \qquad i = 1, \ldots, N\,.$$

   The triangulation $T_h(t_0)$ is then already completely given, because the triangles are completely determined by the position of their nodes and always have the same nodes as vertices. This procedure is equivalent to moving the mesh with $\Phi_h(t_0, \cdot)$. Note that we can use the original mapping $\Phi(t, \cdot)$ instead of the discrete mapping to move the nodes, since $\Phi(t, a_i) = \Phi_h(t, a_i)$, $i = 1, \ldots, N$.

2. Compute the mass matrix $\mathbf{M}_h(t)$ by applying the moved triangulation $T_h(t_0)$ to an algorithm for setting up a mass matrix for the linear FEM.

Since the assembly of the matrices is computationally expensive, we use an algorithm that computes both, $\mathbf{M}_h(t)$ and $\mathbf{S}_h(t)$, at once and shares results as the position of the moved nodes. Assuming that the reader to be familiar with the standard way of assembling mass and stiffness matrices as well as the concept of local-to-global indexing, we state a simplified version of our algorithm without any further explanation.

**Algorithm** Assembly of the Mass and Stiffness Matrix at Time $t_0$

```
Set up M̂ ∈ ℝ^{3×3}                                   ▷ Local mass matrix for K̂
Set up Ŝ(i) ∈ ℝ^2, i = 1,2,3                          ▷ S(i) is the gradient of the ith basis function on K̂
M_h(t_0), S_h(t_0) = 0 ∈ ℝ^{d×d}
a_i(t_0) = Φ(t_0, a_i) for i = 1,...,N
for e = 1,...,E do                                    ▷ Iterate over all triangles K_e(t_0)
    Compute T_{K_e(t_0)} from a_i^e(t), i = 1,2,3     ▷ F_{K_e(t_0)}(ŷ) = T_{K_e(t_0)} ŷ + b_{K_e(t_0)}
    D = |det(T_{K_e(t_0)})|
    for r, s = 1,2.3 do                               ▷ Iterate over all local node and function pairings (r, s)
        i = G(e,r), j = G(e,s)                        ▷ G is the local-to-global index mapping
        M_h(t_0)_{i,j} = M_h(t_0)_{i,j} + D M̂(r,s)
        S_h(t_0)_{i,j} = S_h(t_0)_{i,j} + α (D/2) (Ŝ(r) T^{-1}_{K_e(t_0)}) · (Ŝ(s) T^{-1}_{K_e(t_0)})
    end for
end for
```

*Remark* 5.1. We again want to emphasize an important feature of the spatial discretization: The existence of linear affine mapping $F_{K(t)} : \hat{K} \to K(t)$ that maps the reference triangle $\hat{K}$ to an arbitrary triangle $K(t) \in T_h(t)$, allows us to compute the integrals

$$\int_{K(t)} \varphi_i(t,x)\,\varphi_j(t,x)\,dx$$

and

$$\int_{K(t)} \nabla\varphi_i(t,x) \cdot \nabla\varphi_j(t,x)\,dx$$

easily and without using a numerical quadrature. This is done by evaluating these integrals for the reference element and then transforming them as shown in the above algorithm. The basic idea is described in [11, Kapitel 11].

### 5.1.2 Time Integrators

We apply several numerical time integrators to our problem:

**Runge-Kutta Methods** The deduction of arbitrary Runge-Kutta methods is described in [11, Kapitel 8]. Using the same approach on (5.1), we obtain the iteration

$$\mathbf{M}_{n+1}\mathbf{u}_{n+1} = \mathbf{M}_n\mathbf{u}_n - \tau\sum_{i=1}^{s} b_i\mathbf{S}_{n,i}\mathbf{u}_{n,i}$$
$$\mathbf{M}_{n,i}\mathbf{u}_{n,i} = \mathbf{M}_n\mathbf{u}_n - \tau\sum_{j=1}^{s} a_{ij}\mathbf{S}_{n,j}\mathbf{u}_{n,j}\,.$$

With the abbreviations

$$\mathbf{M}_n = \mathbf{M}_h(t_n), \qquad \mathbf{M}_{n,i} = \mathbf{M}_h(t_n + c_i\tau), \qquad \mathbf{S}_n = \mathbf{S}_h(t_n), \qquad \mathbf{S}_{n,i} = \mathbf{S}_h(t_n + c_i\tau)$$

The parameters $b_i$, $c_i$ and $a_{ij}$ correspond to the notation in the Butcher tableau.

We are using RadauIIA methods of order $p = 3$ and $p = 5$ and the well-known implicit Euler method. All these schemes are described in [9].

**Magnus Integrators** Another type of exponential integrators which are constructed for finite dimensional problems of the form

$$\frac{d}{dt}y(t) = A(t)y(t)$$

are Magnus integrators. These are then of course applied to our formulation (5.2). However, the theory behind Magnus integrators does no fit to our situation and we apply them for comparison. A short introduction into the theory of Magnus integrators, including a proper selection of other references, can be found in [13, Section 3].

The Magnus integrators we tested on our example are two fourth order schemes based on the Gauss nodes and the Simpson's rule and a second order scheme that coincides with the exponential midpoint rule.

**ExpInt** The numerical scheme from (4.7) is applied to (5.2) and was presented in Section 4.1 in detail.

Our problem differs from classical FEM discretizations of parabolic problems, by the time dependency of the matrices. Normally the assembly of the mass and stiffness matrices is done once at the beginning. In our case, the evaluation of the mass and stiffness matrix is needed at least once for every time step in the numerical scheme. Thus the assembly of the matrices is critical for the overall performance of the program.

*Remark* 5.2. The evaluation of $\varphi_k(-\tau\mathbf{A}_h(t))$ for $k = 1, 2, 3$, defined in (4.5), is needed for Magnus integrators and the exponential integrator (4.7). Our implementation is naive: We compute the eigendecomposition of $-\tau\mathbf{A}_h(t)$, apply the $\varphi_k$ to the diagonal matrix and transform the result back. Using e.g. Krylov subspace methods would significantly improve the performance for large scale problems.

## 5.2 The Numerical Test

### 5.2.1 Input Values

We choose $\Omega_0 = [0, 1] \times [0, 3/2]$ and $t \in [0, 1]$. Since linear mappings $\Phi(t, \cdot)$ could imply some consequences, as commutativity of the matrices or uniform transformation of each triangle, that might falsify our results, we consider nonlinear transformation illustrated in Figure 5.1.

This transformation maps the upper part $[0, 1] \times [1/2, 3/2]$ into a bottle neck and back, whereas the lower part remains unchanged. Hence, it is easy to compare the behavior of the diffusion equation on a fixed domain to its behavior on an changing domain.

We do not give a closed representation of our transformations but `Matlab`-files are available. Let us describe each of transformations features separately:

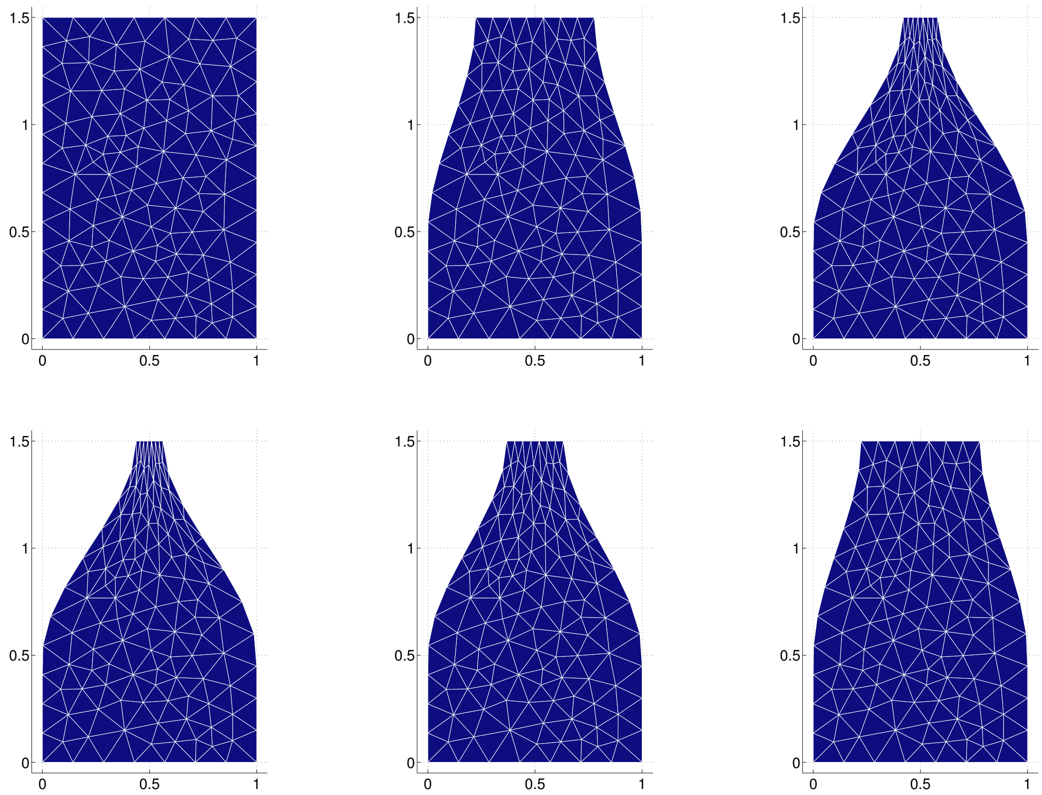


Figure 5.1: The domain $\Omega_t^h$ from our example at $t = 0$ ,$t = 0.15$, $t = 0.4$, $t = 0.5$, $t = 0.7$ and $t = 0.85$ from the top left. This visualization is done on a coarse mesh with 131 nodes. Observe the piecewise approximation of the evolving domain $\Omega_t$.

- The upper part $[0,1] \times [1/2, 3/2]$ is deformed in a quadratic diminution: Given a fixed $t \in [0,1]$, the left boundary part $\{0\} \times [1/2, 3/2]$ is deformed into a quadratic polynomial $(p_l(y), y)$ with $y \in [1/2, 3/2]$. $p_l$ connects smoothly to the lower part (i.e. $p_l'(1/2) = 0$) and keeps the top left vertex rectangular(i.e. $p_l'(3/2) = 0$). The connection to the lower part is fixed with $p_l(1/2) = 0$, while the upper end point is moving in time with $p_l(3/2) = x_l(t)$. The right part boundary part $\{1\} \times [1/2, 3/2]$ is transformed symmetrically with respect to the imaginary line $\{1/2\} \times [1/2, 3/2]$.

- While the boundary is deformed as described above, the inner part of $\Omega_0$ is scaled linearly: Given a segment $[0,1] \times \{y\}$ in $\Omega_0$ along the $x$ axis and the boundary points $x_*(t,y)$ and $x^*(t,y)$ of this segment at time $t$, we map

$$[0, 1/2] \times \{y\} \rightarrow [x_*(t,y), 1/2] \times \{y\}$$

  and

$$[1/2, 1] \times \{y\} \rightarrow [1/2, x^*(t,y)] \times \{y\}$$

  linearly. Observe that this part of the transformation is not differentiable with respect to the space variable, but only continuous.

- The variation in time is now fully described by the position of the upper left $(x_l(t), 3/2)$ and right vertex $(x_r(t), 3/2)$. They are chosen symmetrically with respect to the imaginary line $\{1/2\} \times [1/2, 3/2]$

$$x_r(t) = 1 - x_l(t)\,.$$

To avoid any non-bijective mapping, we only allow $x_l(t) < 1/2$. Furthermore, we have $x_l(t) \geq 0$, since a broadening of the domain diminishes the quality of the mesh.

The actual variation of $x_l(t)$ in time is now also given by a quadratic relation where we chose

$$x_l(0) = 0, \qquad x_l(1/2) = \frac{7}{16}, \qquad x_l(1) = 0\,.$$

Thus the transformations are only non-linear along the $y$ axis and that they show quadratic behavior in time.

The initial value is also chosen to guarantee an easy comparison of the solution on the deforming and the constant part of the domain. It can be described as two concentration walls, orientated along the $y$ axis:

$$u_0(x) = g(x_1)\chi_{[0,1/3]}(x_1) + g(1 - x_1)\chi_{[2/3,1]}(x_1),$$
$$g(\xi) = \sin(6\pi\xi - \pi/2) + 1\,.$$

Given the strong diffusion character of our problem, it is necessary to choose a small diffusion constant

$$\alpha = 0.01$$

since we want the solution not to vanish during the first half of the simulation.

### 5.2.2 Discussion of the Solution

The RadauIIA method with order $p = 5$ shows the fastest convergence rate and is thus used to compute our reference solution with the step size $\tau = 0.001$. We want to point out some noticeable facts of the solution which is shown in Figure 5.2 and in Figure 5.3 in topview. For an easier discussion, we understand $u$ as a distribution of heat:

- The contraction in the upper part of the domain results in an increased heat.
- Since contraction is faster then the diffusion of heat, the two heat walls stay clearly separated. The walls do not merge until $t \approx 0.35$.
- When the domain is most contracted at $t = 0.5$, the diffusion process is happening quite fast, since the homogeneous Dirichlet BC force the solution to vanish on the boundary. Therefore the gradient is quite large and we observe a strong heat flux from the heat peak into the boundary or rather the cold surrounding of the domain. This actual loss of heat can only be avoided if one imposes homogeneous Neumann BC.
- On the other hand, the lower part shows the normal diffusion process on fixed domains unaffected by the solution on the upper part. This is a confirmation that both the numerical method and the PDE behave as they are supposed to.
- In the second half of the simulation, $t \in [0.5, 1]$, the widening of the domain flattens the distribution of heat significantly stronger than the diffusion process would.

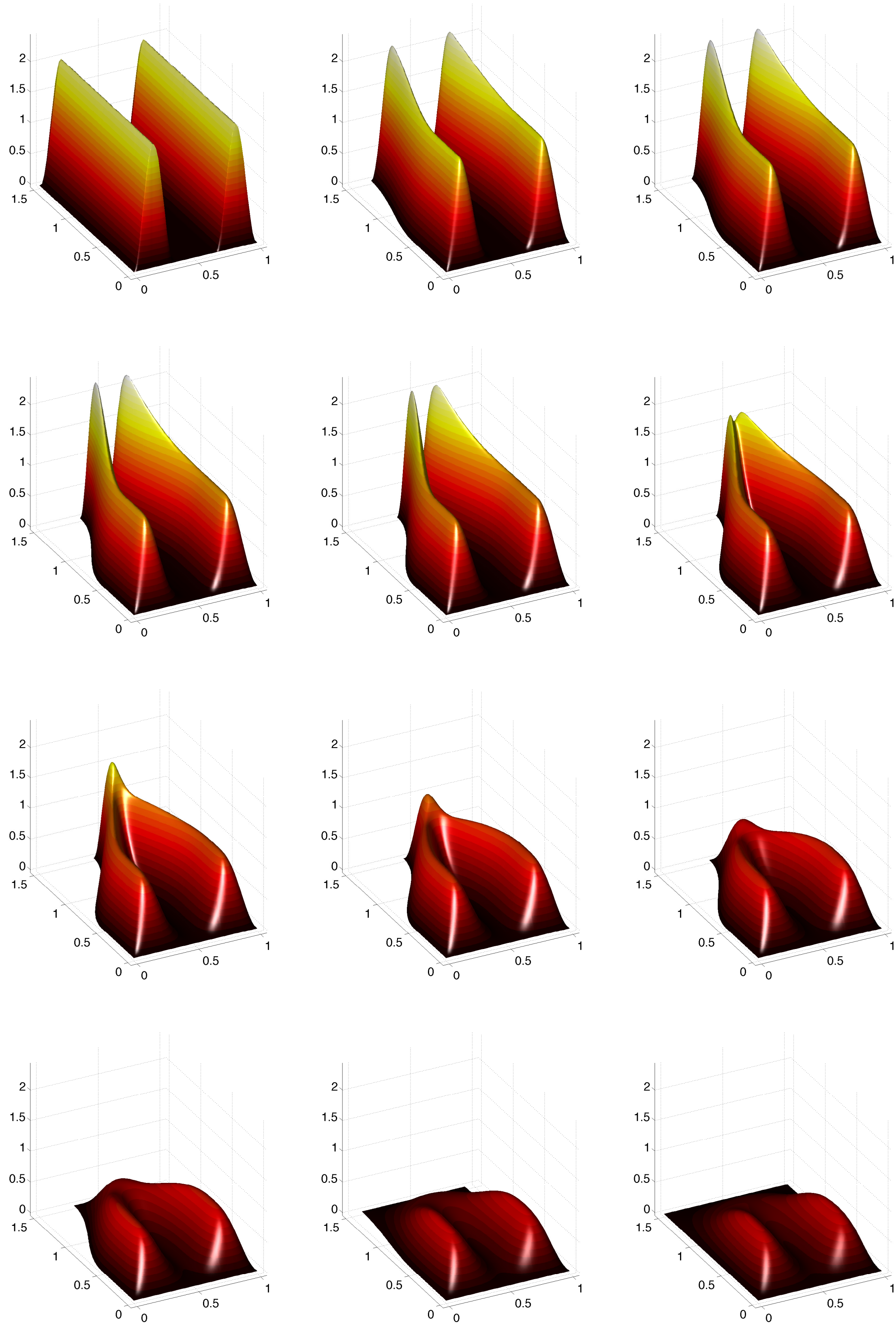


Figure 5.2: Plot of the reference solution computed on a fine mesh with 7341 nodes. We used the RadauIIA method of order 5 and a step size $\tau = 0.001$. The pictures shows the reference solution at almost uniformly distributed times $t$ in $[0, 1]$.

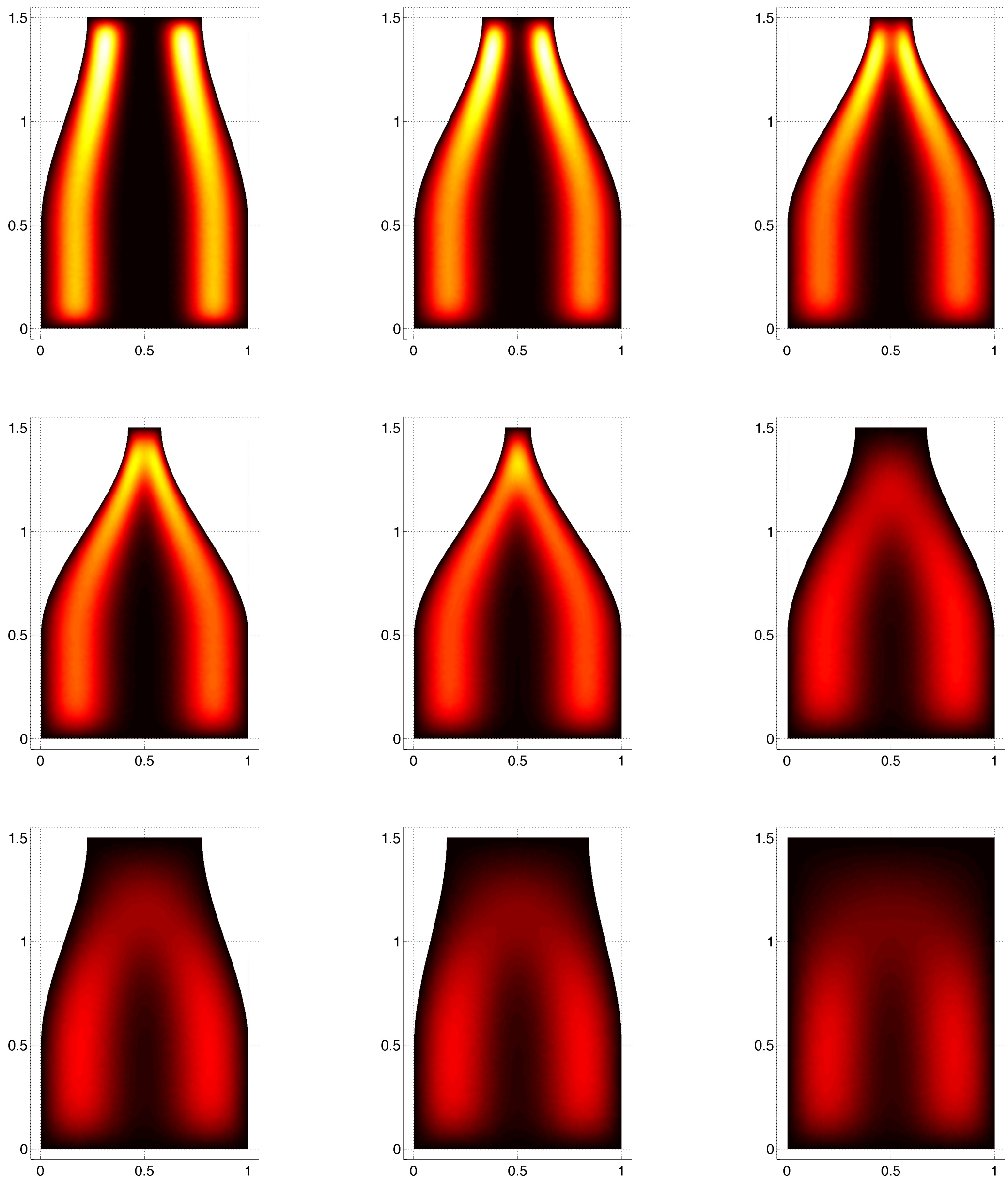


Figure 5.3: Another plot of the reference solution in topview. The pictures are taken an $t = 0.15$, $t = 0.25$, $t = 0.35$, $t = 0.40$, $t = 0.50$, $t = 0.75$, $t = 0.85$, $t = 0.90$ and $t = 1.00$ to discuss and show some features of the solution.

### 5.2.3 Order of Convergence

Finally, we begin with the discussion of the order of convergence tests: To investigate on the order of convergence of a numerical time integrator, we compute a reference solution with the RadauIIA method of order $p = 5$ and the small reference step size $\tau = 0.001$. The test integrator is then applied to (5.1) or (5.2), depending on the integrator, with decreasing step sizes

$$\tau_j = 2^{-j}, \qquad j = 1, \ldots, 5 .$$

The reference solution at time $t = 1$ is denoted by $\mathbf{u}^r$ and the solution of the integrator $I$ with step size $\tau_j$ at the end time $t = 1$ by $\mathbf{u}^I(j)$. The error we consider is then given by

$$\mathbf{e}^I(j) := \mathbf{u}^r - \mathbf{u}^I(j) .$$

We measure the error with respect to the $L^2(\Omega_1^h)$ norm

$$e^I(j) := \left\|\Pi_h(1)\mathbf{e}^I(j)\right\|_{L^2(\Omega_1^h)} = \sqrt{\mathbf{e}^I(j)^*\mathbf{M}_h(1)\mathbf{e}^I(j)}\,. \tag{5.3}$$

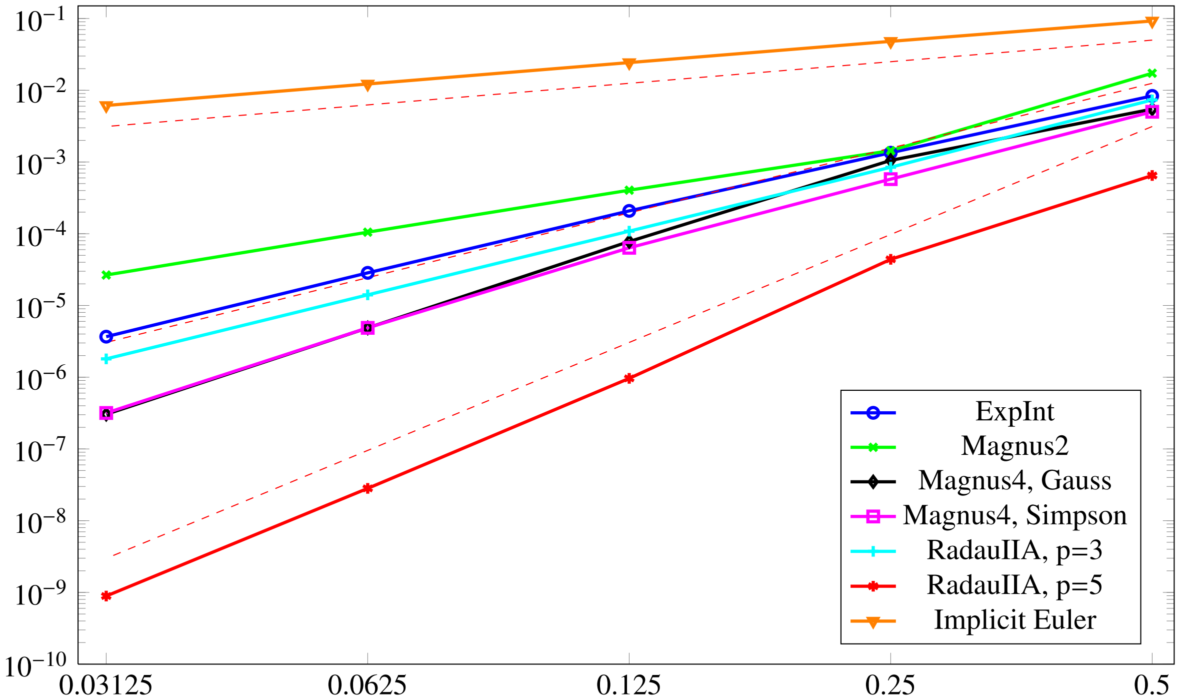


Figure 5.4: Order plots for mesh level 1. For comparison, we added dashed lines of slope 1,3 and 5.

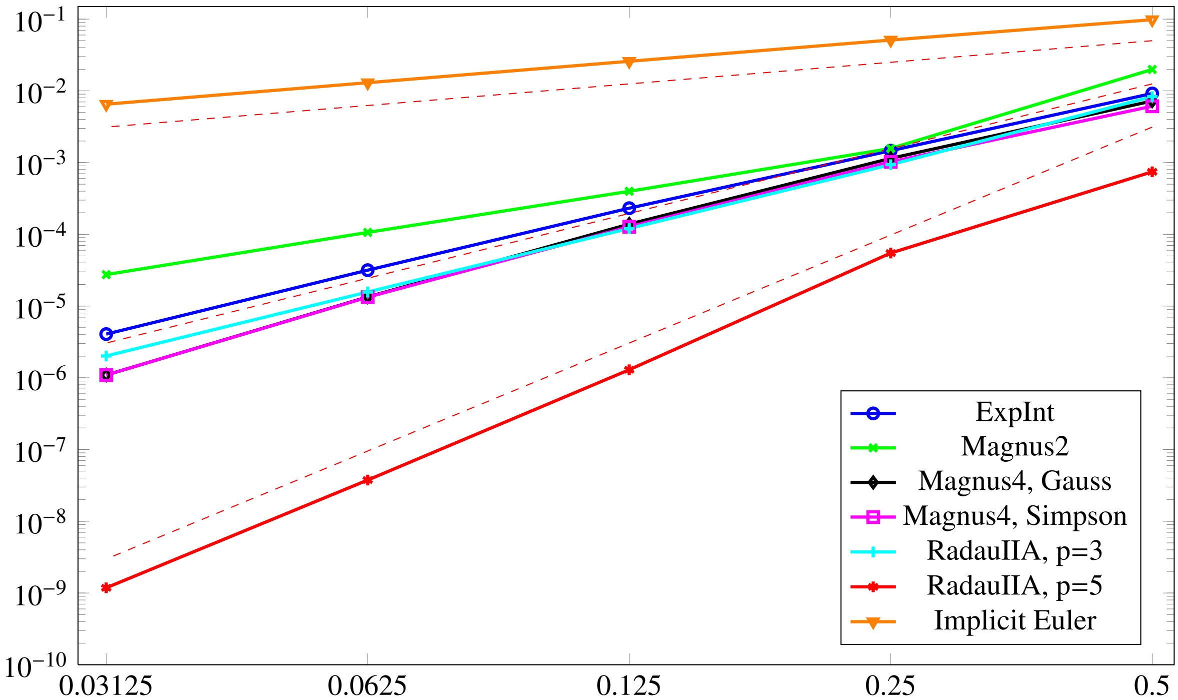


Figure 5.5: Order plots for mesh level 2. For comparison, we added dashed lines of slope 1,3 and 5.

To assure that the convergence rate does not depend on the spatial discretization parameter $h$ we use four triangulations of increasing fineness. We refer to these different meshes by giving them a level: The level 1 mesh has 131 nodes, level 2 has 486 nodes, level 3 has 1871 nodes and level 4 has 7341 nodes. All triangulations were created using the `Partial Differential Equation Toolbox` which is included in `Matlab`.

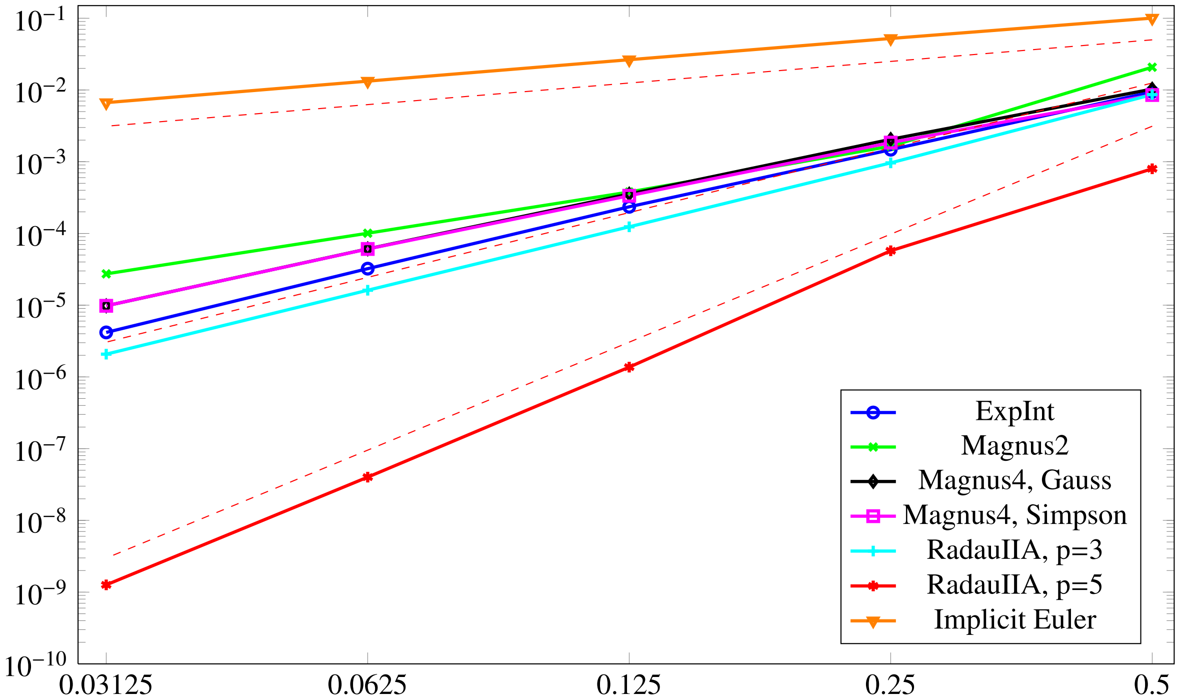


Figure 5.6: Order plots for mesh level 3.For comparison, we added dashed lines of slope 1,3 and 5.

In Figures 5.4, 5.5 and 5.6 we plotted the error (5.3) versus time step for the different numerical integrators at different mesh levels. The numerically observed orders of all but the Magnus integrators conincide with their theoretical convergence order, which can be seen by the dashed slopes we added for comparison. The RadauIIA methods show order 3 ($p = 3$) and order 5 ($p = 5$) convergence and the implicit Euler method shows a convergence rate of order 1. We notice that our exponential integrator (4.7), ExpInt, shows a numerical convergence rate of order 3, as our considerations, although not presented in this thesis, suggest. Obviously ExpInt meets, at least for our test example, the theoretical expectations.

Only the Magnus integrators show order reduction. But since their theory does not apply to our test example, this is not unexpected. The two order 4 Magnus integrators seem to have absolutely the same convergence rate: Their numerically observed order is 2. This rate of convergence is 2 orders less than when they are applied to hyperbolic problems. The Magnus integrator of theoretical order 2, also shows order reduction: For smaller step sizes the progression of the plot suggests that its order in this example is less than 2.

Figure 5.7 shows the plotted error for mesh level 4. Due to our limited computing capacity, we were only able to test the shown integrators. On this mesh level the numerically observed order of the fourth order Magnus integrator shows even stronger order reduction.

Finally, Figure 5.8 shows the error plots of ExpInt for the different mesh levels. The level 1 error is slightly smaller than the error of the other mesh levels, but the strong similarity in the progression as well as the almost identical error for level 2,3 and 4 give reason to expect convergence indepently of fineness $h$ of the spatial discretization.

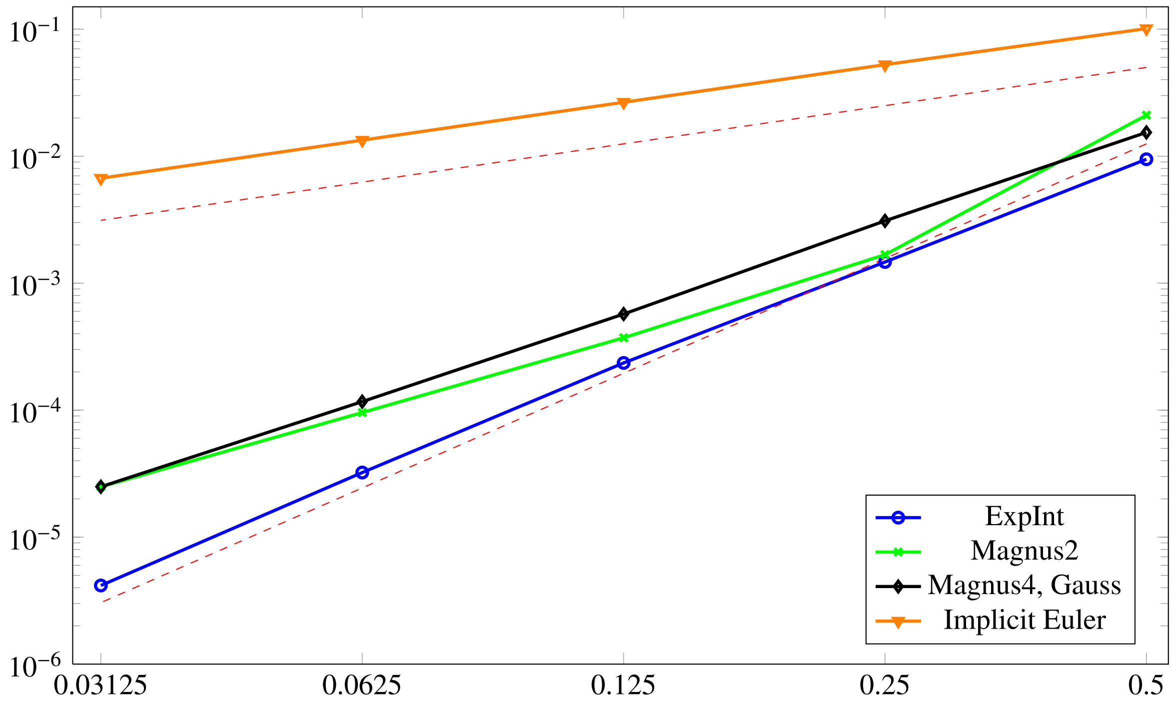


Figure 5.7: Order plots for mesh level 4.For comparison, we added dashed lines of slope 1 and 3.

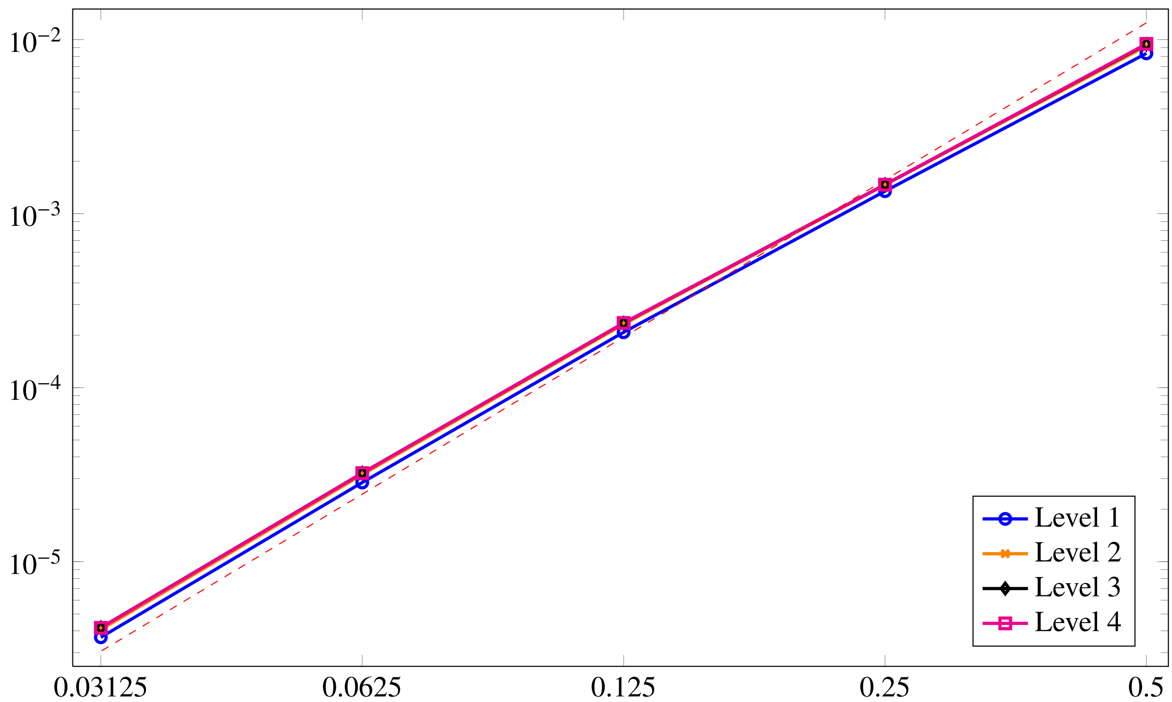


Figure 5.8: Error plot for the exponential integrator ExpInt for multiple mesh levels. The dashed line has slope 3.

# Conclusion

Most of this thesis is concerned with putting up a theoretical framework for our test example, the diffusion equation on evolving domains. The analysis in Chapter 1 as well as the finite element discretization in Chapter 3 required a lot of considerations besides the standard PDE theory. We were able to show that the problem is well-posed and that the semi-discretization converges, bearing in mind that higher order convergence is possible.

The construction of the ExpInt, the numerical exponential integrator for nonautonomous initial value problems, is based on the theory we presented in Chapter 2. The efforts we made to proof the well-posedness of the evolution problem in Theorem 2.8 were necessary to obtain a useful theoretical foundation for the numerical analysis of any approximation. Moreover, we showed that the semi-discrete problem suffices at least some of the assumptions that are necessary to guarantee its the well-posedness as an evolution problem.

Our numerical tests, described in Chapter 5 indicate that ExpInt is in fact a useful numerical scheme. A next step will be numerical tests with varying evolving domains to gather more data.

We also intend to close the theoretical gaps of this thesis soon. But it is still unclear how the proof of $(A_3)$ in Chapter 4 or other supplementary assumptions that are stated in the numerical analysis of ExpInt, is approached best.

These results will be presented elsewhere.

# Appendix A

# Appendix

## A.1 Material Derivative of a Gradient

**Lemma A.1.** *If all of the following quantities exist, then*

$$D_t[\,|\nabla f|^2] = 2\nabla[D_t f]\cdot\nabla f - 2(\nabla f J_x \underline{v})\cdot\nabla f\,.$$

*Proof.* We use these preliminary, formal considerations:

- Let $a, b : \mathbb{R}^2 \to \mathbb{R}^2$ with $a, b \in C^2(\mathbb{R}^2)$. Then

$$\nabla[a(x)\cdot b(x)] = b(x)J_x a(x) + a(x)J_x b(x)\,.$$

- Let all the following quantities exist, then

$$D_t[\nabla f\cdot\nabla f]\circ\Phi = \frac{d}{dt}[\nabla f\circ\Phi\cdot\nabla f\circ\Phi] = 2\frac{d}{dt}[\nabla f\circ\Phi]\cdot\nabla f\circ\Phi = 2D_t[\nabla f]\circ\Phi\cdot\nabla f\circ\Phi$$

  where the material derivative of a vector valued function is understood component wise. Thus

$$D_t[\nabla f\cdot\nabla f] = 2D_t[\nabla f]\cdot\nabla f\,.$$

Let $H_f$ denote the Hessian matrix of $f$. Then, with

$$J_x\nabla f = H_f$$

and the above considerations, we find

$$\begin{aligned} D_t[\,|\nabla f|^2] = \partial_t[\nabla f\cdot\nabla f] + \nabla[\nabla f\cdot\nabla f]\cdot\underline{v} &= 2\partial_t\nabla f\cdot\nabla f + 2(\nabla f\,H_f)\cdot\underline{v} \\ &= 2\nabla[\partial_t f]\cdot\nabla f + 2(\nabla f\,H_f)\cdot\underline{v} \\ &= 2\nabla[D_t f]\cdot\nabla f + 2(H_f\,\underline{v} - \nabla[\nabla f\cdot\underline{v}])\cdot\nabla f \\ &= 2\nabla[D_t f]\cdot\nabla f + 2(H_f\,\underline{v} - H_f\underline{v} - \nabla f J_x\underline{v})\cdot\nabla f \\ &= 2\nabla[D_t f]\cdot\nabla f - 2(\nabla f J_x\underline{v})\cdot\nabla f\,. \end{aligned}$$

□

## A.2 Semigroup Theory

The results in this section are all given in [15]. Let $X$ be a Banach space with respect to $\|\cdot\|$.

The following Theorem states some properties of $C_0$ semigroups:

**Theorem A.2** ([15, Theorem 1.2.4]). *Let $T(t)$ be a $C_0$ semigroup and let $A$ be its infinitesimal generator. Then*

1. *For $x \in X$,*
$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} T(s)x\,ds = T(t)x\,.$$

2. *For $x \in X$*
$$\int_0^t T(s)x\,ds \in D(A)$$
*and*
$$A\left(\int_0^t T(s)x\,ds\right) = T(t)x - x\,.$$

3. *For $x \in D(A)$,*
$$T(t)x \in D(A)$$
*and*
$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax\,.$$

4. *For $x \in D(A)$,*
$$T(t)x - T(s)x = \int_s^t T(\tau)Ax\,d\tau = \int_s^t AT(\tau)x\,d\tau\,.$$

**Theorem A.3** ([15, Theorem 1.7.7]). *Let $A$ be a densely defined operator in $X$ satisfyong the following conditions.*

1. *For some $0 < \delta < \pi/2$, $\rho(A) \supset \Sigma_\delta = \{\lambda \mid \left|\arg\lambda\right| < \pi/2 + \delta\} \cup \{0\}$.*

2. *There exists a constant $M$ such that*
$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|} \qquad \text{for } \lambda \in \Sigma_\delta,\ \lambda \neq 0.$$

*Then, $A$ is the infinitesimal generatpor of a $C_0$ semigroup $T(t)$ satisfying $\|T(t)\| \leq C$. Moreover,*
$$T(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda, A)\,d\lambda$$
*where $\Gamma$ is a smooth curve in $\Sigma_\delta$ running from $\infty e^{-i\vartheta}$ to $\infty e^{i\vartheta}$ for $\pi/2 < \vartheta < \pi/2 + \delta$. The integral converges for $t > 0$ in the uniform operator norm.*

The next result is a characterization of infinitesimal generators of analytic semigroups:

**Theorem A.4** ([15, Theorem 2.5.2]). *Let $T(t)$ be a uniformly bounded $C_0$ semigroup. Let $A$ be the infinitesimal generator of $T(t)$ and assume $0 \in \rho(A)$. The following statements are equivalent:*

1. *$T(t)$ can be extended to an analytic semigroup in a sector $\Delta_\delta = \{z \mid \left|\arg z\right| < \delta\}$ and $\|T(z)\|$ is uniformly bounded in every closed subsector of $\overline{\Delta_{\delta'}}$, $\delta' < \delta$ of $\Delta_\delta$.*

2. *There exists a constant $C$ such that for every $\sigma > 0$, $\tau \neq 0$*
$$\|R(\sigma + i\tau, A)\| \leq \frac{C}{|\tau|}\,.$$

3. *There exist* $0 < \delta < \pi/2$ *and* $M > 0$ *such that*

$$\rho(A) \supset \Sigma = \left\{\lambda \mid \left|\arg \lambda\right| < \tfrac{\pi}{2} + \delta\right\} \cup \{0\}$$

*and*

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|} \qquad \text{for } \lambda \in \Sigma,\ \lambda \neq 0.\ .$$

4. $T(t)$ *is differentiable for* $t > 0$ *and there is a constant* $C$ *such that*

$$\|AT(t)\| \leq \frac{C}{t} \qquad \text{for } t > 0.$$

## A.3 Gronwall Lemmata

The results of this section are taken from [4]. They hold true in a much more general form and setting, but we state the results as we need them.

**Proposition A.5** (Gronwall lemma: integral form, [4, Proposition 2.1])**.** *Let* $T \in \mathbb{R}$, $\beta \in \mathbb{R}$ *and* $a, b \in L^\infty(0, T)$. *Then,*

$$a(t) \leq b(t) + \beta \int_0^t a(s)\, ds \qquad \text{almost everywhere in } [0, T]$$

*implies for allmost all* $t \in [0, T]$

$$a(t) \leq b(t) + \int_0^t e^{\beta(t-s)} b(s)\, ds\ .$$

*If* $b \in W^{1,1}(0, T)$, *it follows*

$$a(t) \leq e^{\beta t}\left(b(0) + \int_0^t e^{-\beta s} b'(s)\, ds\right)\ .$$

**Proposition A.6** (Gronwall lemma: differential form, [4, Proposition 2.2])**.** *Let* $T \in \mathbb{R}$, $a \in W^{1,1}(0, T)$, $\beta \in \mathbb{R}$ *and* $g \in L^1(0, T)$. *Then,*

$$a'(t) \leq g(t) + \beta a(t) \qquad \text{almost everywhere in } [0, T]$$

*implies for allmost all* $t \in [0, T]$

$$a(t) \leq e^{\beta t} a(0) + \int_0^t e^{\beta(t-s)} g(s)\, ds\ .$$

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# Erklärung

Hiermit erkläre ich, David Hipp, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen benutzt, die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht und die Satzung des Karlsruher Instituts für Technologie zur Sicherung guter wissenschaftlicher Praxis beachtet habe.

David Hipp, Karlsruhe den 29.5.2013