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Long time wave propagation in heterogeneous media

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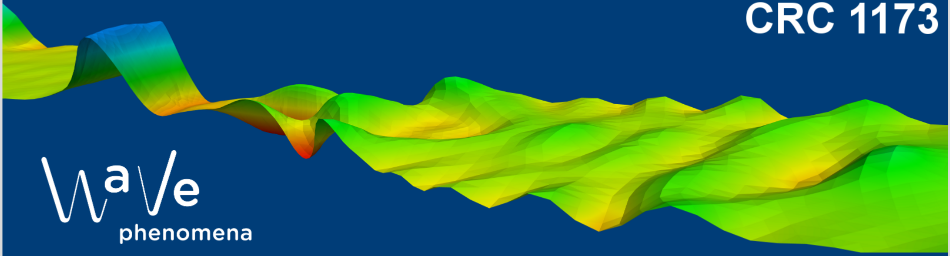

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Wave phenomena

References

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SIAM J. Numer. Anal., 54(3):1507–1534, 2016.

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Setting

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Let $T^\varepsilon = \varepsilon^{-2} T$ and consider $u^\varepsilon : [0, T^\varepsilon] \times \Omega \to \mathbb{R}$ such that

$$\begin{aligned} &\partial_t^2 u^\varepsilon(t,x) = \partial_x \left(a^\varepsilon(x)\partial_x u^\varepsilon(t,x)\right) && \text{in } (0, T^\varepsilon] \times \Omega, \\ &x \mapsto u^\varepsilon(t,x)\ \Omega\text{-periodic} && \text{in } [0, T^\varepsilon], \\ &u^\varepsilon(0,x) = g^0(x), \quad \partial_t u^\varepsilon(0,x) = g^1(x) && \text{in } \Omega, \end{aligned}$$

where for $Y \subseteq \mathbb{R}$, $a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right) = a(y)$ is Y-periodic in y and g^0,g^1 are given initial conditions.

Effective model for short time

$$\partial_t^2 u^0(t,x) = a^0 \partial_x^2 u^0(t,x)$$

where the effective coefficient $a^0 = \langle a(y)(1+\partial\chi)\rangle_Y$ and χ is the periodic solution of an elliptic boundary value problem.

Effective model for time up to $\varepsilon^{-2}T$

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$$\partial_t^2\overline{u}(t,x) = a^0\partial_x^2\overline{u}(t,x) + \varepsilon^2 b^0\partial_x^2\partial_t^2\overline{u}(t,x)$$

where the effective coefficient $a^0 = \langle a(y)(1+\partial\chi)\rangle_Y$ and χ is the periodic solution of an elliptic boundary value problem.
b^0 is obtained by a cascade of cell problems.

Agnes Lamacz, Dispersive effective models for waves in heterogeneous media, Math. Models Methods Appl. Sci., 21 (2011), pp. 1871–1899.

General effective model

The solution $\widetilde{u}$ of

$$\partial_t^2 \widetilde{u} = a^0 \partial_x^2 \widetilde{u} - \varepsilon^2 \left(\widetilde{a}^2 \partial_x^4 \widetilde{u} - \widetilde{b}^0 \partial_x^2 \partial_t^2 \widetilde{u} \right)$$

approximates the true solution with error $\mathcal{O}(\varepsilon)$ up to time $\mathcal{O}(\varepsilon^{-2})$.

Tomas Dohnal, Agnes Lamacz, and Ben Schweizer, Bloch-wave homogenization on large time scales and dispersive effective wave equations, Multiscale Model. Simul., 12 (2014), pp. 488–513.

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Outline

A family of effective equations for the wave equation over long time

The FE-HMM-L

Analysis of the finite element heterogeneous multiscale method over long times

Numerical experiments

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Equivalence class of correctors

Let $\chi \in \mathcal{W}_{\mathrm{per}}(Y)$ be the unique (equivalence class of) solution of the cell problem

$$(a(y)\partial\chi, \partial \boldsymbol{w})_{\mathrm{L}^2(Y)} = -(a(y), \partial \boldsymbol{w})_{\mathrm{L}^2(Y)} \quad \text{for all } \boldsymbol{w} \in \mathcal{W}_{\mathrm{per}}(Y)$$

$$a^0 = \langle a(y)(1+\partial\chi)\rangle_Y$$

Let $\widetilde{b}^0, \widetilde{a}^2 \geq 0$ be non-negative. Find $\widetilde{u} : [0, T^\varepsilon] \times \Omega \to \mathbb{R}$ such that

$$\begin{aligned}
&\partial_t^2 \widetilde{u} = a^0 \partial_x^2 \widetilde{u} - \varepsilon^2 \left(\widetilde{a}^2 \partial_x^4 \widetilde{u} - \widetilde{b}^0 \partial_x^2 \partial_t^2 \widetilde{u} \right) && \text{in } (0, T^\varepsilon] \times \Omega, \\
&x \mapsto \widetilde{u}(t,x) \ \Omega\text{-periodic} && \text{in } [0, T^\varepsilon], \\
&\widetilde{u}(0,x) = g^0(x), \quad \partial_t \widetilde{u}(0,x) = g^1(x) && \text{in } \Omega.
\end{aligned}$$

Theorem

Assume that the tensor $a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right) = a(y)$ *is uniformly* Y*-periodic and* $a(y) \in \mathrm{W}^{1,\infty}(Y)$*. Furthermore, assume that the solution* $\widetilde{u}$ *of the homogeneous equation and the initial conditions satisfy the regularity*

$$\widetilde{u} \in \mathrm{L}^\infty(0, T^\varepsilon; \mathrm{H}^5(\Omega)), \quad \partial_t \widetilde{u} \in \mathrm{L}^\infty(0, T^\varepsilon; \mathrm{H}^4(\Omega)), \quad \partial_t^2 \widetilde{u} \in \mathrm{L}^\infty(0, T^\varepsilon; \mathrm{H}^3(\Omega))$$
$$g^0 \in \mathrm{H}^4(\Omega), \quad g^1 \in \mathrm{H}^3(\Omega).$$

Let χ *be the solution of the cell problem and assume that for a* $\chi \in \chi$*, the coefficients* $\widetilde{b}^0, \widetilde{a}^2$ *satisfy the relation*

$$a^0 \widetilde{b}^0 - \widetilde{a}^2 = a^0 \left\langle \chi^2 \right\rangle_Y - a^0 \left\langle \chi \right\rangle_Y^2$$

Then the following error estimate holds

$$\|u^\varepsilon - \widetilde{u}\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{L}^2(\Omega))} \leq C\varepsilon \left(\left\| g^1 \right\|_{\mathrm{H}^3(\Omega)} + \left\| g^0 \right\|_{\mathrm{H}^4(\Omega)} + \|\widetilde{u}\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^5(\Omega))} + \left\| \partial_t^2 \widetilde{u} \right\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^3(\Omega))} \right)$$

where C *depends only on* Ω, T, Y, a, λ *and* Λ*.*

Set of solutions

$$\widetilde{b}^0 = b^0 + \langle \chi \rangle_Y^2, \quad b^0 = \Big\langle (\chi - \langle \chi \rangle_Y)^2 \Big\rangle_Y, \quad \widetilde{a}^2 = a^0 \langle \chi \rangle_Y^2$$

The set

$$\mathcal{F} = \Big\{ \widetilde{u}_{\langle \chi \rangle} \text{ solution of homogeneous equation with } \widetilde{b}^0, \widetilde{a}^2 \text{ defined above} \Big\}$$

constitutes a family of effective solutions for u^ε.

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Locally periodic media

Now let $f \in \mathrm{L}^2(0, T^\varepsilon; \mathrm{L}^2_0(\Omega))$. Find $u^\varepsilon : [0, T^\varepsilon] \times \Omega \to \mathbb{R}$ such that

$$\begin{aligned} \partial_t^2 u^\varepsilon(t,x) - \partial_x\left(a^\varepsilon(x)\partial_x u^\varepsilon(t,x)\right) &= f(t,x) && \text{in } (0, T^\varepsilon] \times \Omega, \\ x \mapsto u^\varepsilon(t,x) \ \Omega\text{-periodic} & && \text{in } [0, T^\varepsilon], \\ u^\varepsilon(0,x) = g^0(x), \quad \partial_t u^\varepsilon(0,x) &= g^1(x) && \text{in } \Omega, \end{aligned}$$

Here we allow $a^\varepsilon(x) = a\left(x, \frac{x}{\varepsilon}\right)$ where $a(x,y)$ is Ω-periodic in x and Y-periodic in y.
Among the family of effective equations $\mathcal{F}$ we pick $\langle \chi \rangle_Y = 0$ which yields $\widetilde{a}^2 = 0$ and denote the effective solution corresponding to this choice by $\overline{u}$.

Zero mean cell correctors and parameters

For each $x \in \Omega$ define $\chi(x,\cdot) \in \mathrm{W}_{\mathrm{per}}(Y)$ as the unique solution of

$$(a(x,\cdot)\partial_y \chi(x,\cdot), \partial_y w)_{\mathrm{L}^2(Y)} = -(a(x,\cdot), \partial_y w)_{\mathrm{L}^2(Y)} \quad \text{for all } w \in \mathrm{W}_{\mathrm{per}}(Y)$$

For $x \in \Omega$ define

$$a^0(x) = \langle a(x,\cdot)(1 + \partial_y \chi(x,\cdot)) \rangle_Y , \quad b^0(x) = \left\langle (\chi(x,\cdot))^2 \right\rangle_Y$$

The equation for $\overline{u}$ is: Find $\overline{u} : [0, T^\varepsilon] \times \Omega \to \mathbb{R}$ such that

$$\begin{aligned}
&\partial_t^2 \overline{u} - \partial_x \left(a^0(x) \partial_x \overline{u} \right) - \varepsilon^2 \partial_x \left(b^0(x) \partial_x \partial_t^2 \overline{u} \right) = f && \text{in } (0, T^\varepsilon] \times \Omega, \\
&x \mapsto \overline{u}(t,x) \ \Omega\text{-periodic} && \text{in } [0, T^\varepsilon], \\
&\overline{u}(0,x) = g^0(x), \quad \partial_t \overline{u}(0,x) = g^1(x) && \text{in } \Omega.
\end{aligned}$$

Weak formulation

Define bilinear forms

$$A^0(v,w) = \left(a^0(x)\partial_x v, \partial_x w\right)_{\mathrm{L}^2(\Omega)}, \quad B^0(v,w) = \left(b^0(x)\partial_x v, \partial_x w\right)_{\mathrm{L}^2(\Omega)}$$

and the Hilbert space $\mathcal{S}(\Omega) = \left\{v \in \mathrm{L}^2(\Omega) \,:\, \partial_x v \in \mathrm{L}^2(\Omega)\right\}$ equipped with

$$(v,w)_{\mathcal{S}(\Omega)} = (v,w)_{\mathrm{L}^2(\Omega)} + \varepsilon^2 B^0(v,w)$$

$\overline{u}$ satisfies

$$\left(\partial_t^2 \overline{u}(t), v\right)_{\mathcal{S}(\Omega)} + A^0(\overline{u}(t), v) = (f(t), v)_{\mathrm{L}^2(\Omega)}, \quad \forall v \in \mathrm{W}_{\mathrm{per}}(\Omega), \text{ a.e. } t$$

$$\overline{u}(0) = g^0, \quad \partial_t \overline{u}(0) = g^1$$

FE-HMM-L

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Find $u_H : [0, T^\varepsilon] \to \mathrm{V}_H(\Omega)$ such that

$$\left(\partial_t^2 u_H(t), v_H\right)_Q + A_H(u_H(t), v_H) = (f(t), v_H)_{\mathrm{L}^2(\Omega)}\,, \quad \forall v_H \in \mathrm{V}_H(\Omega), \text{ a.e. } t$$

$$u_H(0) = g_H^0, \quad \partial_t u_H(0) = g_H^1$$

where

$$A_H(v_H, w_H) = \sum_{K \in \mathcal{T}_H} \sum_{j=1}^{J} \omega_{K_j} a_K^0(x_{K_j}) \partial_x v_H(x_{K_j}) \partial_x w_H(x_{K_j})$$

$$(v_H, w_H)_Q = (v_H, w_H)_H + (v_H, w_H)_M\,,$$

$$(v_H, w_H)_H = \sum_{K \in \mathcal{T}_H} \sum_{j=1}^{J'} \omega'_{K_j} v_H(x'_{K_j}) w_H(x'_{K_j}),$$

$$(v_H, w_H)_M = \varepsilon^2 \sum_{K \in \mathcal{T}_H} \sum_{j=1}^{J} \omega_{K_j} b_K^0(x_{K_j}) \partial_x v_H(x_{K_j}) \partial_x w_H(x_{K_j})$$

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Let $\psi_{h,K_j} \in \mathrm{V}_h(K_{\delta_j})$ the solution of

$$\left(a^\varepsilon(x)\partial_x\psi_{h,K_j}, \partial_x z_h\right)_{\mathrm{L}^2(K_{\delta_j})} = -(a^\varepsilon(x), \partial_x z_h)_{\mathrm{L}^2(K_{\delta_j})}, \quad \forall z_h \in \mathrm{V}_h(K_{\delta_j})$$

and define approximated parameters at quadrature points as

$$a_K^0(x_{K_j}) = \left\langle a^\varepsilon(x)(1 + \partial_x\psi_{h,K_j})\right\rangle_{K_{\delta_j}}, \quad b_K^0(x_{K_j}) = \varepsilon^{-2}\left\langle \left(\psi_{h,K_j}\right)^2\right\rangle_{K_{\delta_j}}$$

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A priori $\mathrm{L}^\infty(\mathrm{H}^1)$ error

Theorem

Assume that δ satisfies $\frac{\delta}{\varepsilon} \in \mathbb{N}_{>0}$, that the micro mesh size is $h < \varepsilon$ and that the degree of the micro finite element space is $q = 1$. Furthermore, assume that the tensor $a^\varepsilon \in \mathrm{W}^{1,\infty}(\Omega)$ is collocated in the slow variable, i.e. for all $(K, j) \in \mathcal{T}_H \times \{1, \dots, J\}$, $a^\varepsilon(x) = a\left(x_{K_j}, \frac{x}{\varepsilon}\right)$ for a.e. $x \in K_{\delta_j}$. Finally, assume that a^0,$b^0 \in \mathrm{W}^{\ell,\infty}(\Omega)$ and $\partial_t^k \overline{u} \in \mathrm{L}^\infty(0, T^\varepsilon; \mathrm{H}^{\ell+1}(\Omega))$ for $0 \le k \le 4$.
Then the error $e = \overline{u} - u_H$ satisfies the estimate

$$\|\partial_t e\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{L}^2(\Omega))} + \|e\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^1(\Omega))} \le C_1 \left(h/\varepsilon^2\right)^2 + \mathrm{e}^{FE}_{\mathrm{H}^1},$$

where $\mathrm{e}^{FE}_{\mathrm{H}^1}$ is the standard FEM error estimate,

$$\mathrm{e}^{FE}_{\mathrm{H}^1} \le C_2 \left(\left\|g^1 - g^1_H\right\|_{\mathrm{L}^2(\Omega)} + \varepsilon \left\|g^1 - g^1_H\right\|_{\mathrm{H}^1(\Omega)} + \left\|g^0 - g^0_H\right\|_{\mathrm{H}^1(\Omega)} + H^\ell \right),$$

$C_1 = \widetilde{C}_1 \sum_{k=0}^4 \left\|\partial_t^k \overline{u}\right\|_{\mathrm{L}^\infty(\mathrm{H}^{\ell+1}(\Omega))}$ and $C_2 = \widetilde{C}_2 \sum_{k=0}^4 \left\|\partial_t^k \overline{u}\right\|_{\mathrm{L}^1(\mathrm{H}^{\ell+1}(\Omega))}$ with $\widetilde{C}_1$,$\widetilde{C}_2$ independent of H,h,ε and δ.

A priori $\mathrm{L}^\infty(\mathrm{L}^2)$ error

Theorem

As before assume that $h \leq \varepsilon$, $q = 1$ and $a^\varepsilon(x) = a\left(x_{K_j}, \frac{x}{\varepsilon}\right)$ for a.e. $x \in K_{\delta_j}$. Furthermore, assume that $\partial_t^k \overline{u} \in \mathrm{L}^\infty(0, T^\varepsilon; \mathrm{H}^{\ell+1}(\Omega))$ for $0 \leq k \leq 3$, $g_H^1 = I_H g^1$ and $a^0 \in \mathrm{W}^{\ell+1,\infty}(\Omega)$. Then $e = \overline{u} - u_H$ satisfies the estimate

$$\|e\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{L}^2(\Omega))} \leq C_1 \left(h/\varepsilon^2\right)^2 + \mathrm{e}_{\mathrm{L}^2}^{FE},$$

where $\mathrm{e}_{\mathrm{L}^2}^{FE}$ is the standard FEM error estimate,

$$\mathrm{e}_{\mathrm{L}^2}^{FE} \leq C_2 \left(\left\| g^0 - g_H^0 \right\|_{\mathrm{L}^2(\Omega)} + \varepsilon \left\| g^0 - g_H^0 \right\|_{\mathrm{H}^1(\Omega)} + H^{\ell+1} + \varepsilon H^\ell \right),$$

$C_1 = \widetilde{C}_1 \sum_{k=0}^{3} \left\| \partial_t^k \overline{u} \right\|_{\mathrm{L}^\infty(\mathrm{H}^{\ell+1}(\Omega))}$ and $C_2 = \widetilde{C}_2 \sum_{k=0}^{3} \left\| \partial_t^k \overline{u} \right\|_{\mathrm{L}^1(\mathrm{H}^{\ell+1}(\Omega))}$ with $\widetilde{C}_1, \widetilde{C}_2$ independent of H, h, ε and δ.

Corollary

Assume that the tensor $a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right)$ is periodic with $a \in \mathrm{W}^{1,\infty}(\Omega)$ and $f = 0$. Also, assume that $g_H^0 = I_H g^0$, $g_H^1 = I_H g^1$ and let the settings of the FE-HMM-L be such that $\delta/\varepsilon \in \mathbb{N}_{>0}$, $h \leq \varepsilon$, $q = 1$ and $\ell = 1$. Finally assume that the following regularity holds

$$g^0 \in \mathrm{H}^4(\Omega), \quad g^1 \in \mathrm{H}^3(\Omega), \quad \partial_t^k \in \mathrm{L}^\infty(\mathrm{H}^{5-k}(\Omega)) \quad 0 \leq k \leq 3.$$

Then we have the following estimate

$$\|u^\varepsilon - u_H\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{L}^2(\Omega))} \leq C_1 \left(\varepsilon + \left(h/\varepsilon^2\right)^2\right) + C_2 \left(H^2 + \varepsilon H\right)$$

where $C_1 = \widetilde{C}_1 \sum_{k=0}^3 \left\|\partial_t^k \widetilde{u}\right\|_{\mathrm{L}^\infty(\mathrm{H}^{5-k}(\Omega))}$ and $C_2 = \widetilde{C}_2 \sum_{k=0}^3 \left\|\partial_t^k \widetilde{u}\right\|_{\mathrm{L}^1(\mathrm{H}^2(\Omega))}$ independent of H,h,ε and δ.

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$$\mathcal{B}^\varepsilon \widetilde{u}(t,x) = \widetilde{u}(t,x) + \varepsilon\chi\left(\frac{x}{\varepsilon}\right)\partial_x\widetilde{u}(t,x) + \varepsilon^2\theta\left(\frac{x}{\varepsilon}\right)\partial_x^2\widetilde{u}(t,x) + \varepsilon^3\kappa\left(\frac{x}{\varepsilon}\right)\partial_x^3\widetilde{u}(t,x) + \varepsilon^4\rho\left(\frac{x}{\varepsilon}\right)\partial_x^4\widetilde{u}(t,x)$$

$$\mathcal{A}^\varepsilon = -\partial_x\left(a\left(\frac{x}{\varepsilon}\right)\partial_x(\cdot)\right)$$

Now we compute $\left(\partial_t^2 + \mathcal{A}^\varepsilon\right)\mathcal{B}^\varepsilon\widetilde{u}$

$$\partial_t^2(\mathcal{B}^\varepsilon\widetilde{u}) = a^0\partial_x^2\widetilde{u} + \varepsilon a^0\chi\partial_x^3\widetilde{u} + \varepsilon^2\left(a^0(\theta + \widetilde{b}^0) - \widetilde{a}^2\right)\partial_x^4\widetilde{u} + \mathcal{O}(\varepsilon^3)$$

$$\begin{aligned}
-\partial_x\left(a\left(\frac{x}{\varepsilon}\right)\partial_x\left(\mathcal{B}^\varepsilon\widetilde{u}(x)\right)\right) =& \varepsilon^{-1}\left(-\partial_y(a(y)(1+\partial_y\chi))\right)\partial_x\widetilde{u} \\
& \varepsilon^0\left(-\partial_y(a(y)(\chi+\partial_y\theta)) - a(y)(1+\partial_y\chi)\right)\partial_x^2\widetilde{u} \\
& \varepsilon^1\left(-\partial_y(a(y)(\theta+\partial_y\kappa)) - a(y)(\chi+\partial_y\theta)\right)\partial_x^3\widetilde{u} \\
& \varepsilon^2\left(-\partial_y(a(y)(\kappa+\partial_y\rho)) - a(y)(\theta+\partial_y\kappa)\right)\partial_x^4\widetilde{u} + \mathcal{O}(\varepsilon^3)
\end{aligned}$$

Appendix

Lemma
Under the assumptions of Theorem, $\hat{\mathcal{B}}^\varepsilon \widetilde{u}$ satisfies

$$\left(\partial_t^2 + \mathcal{A}^\varepsilon\right) \hat{\mathcal{B}}^\varepsilon \widetilde{u}(t) = \mathcal{R}^\varepsilon \widetilde{u}(t) \quad \text{in } \mathcal{W}^*_{\mathrm{per}}(\Omega) \ \text{ for a.e. } t \in [0, T^\varepsilon],$$

where the right hand side $\mathcal{R}^\varepsilon \widetilde{u} \in \mathrm{L}^2(0, T^\varepsilon; \mathcal{W}^*_{\mathrm{per}}(\Omega))$ satisfies the estimate

$$\|\mathcal{R}^\varepsilon \widetilde{u}\|_{\mathrm{L}^2(0,T^\varepsilon;\mathcal{W}^*_{\mathrm{per}}(\Omega))} \leq C\varepsilon^2 \left(\|\widetilde{u}\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^5(\Omega))} + \left\|\partial_t^2 \widetilde{u}\right\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^3(\Omega))} \right)$$

for a constant C that only depends on T,Y,a,λ and Λ.

Lemma
Under the assumptions of Theorem, $\eta^\varepsilon = \hat{\mathcal{B}}^\varepsilon \widetilde{u} - [u^\varepsilon]$ satisfies

$$\begin{aligned}
&\|\partial_t \eta^\varepsilon\|_{\mathrm{L}^\infty(\mathcal{W}^*_{\mathrm{per}}(\Omega))} + \|\eta^\varepsilon\|_{\mathrm{L}^\infty(\mathcal{L}^2)} \\
&\quad \leq C \left(\|\partial_t \eta^\varepsilon(0)\|_{\mathcal{W}^*_{\mathrm{per}}(\Omega)} + \|\eta^\varepsilon(0)\|_{\mathcal{L}^2} + \varepsilon^{-1} \|\mathcal{R}^\varepsilon \widetilde{u}\|_{\mathrm{L}^2(\mathcal{W}^*_{\mathrm{per}}(\Omega))} \right),
\end{aligned}$$

where C depends only on λ,Λ and T and $\mathcal{R}^\varepsilon \widetilde{u}$ is given above.

Appendix

Lemma

$$e_{a^0} = \sup_{K,j} \left| a^0(x_{K_j}) - a^0_K(x_{K_j}) \right| \leq C \left(\frac{h}{\varepsilon} \right)^2, \quad e_{b^0} = \sup_{K,j} \varepsilon^2 \left| a^0(x_{K_j}) - a^0_K(x_{K_j}) \right| \leq C\varepsilon \left(\frac{h}{\varepsilon} \right)^2$$