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An introduction to homogenization and Maxwell's equations in dielectric media

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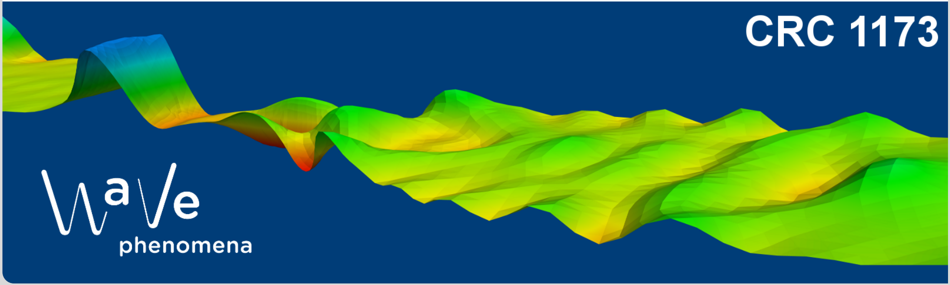

Outline

Homogenization

- 1D example
- Maxwell's equations

Implementation

Outline

Homogenization

1D example

Maxwell's equations

Implementation

Homogenization: 1D example

In $\Omega = (0, 1)$ consider

$$-\partial_x \left(A_\delta(x) \partial_x u_\delta(x) \right) = 1, \qquad u_\delta(0) = u_\delta(1) = 0$$

$$A_\delta(x) = A\left(\frac{x}{\delta}\right) = \left(2 - \cos\left(2\pi \frac{x}{\delta}\right)\right)^{-1} \qquad \text{for } \delta = 2^{-k} \quad k \in \mathbb{N}$$

Homogenization: 1D example

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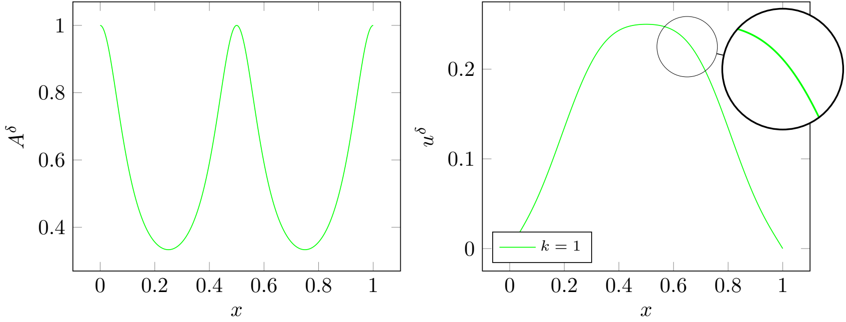

Homogenization: 1D example

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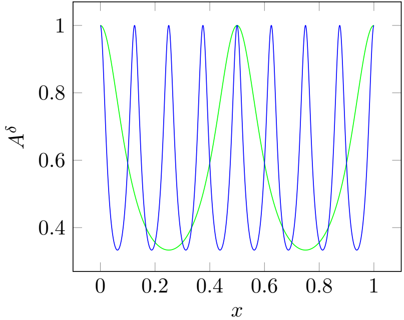

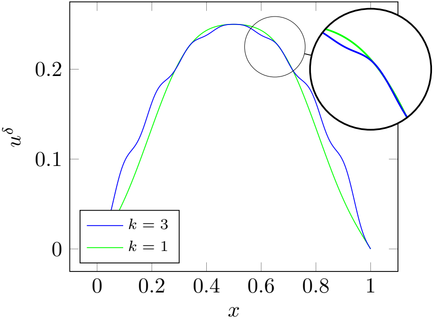

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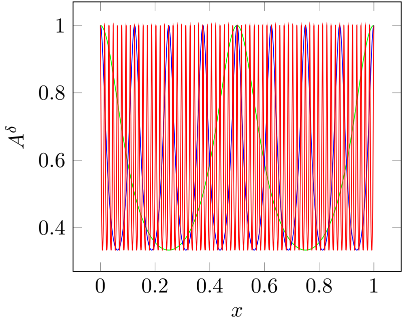

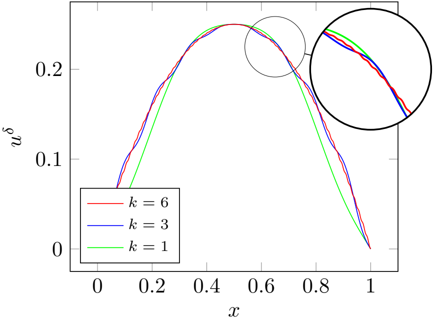

Homogenization: 1D example

$$
\begin{aligned}
u_\delta(x) &= x - x^2 \\
&- \delta \left(\frac{1}{4\pi} \sin\left(2\pi \frac{x}{\delta}\right) - \frac{x}{2\pi} \sin\left(2\pi \frac{x}{\delta}\right) - \delta \left(\frac{1}{4\pi^2} \cos\left(2\pi \frac{x}{\delta}\right) - \frac{1}{4\pi^2} \right) \right) \\
&\to x - x^2 \quad \text{, as } \delta \to 0
\end{aligned}
$$

Homogenization: 1D example

$$u_\delta(x) = x - x^2$$
$$- \delta \left(\frac{1}{4\pi} \sin\left(2\pi \frac{x}{\delta}\right) - \frac{x}{2\pi} \sin\left(2\pi \frac{x}{\delta}\right) - \delta \left(\frac{1}{4\pi^2} \cos\left(2\pi \frac{x}{\delta}\right) - \frac{1}{4\pi^2} \right)\right)$$
$$\to x - x^2 \quad \text{, as } \delta \to 0$$

- Idea:

$$u_\delta(x) = u_0(x) + \delta u^{(1)}(x, \frac{x}{\delta}) + \delta^2 u^{(2)}(x, \frac{x}{\delta}) + \dots$$

Homogenization: 1D example

$$
\begin{aligned}
u_\delta(x) &= x - x^2 \\
&\quad - \delta\left(\frac{1}{4\pi}\sin\left(2\pi\frac{x}{\delta}\right) - \frac{x}{2\pi}\sin\left(2\pi\frac{x}{\delta}\right) - \delta\left(\frac{1}{4\pi^2}\cos\left(2\pi\frac{x}{\delta}\right) - \frac{1}{4\pi^2}\right)\right) \\
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Homogenization: 1D example

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- Idea:

$$u_\delta(x) = u_0(x) + \delta u^{(1)}(x, \frac{x}{\delta}) + \delta^2 u^{(2)}(x, \frac{x}{\delta}) + \dots$$

- Goal: Find A_0 such that u_0 is solution of

$$-\partial_x\left(A_0(x)\partial_x u_0(x)\right) = 1, \qquad u_0(0) = u_0(1) = 0$$

Asymptotic expansion

Problem

$$\int\limits_{\Omega} A(\frac{x}{\delta})\partial_x u_\delta(x)\partial_x v(x)\,\mathrm{d}x = \int\limits_{\Omega} v(x)\,\mathrm{d}x \quad \forall v \in \mathrm{H}_0^1(\Omega)$$

Asymptotic expansion

Problem

$$\int_\Omega A(\frac{x}{\delta})\partial_x u_\delta(x)\partial_x v(x)\,\mathrm{d}x = \int_\Omega v(x)\,\mathrm{d}x \quad \forall v \in \mathrm{H}^1_0(\Omega)$$

Ansatz

$$u_\delta(x) = u^{(0)}(x, \frac{x}{\delta}) + \delta u^{(1)}(x, \frac{x}{\delta})$$
$$v(x) = v^{(0)}(x) + \delta v^{(1)}(x, \frac{x}{\delta})$$

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$$\partial_x u_\delta(x) = \delta^{-1}\partial_y u^{(0)}(x, \frac{x}{\delta}) + \partial_x u^{(0)}(x, \frac{x}{\delta}) + \partial_y u^{(1)}(x, \frac{x}{\delta}) + \delta\partial_x u^{(1)}(x, \frac{x}{\delta})$$
$$\partial_x v(x) = \partial_x v^{(0)}(x) + \partial_y v^{(1)}(x, \frac{x}{\delta}) + \delta\partial_x v^{(1)}(x, \frac{x}{\delta})$$

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Comparison

$\mathcal{O}(\delta^{-1})$

$$\int_\Omega A(\frac{x}{\delta})\partial_y u^{(0)}(x,\frac{x}{\delta})\left[\partial_x v^{(0)}(x)+\partial_y v^{(1)}(x,\frac{x}{\delta})\right]\,\mathrm{d}x=0$$

choose $v^{(0)}=0$, $v^{(1)}=u^{(0)}$

$$0=\int_\Omega A(\frac{x}{\delta})\partial_y u^{(0)}(x,\frac{x}{\delta})\cdot\partial_y u^{(0)}(x,\frac{x}{\delta})\,\mathrm{d}x\geq\alpha\left\|\partial_y u^{(0)}(\cdot,\frac{\cdot}{\delta})\right\|_{L^2(\Omega)}$$

- $u^{(0)}(x,\frac{x}{\delta})=u_0(x)$ with $\partial_y u_0=0$

Asymptotic expansion

Problem

$$\int_\Omega A(\frac{x}{\delta})\partial_x u^\delta(x)\partial_x v(x)\,\mathrm{d}x = \int_\Omega v(x)\,\mathrm{d}x \quad \forall v \in \mathrm{H}_0^1(\Omega)$$

Ansatz

$$u_\delta(x) = u_0(x) + \delta u^{(1)}(x, \frac{x}{\delta})$$
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Comparison

$\mathcal{O}(\delta^0)$

$$\int_\Omega A(\frac{x}{\delta}) \left[\partial_x u_0(x) + \partial_y u^{(1)}(x, \frac{x}{\delta})\right] \left[\partial_x v^{(0)}(x) + \partial_y v^{(1)}(x, \frac{x}{\delta})\right] \, \mathrm{d}x$$

$$= \int_\Omega v^{(0)}(x) \, \mathrm{d}x$$

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$$= \int_\Omega v^{(0)}(x) \,\mathrm{d}x$$

choose $v^{(0)} = 0$, $v^{(1)} = w(x)z(\frac{x}{\delta})$ with z 1-periodic, $Y = (0, 1)$

$$0 = \int_\Omega A(\frac{x}{\delta}) \left[\partial_x u_0(x) + \partial_y u^{(1)}(x, \frac{x}{\delta})\right] \partial_y z(\frac{x}{\delta}) w(x) \,\mathrm{d}x$$

Comparison

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$$\to \int_\Omega \int_Y A(y) \left[\partial_x u_0(x) + \partial_y u^{(1)}(x, y)\right] \partial_y z(y)\,\mathrm{d}y\, w(x)\,\mathrm{d}x$$

Cell Problem

$$\int_{Y} A(y)(\partial_x u_0(x) + \partial_y u^{(1)}(x,y))\partial_y z(y)\,\mathrm{d}y = 0 \quad \forall z \in \mathrm{H}^1_{\#}(Y)$$

$\mathrm{H}^1_{\#}(Y)$ space of $\mathrm{H}^1(Y)$-functions with periodic boundary conditions and zero mean

Cell Problem

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$\mathrm{H}^1_\#(Y)$ space of $\mathrm{H}^1(Y)$-functions with periodic boundary conditions and zero mean

Ansatz

$$u^{(1)}(x,y) = \partial_x u_0(x)\chi(y)$$

$$\partial_y u^{(1)}(x,y) = \partial_x u_0(x)\partial_y \chi(y)$$

$$\int_Y A(y)(1 + \partial_y \chi(y))\partial_y z(y)\,\mathrm{d}y = 0 \quad \forall z \in \mathrm{H}^1_\#(Y)$$

Homogeneous equation

$\mathcal{O}(\delta^0)$

$$\int\limits_{\Omega} A(\frac{x}{\delta}) \left[\partial_x u_0(x) + \partial_y u^{(1)}(x, \frac{x}{\delta})\right] \left[\partial_x v^{(0)}(x) + \partial_y v^{(1)}(x, \frac{x}{\delta})\right] \, \mathrm{d}x$$

$$= \int\limits_{\Omega} v^{(0)}(x) \, \mathrm{d}x$$

choose $v^{(0)}$ arbitrary, $v^{(1)} = 0$

$$\int\limits_{Y} v^{(0)}(x) \, \mathrm{d}x = \int\limits_{\Omega} A(\frac{x}{\delta}) \left[\partial_x u_0(x) + \partial_x u_0(x) \partial_y \chi(\frac{x}{\delta})\right] \partial_x v^{(0)}(x) \, \mathrm{d}x$$

Homogeneous equation

$\mathcal{O}(\delta^0)$

$$\int_\Omega A(\frac{x}{\delta})\left[\partial_x u_0(x)+\partial_y u^{(1)}(x,\frac{x}{\delta})\right]\left[\partial_x v^{(0)}(x)+\partial_y v^{(1)}(x,\frac{x}{\delta})\right]\,\mathrm{d}x$$

$$=\int_\Omega v^{(0)}(x)\,\mathrm{d}x$$

choose $v^{(0)}$ arbitrary, $v^{(1)}=0$

$$\int_Y v^{(0)}(x)\,\mathrm{d}x=\int_\Omega A(\frac{x}{\delta})\left[\partial_x u_0(x)+\partial_x u_0(x)\partial_y\chi(\frac{x}{\delta})\right]\partial_x v^{(0)}(x)\,\mathrm{d}x$$

$$\to\int_\Omega \underbrace{\int_Y A(y)(1+\partial_y\chi(y))\,\mathrm{d}y}_{A_0}\;\partial_x u_0(x)\partial_x v^{(0)}(x)\,\mathrm{d}x$$

Homogenization result

Theorem

Let A_δ *be locally* δ*-periodic,* $A \in \mathrm{L}^\infty(\Omega; \mathrm{C}_\#(Y))^{d\times d}$ *uniformly coercive and bounded. The sequence* $(u_\delta)_{\delta>0}$ *converges weakly in* $\mathrm{H}^1_0(\Omega)$ *to* u_0 *and* (∇u_δ) *converges to* $\nabla u_0 + \nabla_y u^{(1)}$ *in the two-scale sense, where* $(u_0, u^{(1)}) \in \mathrm{H}^1_0(\Omega) \times \mathrm{L}^2(\Omega; \mathrm{H}^1_\#(Y))$ *is the unique solution of*

$$\int_\Omega \int_Y A(x,y)(\nabla u_0(x) + \nabla_y u^{(1)}(x,y)) \cdot (\nabla v_0(x) + \nabla_y v^{(1)}(x,y)) \,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_\Omega f(x) v_0(x)\,\mathrm{d}x \quad \textit{for all } (v_0, v^{(1)}) \in \mathrm{H}^1_0(\Omega) \times \mathrm{L}^2(\Omega; \mathrm{H}^1_\#(Y))$$

The example

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Problem

In $\Omega = (0,1)$ *consider*

$$-\frac{d}{dx}\left(A^\delta(x)\frac{d}{dx}u^\delta(x)\right) = 1, \qquad u^\delta(0) = u^\delta(1) = 0$$

$$A^\delta(x) = A\left(\frac{x}{\delta}\right) = \left(2 - \cos\left(2\pi\frac{x}{\delta}\right)\right)^{-1} \qquad \textit{for } \delta = 2^{-k} \quad k \in \mathbb{N}$$

The cell corrector is given by $\chi(y) = -\frac{\sin(2\pi y)}{4\pi}$

$$A_0 = \int_Y A(y)(1 + \partial_y \chi(y))\,\mathrm{d}y = \frac{1}{2}$$

Back to asymptotic expansion

Solution of homogeneous system

$$-\frac{1}{2}\partial_x\partial_x u_0(x) = 1, \quad u_0(0) = u_0(1) = 0$$

is $u_0(x) = x - x^2$ and with the first corrector we get

$$\begin{aligned} u_\delta(x) &= u_0(x) + \delta u^{(1)}(x, \frac{x}{\delta}) = u_0(x) + \delta \partial_x u_0(x) \chi(\frac{x}{\delta}) \\ &= x - x^2 - \delta\,(1-2x)\,\frac{\sin(2\pi\frac{x}{\delta})}{4\pi} \\ &= x - x^2 - \delta \left(\frac{1}{4\pi} \sin(2\pi\frac{x}{\delta}) + \frac{x}{2\pi} \sin(2\pi\frac{x}{\delta}) \right) \end{aligned}$$

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1D example

Maxwell's equations

Implementation

Homogenization of Maxwell's equations

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Problem

$$\varepsilon^\delta(x)\partial_t \mathbf{E}^\delta(t,x) = \operatorname{curl} \mathbf{H}^\delta(t,x) - j(t,x)$$
$$\mu^\delta(x)\partial_t \mathbf{H}^\delta(t,x) = -\operatorname{curl} \mathbf{E}^\delta(t,x)$$

Ansatz

$$\mathbf{E}^\delta(t,x) = \mathbf{E}^{(0)}(t,x,\frac{x}{\delta}) + \delta \mathbf{E}^{(1)}(t,x,\frac{x}{\delta})$$
$$\mathbf{H}^\delta(t,x) = \mathbf{H}^{(0)}(t,x,\frac{x}{\delta}) + \delta \mathbf{H}^{(1)}(t,x,\frac{x}{\delta})$$

Homogeneous Maxwell system

$$\begin{aligned}\varepsilon^{\text{eff}}(x)\partial_t \mathbf{E}^{\text{eff}}(t,x) &= \operatorname{curl}\mathbf{H}^{\text{eff}}(t,x) - j(t,x)\\ \mu^{\text{eff}}(x)\partial_t \mathbf{H}^{\text{eff}}(t,x) &= -\operatorname{curl}\mathbf{E}^{\text{eff}}(t,x)\end{aligned}$$

where the homogeneous parameters $\alpha = \varepsilon, \mu$ are given as

$$\alpha^{\text{eff}}(x)_k := \int_Y \alpha(x,y)(\mathrm{e}_k + \nabla_y \chi_k^\alpha(x,y))\,\mathrm{d}y$$

and the cell correctors χ_α are solutions of the cell problems

$$\int_Y \alpha(x,y)(\mathrm{e}_k + \nabla_y \chi_k^\alpha(x,y)) \cdot \nabla_y v(y)\,\mathrm{d}y = 0 \quad \text{for all } v \in \mathrm{H}^1_\#(Y)$$

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Polarization model

Domain $\Omega \subseteq \mathbb{R}^3$, $t \in (0,T)$, ε,μ bounded, uniformly positive definite

$$
\begin{aligned}
\partial_t \mathbf{D} &= \operatorname{curl} \mathbf{H} - \mathbf{j} && && (0,T) \times \Omega \\
\mu \partial_t \mathbf{H} &= -\operatorname{curl} \mathbf{E} && && (0,T) \times \Omega \\
\mathbf{n} \times \mathbf{E} &= \mathbf{0} && && (0,T) \times \partial\Omega \\
\mathbf{E}(0) &= \mathbf{E}_0, && \mathbf{H}(0) = \mathbf{H}_0 && \Omega \\
\operatorname{div} \mathbf{D}(0) &= \varrho(0), && \operatorname{div}(\mu \mathbf{H}_0)) = \mathbf{0}, && \mathbf{n} \cdot (\mu \mathbf{H}_0) = \mathbf{0} \qquad \Omega
\end{aligned}
$$

$$
\mathbf{D}(t,x) = \varepsilon(x)\mathbf{E}(t,x)
$$

Polarization model

Domain $\Omega \subseteq \mathbb{R}^3$, $t \in (0,T)$, ε,μ bounded, uniformly positive definite

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\mathbf{n}\times\mathbf{E} &= \mathbf{0} && && (0,T)\times\partial\Omega\\
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$$

$$
\mathbf{D}(t,x) = \varepsilon(x)\mathbf{E}(t,x) + \mathbf{P}(t,x)
$$

Polarization model

Domain $\Omega \subseteq \mathbb{R}^3$, $t \in (0,T)$, ε,μ bounded, uniformly positive definite

$$\begin{aligned}
\partial_t \mathbf{D} &= \operatorname{curl} \mathbf{H} - \mathbf{j} && && (0,T)\times\Omega \\
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\end{aligned}$$

$$\mathbf{D}(t,x) = \varepsilon(x)\mathbf{E}(t,x) + \mathbf{P}(t,x)$$

Debye-Model for dielectric material

$$\nu(x)^{-1}\partial_t \mathbf{P}(t,x) = \tau(x)^{-1}\mathbf{E}(t,x) - \tau(x)^{-1}\nu(x)^{-1}\mathbf{P}(t,x)$$

$$\mathbf{P}(0,x) = \mathbf{P}_0(x)$$

ν,τ bounded, uniformly positive definite

Abstarct Cauchy system

$$\mathbf{u}(t,x) = \begin{pmatrix} \mathbf{E}(t,x) \\ \mathbf{H}(t,x) \\ \mathbf{P}(t,x) \end{pmatrix}$$

solves

$$\begin{aligned} \mathbf{M}\partial_t \mathbf{u}(t) &= (\mathbf{A} + \mathbf{B})\,\mathbf{u}(t) + \mathbf{g} \\ \mathbf{u}(0) &= \mathbf{u}_0 \end{aligned}$$

$\mathbf{M}$ symmetric positive definite and bounded

$$\mathbf{M} = \begin{pmatrix} \varepsilon & & \\ & \mu & \\ & & \nu^{-1} \end{pmatrix}, \mathbf{A} = \begin{pmatrix} & \operatorname{curl} & \\ -\operatorname{curl} & & \\ & & \end{pmatrix}, \mathbf{B} = \tau^{-1} \begin{pmatrix} -\nu & & \mathcal{I}_3 \\ & & \\ \mathcal{I}_3 & & -\nu^{-1} \end{pmatrix}$$

Then $\mathbf{S} := \mathbf{M}^{-1}(-\mathbf{A} - \mathbf{B})$ is a maximal monotone operator

Well posedness of Cauchy problem

$$X := L^2(\Omega)^3 \times L^2(\Omega)^3 \times L^2(\Omega)^3, \qquad (\phi,\psi)_X = (\mathbf{M}\phi,\psi)$$

$$\mathcal{D}(\mathbf{S}) := H_0(\mathrm{curl},\Omega) \times H(\mathrm{curl},\Omega) \times L^2(\Omega)^3$$

Theorem (Hipp, Hochbruck, Stohrer)

For $\mathbf{u}_0 \in \mathcal{D}(\mathbf{S})$ and $\mathbf{g} \in C(0,T;\mathcal{D}(\mathbf{S}))$ or $\mathbf{g} \in C^1(0,T;X)$ the abstract Cauchy problem has a unique solution $\mathbf{u} \in C^1(0,T;X) \cap C(0,T;\mathcal{D}(\mathbf{S}))$ and there exist a constant $C > 0$ with

$$\|\mathbf{u}(t)\|_X \le C \left(\|\mathbf{u}_0\|_X + t\,\|\mathbf{g}\|_{L^\infty(0,t;X)} \right)$$

David Hipp, Marlis Hochbruck, and Christian Stohrer. "Unified error analysis for non-conforming space discretizations of wave-type equations."

Discrete Model

$\mathbf{E} \in \mathrm{H}_0(\mathrm{curl}, \Omega)$, $\mathbf{H} \in \mathrm{H}(\mathrm{curl}, \Omega)$, $\mathbf{P} \in \mathrm{L}^2(\Omega)$ such that

$$\begin{pmatrix} \varepsilon & & \\ & \mu & \\ & & \nu^{-1} \end{pmatrix} \partial_t \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \mathbf{P} \end{pmatrix} = \begin{pmatrix} -\tau^{-1}\nu & \mathrm{curl} & \tau^{-1} \\ -\,\mathrm{curl} & & \\ \tau^{-1} & & -\tau^{-1}\nu^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \\ \mathbf{P} \end{pmatrix} + \begin{pmatrix} -\mathbf{j} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\rightsquigarrow \mathbf{M}_H \partial_t \mathbf{u}_H = (\mathbf{A}_H + \mathbf{B}_H)\, \mathbf{u}_H + \mathbf{j}_H$$

where

$$\mathbf{M}_H := \begin{pmatrix} \mathbf{M}_\varepsilon & & \\ & \mathbf{M}_\mu & \\ & & \mathbf{M}_{\nu^{-1}} \end{pmatrix}, \quad \mathbf{A}_H + \mathbf{B}_H := \begin{pmatrix} -\mathbf{M}_{\tau^{-1}\nu} & \mathbf{C} & \mathbf{M}_{\tau^{-1}} \\ -\mathbf{C} & & \\ \mathbf{M}_{\tau^{-1}} & & -\mathbf{M}_{\tau^{-1}\nu^{-1}} \end{pmatrix}$$

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$\mathbf{E} \in \mathrm{H}_0(\mathrm{curl}, \Omega)$, $\mathbf{H} \in \mathrm{H}(\mathrm{curl}, \Omega)$, $\mathbf{P} \in \mathrm{L}^2(\Omega)$ such that

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Thank you for your attention!