Karlsruhe Institute of Technology

A unified error analysis for spatial discretizations of wave-type equations

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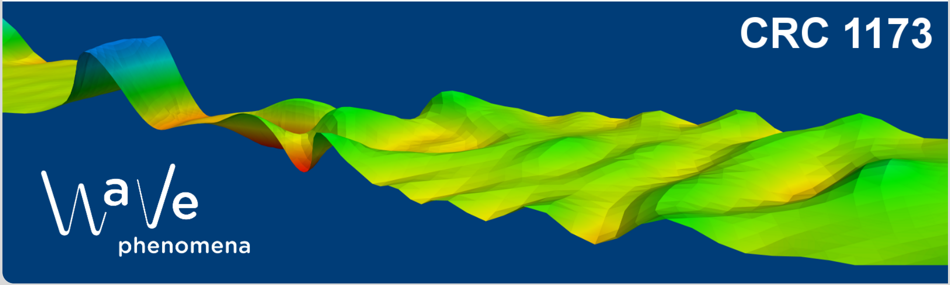
joint work with Marlis Hochbruck and Christian Stohrer

AG Numerik



Dynamic boundary conditions

- $\Omega \subset \mathbb{R}^d$ bounded and open domain with Lipschitz-boundary $\Gamma := \partial\Omega$

$$u_{tt}(t,\mathrm{x}) - \Delta u(t,\mathrm{x}) = 0, \quad (t,\mathrm{x}) \in [0,T] \times \Omega \qquad + \text{ ivs } + \text{ bcs}$$

Definition: Dynamic boundary conditions are differential or evolution equations on the boundary.

Model problem

kinetic boundary conditions

$$u_{tt} - \Delta_\Gamma u = -\partial_n u \qquad \text{on } \Gamma$$

acoustic boundary conditions

$$m\delta_{tt} + d\delta_t + k\delta = -u_t \qquad \text{on } \Gamma$$

$$\delta_t = \partial_n u \qquad \text{on } \Gamma$$

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Goal: Convergence rates for finite element space discretizations

Analysis of wave eq with kinetic bcs

$$
\begin{aligned}
u_{tt} - \Delta u &= 0 && \text{in } \Omega \\
u_{tt} + u - \Delta_\Gamma u &= -\partial_n u && \text{on } \Gamma \in C^2
\end{aligned}
$$

Variational formulation: find $u\colon (0,T) \to V$ s.t.

$$
m(u''(t), \varphi) + a(u(t), \varphi) = 0 \qquad \forall \varphi \in V,
$$

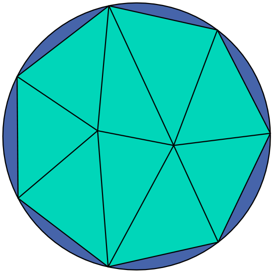
where

$$
\begin{aligned}
V &= \{ v \in H^1(\Omega) \mid \gamma(v) \in H^1(\Gamma) \} \\
m(v, \varphi) &= \int_\Omega v\varphi \,\mathrm{d}x + \int_\Gamma v\varphi \,\mathrm{d}s, \\
a(v, \varphi) &= \int_\Omega \nabla v \cdot \nabla \varphi \,\mathrm{d}x + \int_\Gamma v\varphi + \nabla_\Gamma v \cdot \nabla_\Gamma \varphi \,\mathrm{d}s.
\end{aligned}
$$

Non-conforming finite elements

- boundary conditions with Δ_Γ imposed on smooth Γ (e.g. C^2)
- triangulations leads to $\Omega_h \approx \Omega$
- discretization is non-conforming since

$$V_h \not\subset V, \qquad m_h \neq m, \qquad a_h \neq a$$



$$a_h(u_h, \varphi_h) = \int_{\Omega_h} \nabla u_h \cdot \nabla \varphi_h \,\mathrm{dx} + \int_{\Gamma_h} u_h \varphi_h + \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} \varphi_h \,\mathrm{d}s$$

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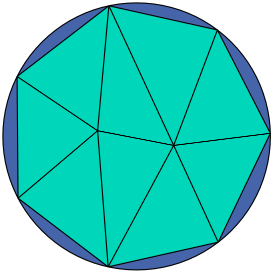
Idea: use pw smooth homeomorphism

$$G_h \colon \Omega_h \to \Omega$$

from [Elliott, Ranner '13] to define

$$u_h^\ell(\mathrm{x}) := u_h\big(G_h^{-1}(\mathrm{x})\big), \qquad \mathrm{x} \in \Omega$$



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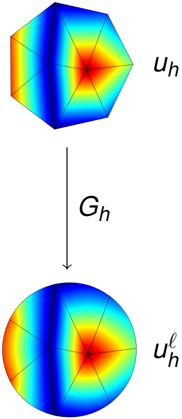

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Challenges

Goal: convergence rates for finite element discretizations

Strategies:

1. use general from literature ✗   (non-conforming FEs)
2. develop error analysis by using ideas from related situation (✗) (multiple pdes $\rightsquigarrow$ repetitive work)

My fear of large numbers

wave eq with Robin bcs

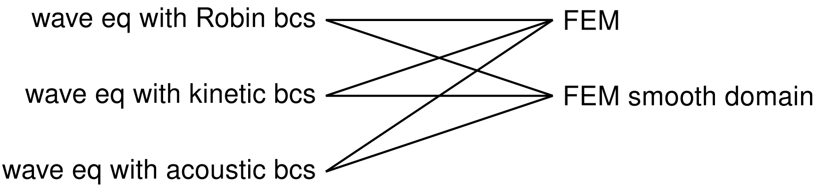
wave eq with kinetic bcs

wave eq with acoustic bcs

FEM

FEM smooth domain

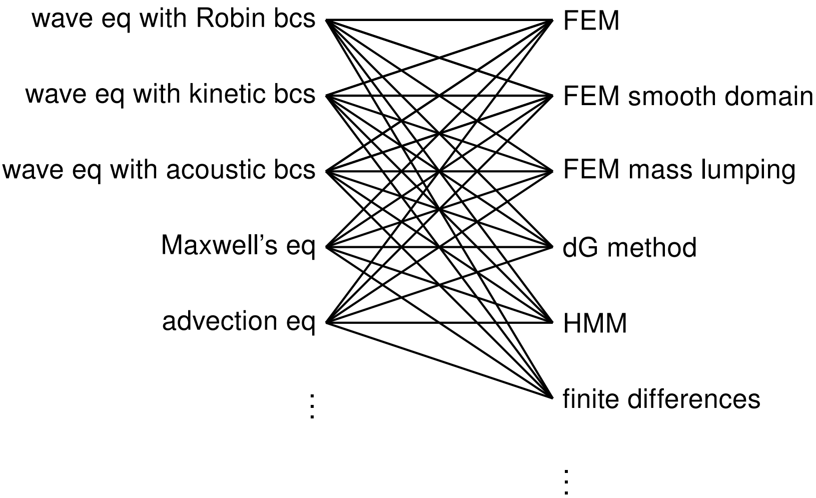
My fear of large numbers



My fear of large numbers

wave eq with Robin bcs	FEM
wave eq with kinetic bcs	FEM smooth domain
wave eq with acoustic bcs	FEM mass lumping
Maxwell's eq	dG method
advection eq	HMM
⋮	finite differences
	⋮

My fear of large numbers



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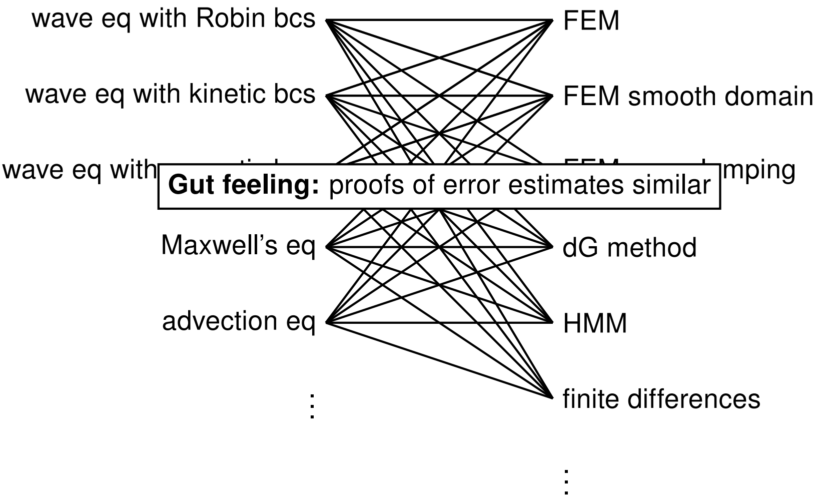
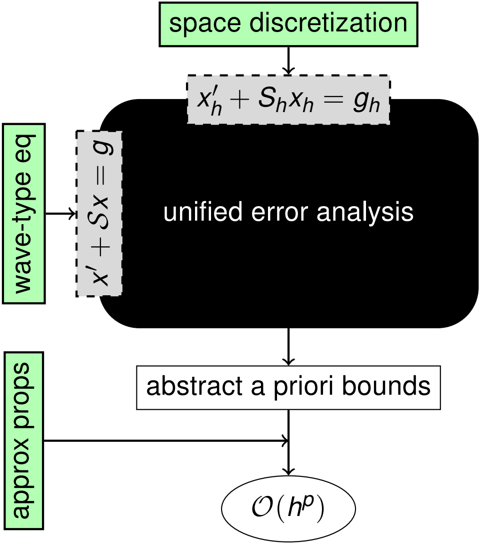


Gut feeling: proofs of error estimates similar

Modular approach



Abstract non-conforming space discretizations

$$\boxed{x'(t) + \mathcal{S}x(t) = g(t)}$$

$$\begin{tikzcd}
X \arrow[d, dashed, "Q_h^*"] & Q_h(X_h) \arrow[l, hook'] \\
X_h \arrow[ur, "Q_h"'] &
\end{tikzcd}$$

$$\boxed{x_h'(t) + S_h x_h(t) = g_h(t)}$$

Interpretation:

- "$X = L^2(\Omega)$"
- "$X_h =$ FEs in Ω_h"
- "$Q_h x_h = x_h^\ell$"

Abstract non-conforming space discretizations

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Interpretation:

- "$X = L^2(\Omega)$"
- "X_h = FEs in Ω_h"
- "$Q_h x_h = x_h^\ell$"
- "J_h = interpolation"
- "$Z = H^2(\Omega)$"

General error bound

Theorem

$$\|Q_h x_h(t) - x(t)\|_X \leq C(1+t)(E_1 + E_2 + E_3)$$

where

$$E_1 = \|x_h^0 - J_h x^0\|_{X_h} + \|g_h - Q_h^* g\|_{L^\infty(X_h)}$$

$$E_2 = \|(Q_h^* - J_h)x'\|_{L^\infty(X_h)} + \|(Q_h^* \mathcal{S} - S_h J_h)x\|_{L^\infty(X_h)}$$

$$E_3 = \|(\mathrm{I} - Q_h J_h)x\|_{L^\infty(X)}$$

$$\boxed{x'(t) + \mathcal{S}x(t) = g(t)}$$

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$$\boxed{x_h'(t) + S_h x_h(t) = g_h(t)}$$

Idea of proof. split error into

$$Q_h x_h - x = Q_h e_h + (Q_h J_h x - x), \qquad e_h := x_h - J_h x$$

and use discrete stability in

$$e_h{}' + S_h e_h = g_h - Q_h^* g + (Q_h^* \mathcal{S} - S_h J_h)x + (Q_h^* - J_h)x'$$

Towards the complete estimate

Lemma

$$\|(Q_h^* - J_h)x'\|_{L^\infty(X_h)} \leq C\Big(\|(\mathrm{I} - Q_h J_h)x'\|_{L^\infty(X)} + \|\Delta p(J_h x')\|_{L^\infty(X_h^*)}\Big)$$

where

$$\Delta p(z_h, y_h) := p(Q_h z_h, Q_h y_h) - p_h(z_h, y_h)$$

Proof. Use

- p inner product on X
- p_h inner product on X_h
- $\|z_h\|_{X_h} = \max_{\|y_h\|_{X_h}=1} p_h(z_h, y_h)$
- $\|\cdot\|_{X_h} \sim \|Q_h \cdot\|_X$

Error bound for symmetric hyperbolic systems

For symmetric hyperbolic systems use $\mathcal{S} \in \mathcal{L}(Y, X)$

$$E_2 = \|(Q_h^* - J_h)x'\|_{L^\infty(X_h)} + \|(Q_h^*\mathcal{S} - S_h J_h)x\|_{L^\infty(X_h)}$$
$$\leq C\Big(\|(\mathrm{I} - Q_h J_h)x'\|_{L^\infty(X)} + \|\Delta p(J_h x')\|_{L^\infty(X_h^*)}$$
$$+ \|(\mathrm{I} - Q_h J_h)x\|_{L^\infty(Y)} + \|\Delta s(J_h x)\|_{L^\infty(X_h^*)}\Big)$$

Theorem

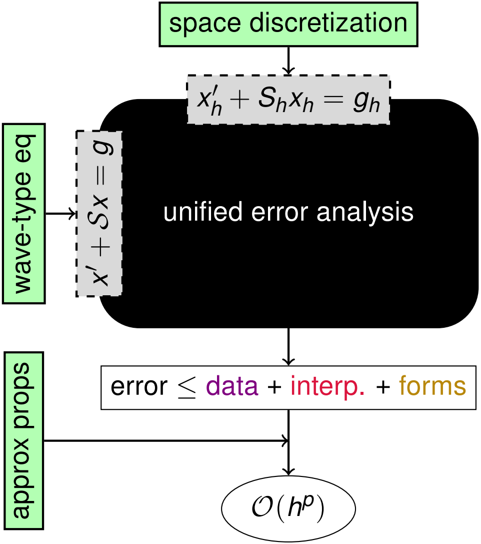
$$\|Q_h x_h(t) - x(t)\|_X$$
$$\leq C(1+t)\big(\text{error in data} + \text{interp. of } x, x' + \text{"}p - p_h\text{"} + \text{"}s - s_h\text{"}\big)$$

Proof.

- use general error bound
- choose $J_h = I_h$

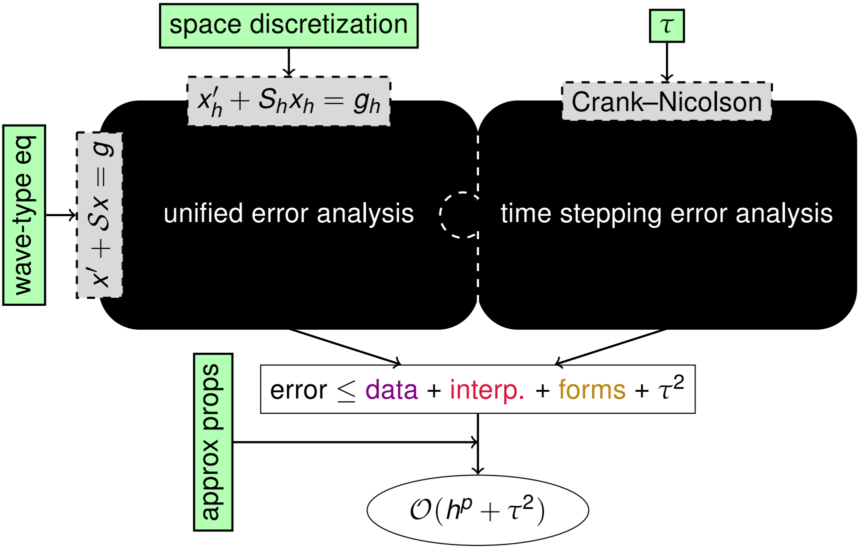
$E_1 = \|x_h^0 - J_h x^0\|_{X_h} + \|g_h - Q_h^* g\|_{L^\infty(X_h)}$ and $E_3 = \|(\mathrm{I} - Q_h J_h)x\|_{L^\infty(X)}$

Modularity



space discretization
$x_h' + S_h x_h = g_h$
wave-type eq
$x' + \mathcal{S}x = g$
unified error analysis
error $\leq$ data + interp. + forms
approx props
$\mathcal{O}(h^p)$

Modularity and full discretization



Exponential quadrature

Variation-of-constants formula:

$$x(t_{n+1}) = e^{-\tau\mathcal{S}} x(t_n) + \int_0^\tau e^{-(\tau-\theta)\mathcal{S}} g(t_n+\theta)\,\mathrm{d}\theta, \qquad t_n = n\tau$$

Idea: for $t \in [t_n, t_{n+1}]$ use

$$g(t) \approx \sum_{i=1}^{s} g(t_n + c_i\tau)\ell_i(t), \qquad \ell_i(t) = \prod_{\substack{m=1\\ m\neq i}}^{s} \frac{t-c_m}{c_i-c_m}$$

Exponential quadrature rule

$$x^{n+1} = e^{-\tau\mathcal{S}} x^n + \sum_{i=1}^{s} b_i(-\tau\mathcal{S}) g(t_n + c_i\tau), \qquad n \geq 0,$$

where $b_i(-\tau\mathcal{S}) := \int_0^\tau e^{-(\tau-\theta)\mathcal{S}} \ell_i(\theta)\,\mathrm{d}\theta$

- order q conditions $\iff$ exact integration for $g \in \mathcal{P}_q$

Full discretization

Fully discrete scheme:

$$x_h^{n+1} = e^{-\tau S_h} x_h^n + \sum_{i=1}^{s} b_i(-\tau S_h) g_h(t_n + c_i \tau), \qquad n \geq 0$$

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Error analysis:

- consider $e_h^n := x_h^n - J_h x(t_n)$
- subtract $J_h x(t_{n+1})$ from scheme

$$\begin{aligned} e_h^{n+1} &= e^{-\tau S_h} e_h^n + e^{-\tau S_h} J_h x(t_n) + \sum_{i=1}^{s} b_i(-\tau S_h) g_h(t_n + c_i \tau) - J_h x(t_{n+1}) \\ &= e^{-\tau S_h} e_h^n + \Delta_n \end{aligned}$$

- with stability of exponential schemes

$$\| e_h^n \|_{X_h} \leq C \Big(\| e_h^0 \|_{X_h} + t_n \sup_{k=0,\dots,n} \tau^{-1} \| \Delta_k \|_{X_h} \Big)$$

Full error analysis

- for schemes of order q

$$\Delta_n = e^{-\tau S_h} J_h x(t_n) + \sum_{i=1}^{s} b_i(-\tau S_h) g_h(t_n + c_i\tau) - J_h x(t_{n+1})$$
$$= \widetilde{x}_h(\tau) - J_h x(t_{n+1})$$

where

$$\widetilde{x}_h' + S_h \widetilde{x}_h = \mathcal{I}_q g_h, \qquad \widetilde{x}_h(0) = J_h x(t_n)$$

- from general error bound

$$\begin{aligned} \|\Delta_n\|_{X_h} &= \|\widetilde{x}_h(\tau) - J_h x(t_{n+1})\|_{X_h} \\ &\leq C\tau \left(\|\mathcal{I}_q g_h - J_h g\|_{L^\infty(X_h)} + \text{spatial error} \right) \\ &\leq C\tau \left(\tau^q + \text{spatial error} \right) \end{aligned}$$

Error bound

Theorem

$$\|Q_h x_h^n - x(t_n)\|_X \leq C(1+t_n)\Big(\tau^q + \textit{spatial error}\Big)$$

Proof. Split error

$$\begin{aligned}\|Q_h x_h^n - x(t_n)\|_X &\leq \|Q_h(x_h^n - J_h x(t_n))\|_X \qquad + \|(\mathrm{I} - Q_h J_h)x(t_n)\|_X \\ &\leq C\|e_h^n\|_X \qquad\qquad\qquad\quad + \|(\mathrm{I} - Q_h J_h)x(t_n)\|_X\end{aligned}$$

and use

$$\|e_h^n\|_{X_h} \leq C\Big(\|e_h^0\|_{X_h} + t_n \sup_{k=0,\dots,n} \quad \tau^{-1}\|\Delta_k\|_{X_h} \qquad\Big)$$

Error bound

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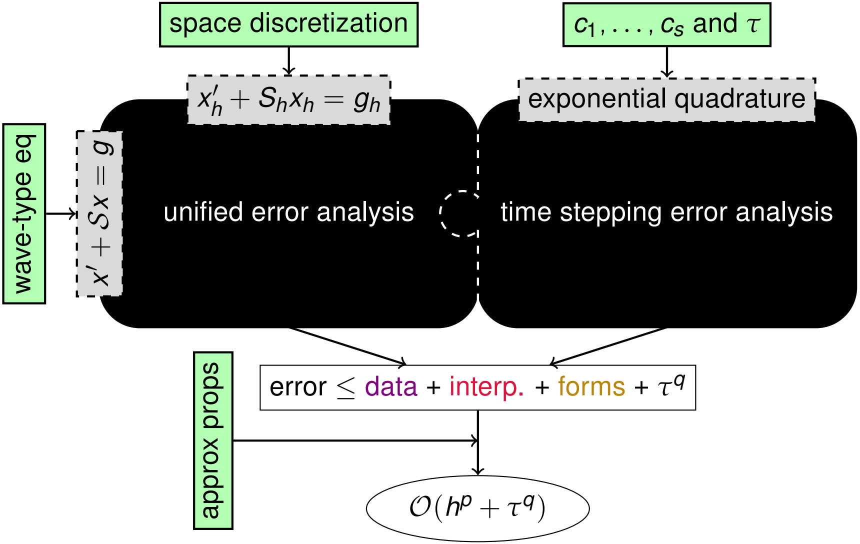
and use

$$\|e_h^n\|_{X_h} \le C\Big(\|e_h^0\|_{X_h} + t_n \sup_{k=0,\dots,n} (\tau^q + \text{spatial error})\Big)$$

Modularity and full discretization



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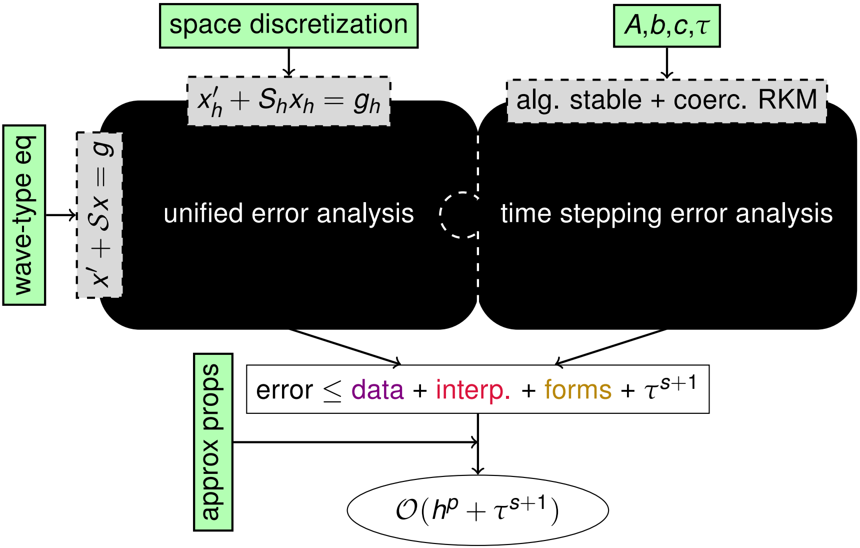

Benefits of the unified error analysis

- formalized derivation of (new) convergence rates
- error bounds for full discretization with
  - algrebaically stable and coercive Runge–Kutta methods
  - exponential quadrature rules
- modularization s.t. extensions have wide range of applications

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