A Convergence Analysis of the Peaceman-Rachford Scheme for Semilinear Evolution Equations

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Problem Set-up

Semilinear evolution equation

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where

- A is linear
- F is possibly nonlinear

Example Problem - Allan Cahn equation

Reaction-diffusion equation:

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(last term from potential: $\frac{1}{4}(1 - u^2)^2$)

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Model of of certain phase separation processes

Peaceman-Rachford Scheme

Compute one time step by applying the operator

$$S = \underbrace{(I - \frac{\tau}{2}F)^{-1}}\ (I + \frac{\tau}{2}A)\ (I - \frac{\tau}{2}A)^{-1}\ \underbrace{(I + \frac{\tau}{2}F)}$$

to last solution.

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Peaceman-Rachford Scheme

Compute one time step by applying the operator

$$S = \underbrace{(I - \frac{\tau}{2}F)^{-1}}_{\text{imp. Euler}} (I + \frac{\tau}{2}A)\,(I - \frac{\tau}{2}A)^{-1} \underbrace{(I + \frac{\tau}{2}F)}_{\text{exp. Euler}}$$

to last solution.

Yields approximation $u(n\tau) \approx S^n u_0$

Advantage: A and F separated

Goal

Additional Regularity of solution

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$$\Downarrow$$

Error bound depending on τ

Existence and Uniqueness of Solutions

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Main assumption:
Operators A, F and $A + F$ are maximal dissipative.

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Maximal Dissipative

A operator $G : D(G) \mapsto H$ (real HS) is called maximal dissipative if there exists $M[G] \geq 0$ such that

- $R(I - \tau G) = H \qquad$ for all $\tau > 0$ with $\tau M[G] < 1$
- $\big(Gu - Gv, u - v\big) \leq M[G]\|u - v\|^2 \qquad$ for all $u, v \in D(G)$

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Operators A, F and $A + F$ are maximal dissipative. Then:

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- initial value u_0 (regularity)
- $A + F \to$ exact solution
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Main assumption:
Operators A, F and $A + F$ are maximal dissipative. Then:

- numerical solution exists
- theory of nonlinear contractive semigroups $\to$ exact solution

Main Result

Under certain further assumptions we have the estimate

$$\|u(nh) - S^n u_0\| \leq \frac{5}{2}\tau^p \; e^{3/2T(M[A]+M[F])} \sum_{j=0}^{p} \|A^{p-j} u^{(j+1)}\|_{L^1(0,T,H)}$$

for $n\tau \leq T$ where either $p = 1$ or $p = 2$.

Strategy of the Proof

Step 1: stability

Recall : $S = (I - \frac{\tau}{2}F)^{-1}\ (I + \frac{\tau}{2}A)\ (I - \frac{\tau}{2}A)^{-1}\ (I + \frac{\tau}{2}F)$

Applying S n times:

$$(I + \frac{\tau}{2}F)\ (I - \frac{\tau}{2}F)^{-1} \text{ and } (I + \frac{\tau}{2}A)\ (I - \frac{\tau}{2}A)^{-1}$$

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Obtain (exponential) Lipschitz constants

$$e^{3/2\tau M[A]} \text{ and } e^{3/2\tau M[F]}$$

Strategy of the Proof

Step 2: local error

Regularity assumptions $\rightarrow$ fundamental theorem of calculus

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Split the remaining term:

- quadrature error
- splitting error

Possible Extensions

Add another operator

Consider

$$u'(t) = (A + B + F)u(t), \quad u(0) = u_0$$

where B might satisfy some stronger assumptions.

$$\tilde{S} = (I - \frac{\tau}{2}F)^{-1} (I + \frac{\tau}{2}A) (I - \frac{\tau}{2}A)^{-1} (I + \frac{\tau}{2}B) (I - \frac{\tau}{2}B)^{-1} (I + \frac{\tau}{2}F) \ ?$$

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Consider complex Hilbert space

Consider Schrödinger equation $\rightarrow i\Delta$

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$$\Downarrow$$

change condition:

$$Re\big(Gu - Gv, u - v\big) \leq M[G]\|u - v\|^2 \qquad \text{for all } u, v \in D(G)$$

Numerical Example

Numerical Example - Convergence of Lie

Numerical Example - Convergence of Peacemann-Rachford