A Convergence Analysis of the Peaceman-Rachford Scheme for Semilinear Evolution Equations

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Problem Set-up

Semilinear evolution equation

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where

- A is linear
- F is possibly nonlinear

Example Problem - Allan Cahn equation

Reaction-diffusion equation:

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(last term from potential: $\frac{1}{4}(1-u^2)^2$)

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Model of of certain phase separation processes

Peaceman-Rachford Scheme

Compute one time step by applying the operator

$$S = \underbrace{(I - \frac{\tau}{2}F)^{-1}}_{} (I + \frac{\tau}{2}A) (I - \frac{\tau}{2}A)^{-1} \underbrace{(I + \frac{\tau}{2}F)}_{}$$

to last solution.

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Advantage: A and F separated

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Error bound depending on au

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Main assumption:

Operators A, F and A + F are maximal dissipative.

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Maximal Dissipative

A operator $G: D(G) \mapsto H$ (real HS) is called maximal dissipative if there exists $M[G] \geq 0$ such that

- $\blacksquare R(I \tau G) = H$ for all $\tau > 0$ with $\tau M[G] < 1$
- $(Gu Gv, u v) \le M[G] ||u v||^2 \quad \text{for all } u, v \in D(G)$

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Main Result

Under certain further assumptions we have the estimate

$$||u(nh) - S^n u_0|| \le \frac{5}{2} \tau^p e^{3/2T(M[A]+M[F])} \sum_{j=0}^p ||A^{p-j} u^{(j+1)}||_{L^1(0,T,H)}$$

for $n\tau \leq T$ where either p=1 or p=2.

Step 1: stability

Recall :
$$S = (I - \frac{\tau}{2}F)^{-1} (I + \frac{\tau}{2}A) (I - \frac{\tau}{2}A)^{-1} (I + \frac{\tau}{2}F)$$

Applying *S n* times:

$$(I + \frac{\tau}{2}F) (I - \frac{\tau}{2}F)^{-1}$$
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Obtain (exponential) Lipschitz constants

$$e^{3/2\tau M[A]}$$
 and $e^{3/2\tau M[F]}$

Step 2: local error

Regularity assumptions \rightarrow fundamental theorem of calculus

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Split the remaining term:

- quadrature error
- splitting error

Possible Extensions

Add another operator

Consider

$$u'(t) = (A + B + F)u(t), \quad u(0) = u_0$$

where B might satisfy some stronger assumptions.

$$\tilde{S} = (I - \frac{\tau}{2}F)^{-1} \left(I + \frac{\tau}{2}A\right) \left(I - \frac{\tau}{2}A\right)^{-1} \left(I + \frac{\tau}{2}B\right) \left(I - \frac{\tau}{2}B\right)^{-1} \left(I + \frac{\tau}{2}F\right) ?$$

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Consider complex Hilbert space

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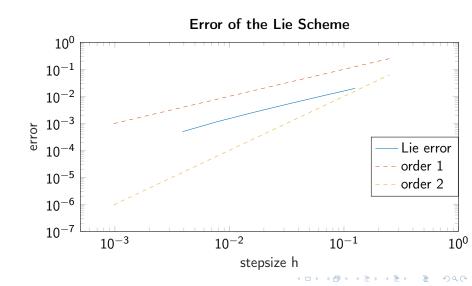
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change condition:

$$Re(Gu - Gv, u - v) \le M[G]||u - v||^2$$
 for all $u, v \in D(G)$

Numerical Example

Numerical Example - Convergence of Lie



Numerical Example - Convergence of Peacemann-Rachford

