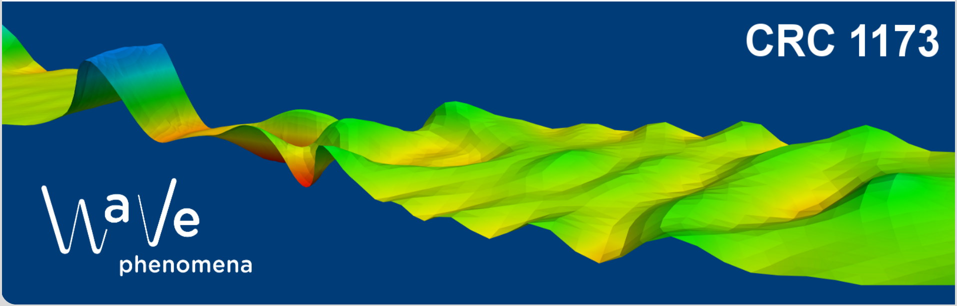
Karlsruhe Institute of Technology

Uniformly accurate methods for Klein-Gordon-Schrödinger and Klein-Gordon-Zahkarov systems

joint work with G. Kokkala and K. Schratz

Simon Baumstark | October 12, 2017



Outline

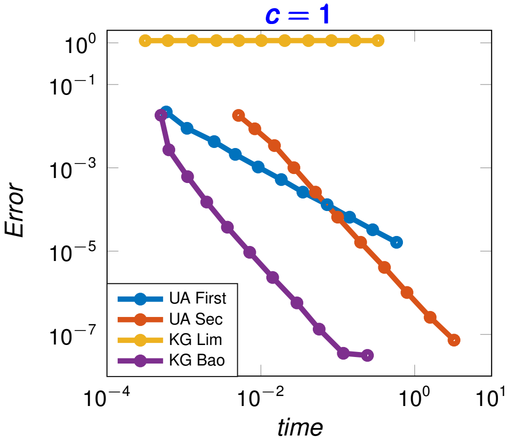
1. Update and introduction
2. Uniformly accurate scheme for the Klein-Gordon-Schrödinger system
3. Numerical experiments
4. Uniformly accurate scheme for the Klein-Gordon-Zakharov system
5. Outlook

Last year: UA scheme for KG equations

joint work with E. Faou and K. Schratz

Last year: UA scheme for KG equations

- **Work-precision plot:**



Simulation on $x \in [0, 2\pi]$, $t \in [0, 1]$, $\tau_{ref} \approx 10^{-6}$ and $M = 256$.

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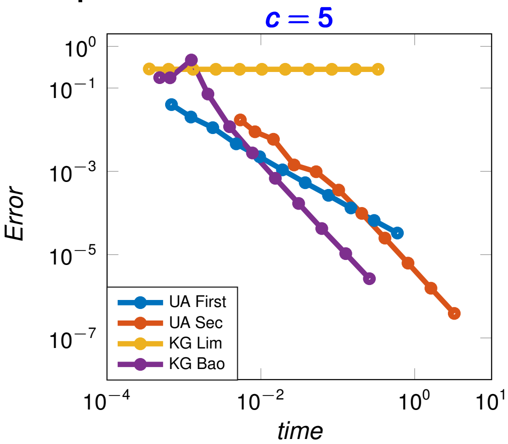

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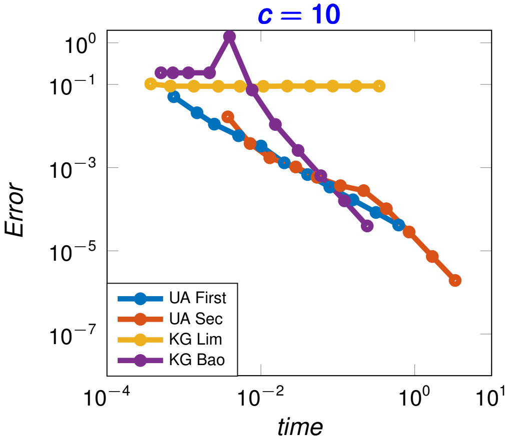

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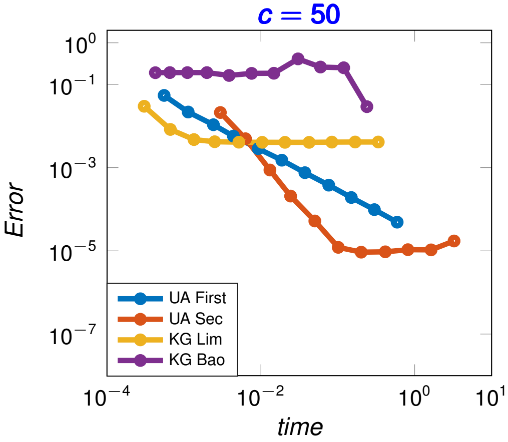

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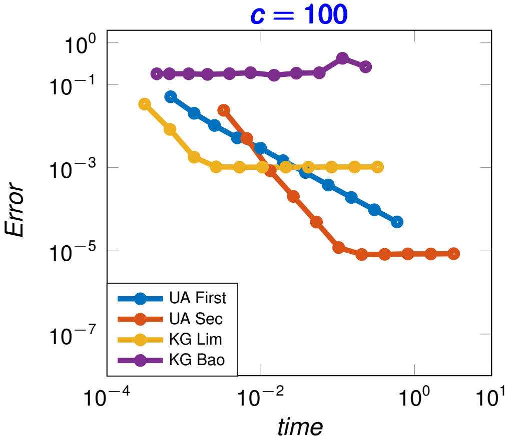

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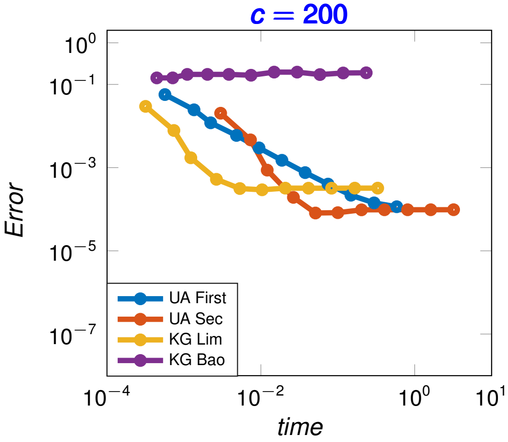

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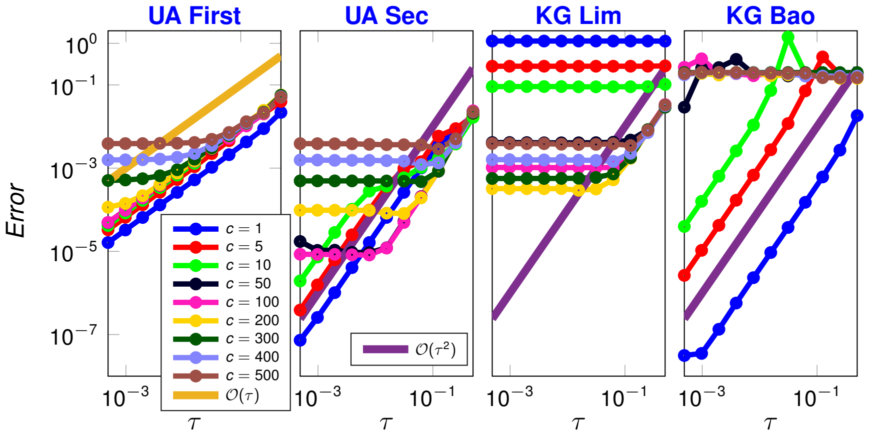

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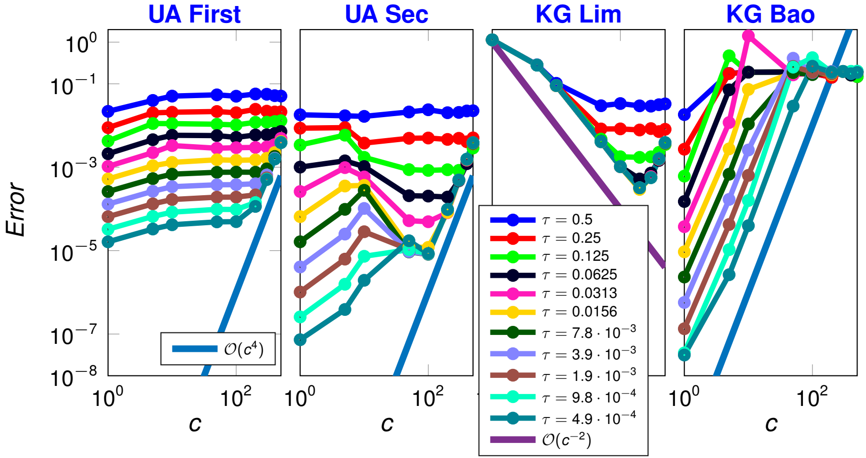

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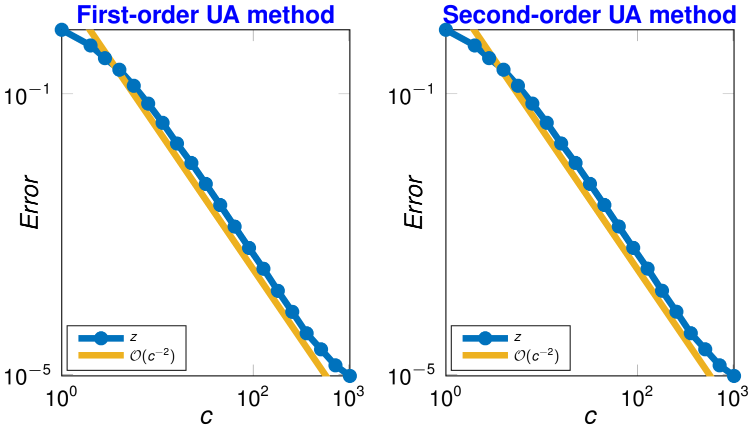

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i\partial_t n(t,x) + \Delta n(t,x) &= -n(t,x) z(t,x),
\end{aligned}
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with initial conditions

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z(0,x) = z_0(x), \quad \partial_t z(0,x) = c^2 z_1(x), \qquad n(0,x) = n_0(x),
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- **Numerical Challenge:**
  Highly oscillatory (non-relativistic) limit regime, i.e. $c \gg 1$.
- Goal: Search numerical approx. $z^n \approx z(t_n)$, $n^n \approx n(t_n)$ with $t_n = n\tau$.

Numerical methods

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1.) Gautschi-type method:

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Idea: Discretize Duhamel's formula (variation of constants formula).

Gautschi-type methods

- **KGS equation** $\left(\langle\nabla\rangle_c := \sqrt{-\Delta + c^2}\right)$**:**

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- **Duhamel's formula** $(\bullet := c\langle\nabla\rangle_c \tau)$:

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- Problem: large derivative $\partial_t z = \mathcal{O}(c^2)$.

Gautschi-type methods

- **Gautschi-type method applied to KGS system at $t_n = 0.6$:**

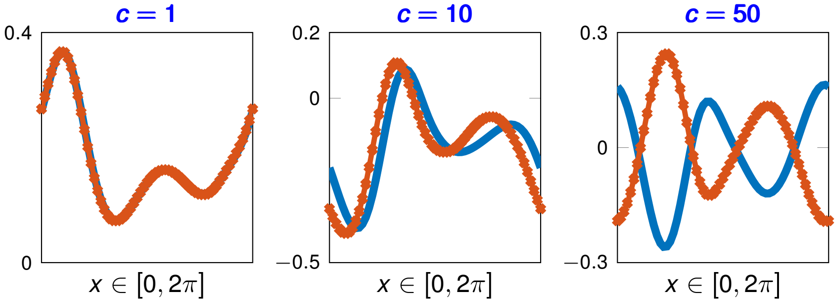

Figure: **blue line:** reference solution of z ($\tau_{\text{ref}} \approx 10^{-5}$),
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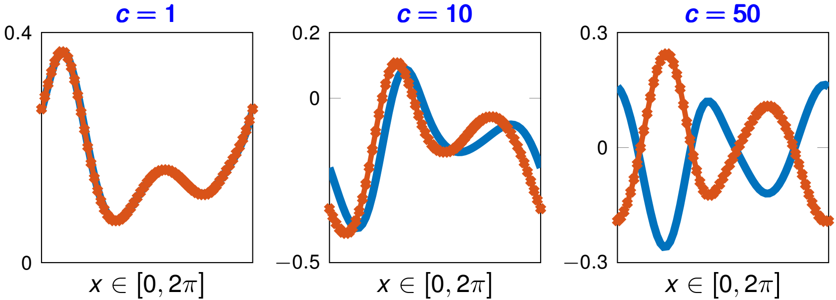

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Problem: Time step restriction for large $\boldsymbol{c}$!

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Multiscale expansion yields **decoupled free Schrödinger limit system**

$$\partial_t u_\infty(t,x) = -\tfrac{i}{2}\Delta u_\infty(t,x), \qquad u_\infty(0) = z_0 - iz_1,$$
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such that (for sufficiently smooth solutions)

$$z = \frac{1}{2}\left(u_\infty \mathrm{e}^{ic^2 t} + c.c.\right) + \mathcal{O}(c^{-2}).$$

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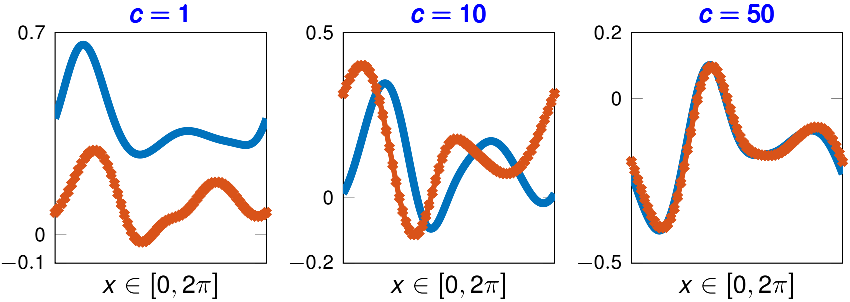
Advantage:

Non-oscillatory limit system can be solved exactly in Fourier space!

Limit approximation

- **Limit approximation vs. reference solution at $t_n = 0.7$:**



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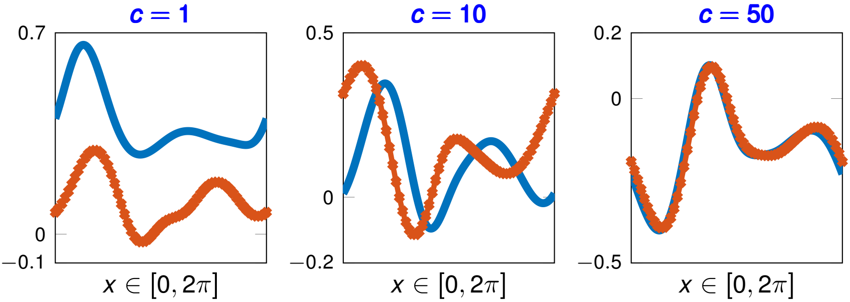

Figure: **blue line:** reference solution of z ($\tau_{\mathrm{ref}} \approx 10^{-5}$),
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Problem: Good approximation only for $\boldsymbol{c} \gg \mathbf{1}$!

Uniformly accurate scheme

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Other UA scheme:

Bao/Zhao 2013: only linear convergence rate $\mathcal{O}(\tau)$ for all $c \in [1, \infty)$

Uniformly accurate scheme

- KGS as first-order system in time with $z = \frac{1}{2}(u + \overline{u})$

$$i\partial_t u = -c\langle\nabla\rangle_c u + c\langle\nabla\rangle_c^{-1}|n|^2,$$
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- **Twisted variable** $u_*(t) = \mathrm{e}^{-ic^2 t} u(t)$ with $\mathcal{A}_c := c\langle\nabla\rangle_c - c^2$ satisfies

$$i\partial_t u_* = -\mathcal{A}_c u_* + c\langle\nabla\rangle_c^{-1}|n|^2,$$

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- $\mathcal{A}_c$ and $c\langle\nabla\rangle_c^{-1}$ are uniformly bounded in c:

$$\|\mathcal{A}_c u\|_r^2 \leq \frac{1}{2}\|u\|_{r+2}^2, \qquad \|c\langle\nabla\rangle_c^{-1} u\|_r \leq \|u\|_r.$$

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Advantage:

All operators uniformly bounded in c $\rightsquigarrow$ $\boldsymbol{\partial_t u_*}$ **uniformly bounded in** $\boldsymbol{c}$**!**

Uniformly accurate scheme

A first-order UA scheme

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A first-order UA scheme

- Duhamel's formula yields

$$u_*(t_n+\tau) = \mathrm{e}^{i\tau\mathcal{A}_c} u_*(t_n) - ic\langle\nabla\rangle_c^{-1} \mathrm{e}^{i\tau\mathcal{A}_c} \int_0^\tau \mathrm{e}^{-is\mathcal{A}_c} \; \mathrm{e}^{-ic^2(t_n+s)} \; |n(t_n+s)|^2 \; \mathrm{d}s,$$

$$n(t_n+\tau) = \mathrm{e}^{i\tau\Delta} n(t_n) + \frac{i}{2}\mathrm{e}^{i\tau\Delta} \int_0^\tau \mathrm{e}^{-is\Delta} \; \left[\mathrm{e}^{ic^2(t_n+s)} \; u_*(t_n+s) \; + c.c.\right] \; n(t_n+s) \; \mathrm{d}s.$$

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$$\mathrm{e}^{-is\mathcal{A}_c} = 1 + \mathcal{O}(s\Delta), \qquad u_*(t_n+s) = u_*(t_n) + \mathcal{O}\left(s\,\partial_t u_*\right)$$

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- We obtain:

$$u_*(t_n+\tau) = \mathrm{e}^{i\tau\mathcal{A}_c} u_*(t_n) - ic\langle\nabla\rangle_c^{-1}\mathrm{e}^{i\tau\mathcal{A}_c}\int_0^\tau \mathrm{e}^{-ic^2(t_n+s)}|n(t_n)|^2\mathrm{d}s + \mathcal{O}(\tau^2),$$

$$n(t_n+\tau) = \mathrm{e}^{i\tau\Delta} n(t_n) + \frac{i}{2}\mathrm{e}^{i\tau\Delta}\int_0^\tau \left[\mathrm{e}^{ic^2(t_n+s)}u_*(t_n) + c.c.\right] n(t_n)\mathrm{d}s + \mathcal{O}(\tau^2).$$

Uniformly accurate scheme

A first-order UA scheme

- Duhamel's formula yields

$$u_*(t_n+\tau) = \mathrm{e}^{i\tau\mathcal{A}_c} u_*(t_n) - ic\langle\nabla\rangle_c^{-1}\mathrm{e}^{i\tau\mathcal{A}_c}\int_0^\tau \underbrace{\mathrm{e}^{-is\mathcal{A}_c}}_{=1+s\cdot\text{“nice”}}\ \mathrm{e}^{-ic^2(t_n+s)}\ \underbrace{|n(t_n+s)|^2}_{=|n(t_n)|^2+s\cdot\text{“nice”}}\ \mathrm{d}s,$$

$$n(t_n+\tau) = \mathrm{e}^{i\tau\Delta} n(t_n) + \frac{i}{2}\mathrm{e}^{i\tau\Delta}\int_0^\tau \underbrace{\mathrm{e}^{-is\Delta}}_{=1+s\cdot\text{“nice”}} \left[\mathrm{e}^{ic^2(t_n+s)}\ \underbrace{u_*(t_n+s)}_{=u_*(t_n)+s\cdot\text{“nice”}}\ + c.c.\right] \underbrace{n(t_n+s)}_{=n(t_n)+s\cdot\text{“nice”}}\ \mathrm{d}s.$$

- We obtain:

$$u_*(t_n+\tau) = \mathrm{e}^{i\tau\mathcal{A}_c} u_*(t_n) - ic\langle\nabla\rangle_c^{-1}\mathrm{e}^{i\tau\mathcal{A}_c}\int_0^\tau \mathrm{e}^{-ic^2(t_n+s)}|n(t_n)|^2\mathrm{d}s + \mathcal{O}(\tau^2),$$

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- Now we integrate the highly-oscillatory phases $\mathrm{e}^{\pm ic^2(t_n+s)}$ exactly.

First-order UA scheme

- Yields **first-order UA scheme:**

$$u_*^{n+1} = \mathrm{e}^{i\tau\mathcal{A}_c} u_*^n - i\tau \mathrm{e}^{-ic^2 t_n}\varphi_1(-i\tau c^2) c \langle\nabla\rangle_c^{-1} \mathrm{e}^{i\tau\mathcal{A}_c} |n^n|^2,$$

$$n^{n+1} = \mathrm{e}^{i\tau\Delta} n^n + \frac{i}{2}\tau \mathrm{e}^{i\tau\Delta}\left[\mathrm{e}^{ic^2 t_n}\varphi_1(ic^2\tau) u_*^n n^n + \mathrm{e}^{-ic^2 t_n}\varphi_1(-ic^2\tau)\overline{u_*^n} n^n\right]$$

with

$$u_*^0 = z_0 - ic^{-1}\langle\nabla\rangle_c^{-1} z_1,$$
$$n^0 = n_0,$$

and $\varphi_1(x) := \frac{\mathrm{e}^x - 1}{x}$.

First-order UA scheme

Asymptotic convergence to the limit scheme

First-order UA scheme

Asymptotic convergence to the limit scheme

- Iteration scheme for the limit system

$$u_\infty^{n+1} = \mathrm{e}^{-\frac{i\tau}{2}\Delta} u_\infty^n,$$
$$n_\infty^{n+1} = \mathrm{e}^{i\Delta\tau} n_\infty^n.$$

First-order UA scheme

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- First-order uniformly accurate scheme

$$u_*^{n+1} = \mathrm{e}^{i\tau\mathcal{A}_c} u_*^n - i\tau \mathrm{e}^{-ic^2 t_n}\varphi_1(-i\tau c^2) c\langle\nabla\rangle_c^{-1} \mathrm{e}^{i\tau\mathcal{A}_c}|n^n|^2,$$
$$n^{n+1} = \mathrm{e}^{i\tau\Delta} n^n + \frac{i}{2}\tau \mathrm{e}^{i\tau\Delta}\left[\mathrm{e}^{ic^2 t_n}\varphi_1(ic^2\tau) u_*^n n^n + \mathrm{e}^{-ic^2 t_n}\varphi_1(-ic^2\tau)\overline{u_*^n} n^n\right].$$

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$$

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\begin{aligned}
u_*^{n+1} &= \mathrm{e}^{i\tau\mathcal{A}_c} u_*^n - i\tau \mathrm{e}^{-ic^2 t_n}\varphi_1(-i\tau c^2) c\langle\nabla\rangle_c^{-1} \mathrm{e}^{i\tau\mathcal{A}_c} |n^n|^2,\\
n^{n+1} &= \mathrm{e}^{i\tau\Delta} n^n + \frac{i}{2}\tau \mathrm{e}^{i\tau\Delta}\left[\mathrm{e}^{ic^2 t_n}\varphi_1(ic^2\tau) u_*^n n^n + \mathrm{e}^{-ic^2 t_n}\varphi_1(-ic^2\tau)\overline{u_*^n} n^n\right].
\end{aligned}
$$

With $\|\mathcal{A}_c + \frac{1}{2}\Delta\|_r = \mathcal{O}(c^{-2})$

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$$u_*^{n+1} = \mathrm{e}^{-\frac{i\tau}{2}\Delta} u_*^n - i\tau \mathrm{e}^{-ic^2 t_n} \varphi_1(-i\tau c^2) c \langle \nabla \rangle_c^{-1} \mathrm{e}^{-\frac{i\tau}{2}\Delta} |n^n|^2,$$
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n^{n+1} &= \mathrm{e}^{i\tau\Delta}n^n + \frac{i}{2}\tau\mathrm{e}^{i\tau\Delta}\left[\mathrm{e}^{ic^2 t_n}\varphi_1(ic^2\tau)u_*^n n^n + \mathrm{e}^{-ic^2 t_n}\varphi_1(-ic^2\tau)\overline{u_*^n}n^n\right].
\end{aligned}
$$

With $\|\tau\varphi_1(ilc^2\tau)\|_r = \mathcal{O}(c^{-2})$, for $l \neq 0$

First-order UA scheme

Asymptotic convergence to the limit scheme

- Iteration scheme for the limit system

$$u_\infty^{n+1} = \mathrm{e}^{-\frac{i\tau}{2}\Delta} u_\infty^n,$$
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- First-order uniformly accurate scheme

$$u_*^{n+1} = \mathrm{e}^{-\frac{i\tau}{2}\Delta} u_*^n + \mathcal{O}(c^{-2}),$$
$$n^{n+1} = \mathrm{e}^{i\tau\Delta} n^n + \mathcal{O}(c^{-2}).$$

First-order UA scheme

Asymptotic convergence to the limit scheme

- Iteration scheme for the limit system

$$u_\infty^{n+1} = \mathrm{e}^{-\frac{i\tau}{2}\Delta} u_\infty^n,$$
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- First-order uniformly accurate scheme

$$u_*^{n+1} = u_\infty^{n+1} + \mathcal{O}(c^{-2}),$$

$$n^{n+1} = n_\infty^{n+1} + \mathcal{O}(c^{-2}).$$

First-order UA scheme

Theorem (Convergence bound for the first-order UA scheme)

Fix $r > d/2$ and assume that

$$\sup_{0 \leq t \leq T} \|u_*(t)\|_{r+2} + \|n_*(t)\|_{r+2} \leq M.$$

For u_* defined in the first-order scheme we set

$$z^n := \frac{1}{2}\left(\mathrm{e}^{ic^2 t_n} u_*^n + \mathrm{e}^{-ic^2 t_n} \overline{u_*^n}\right).$$

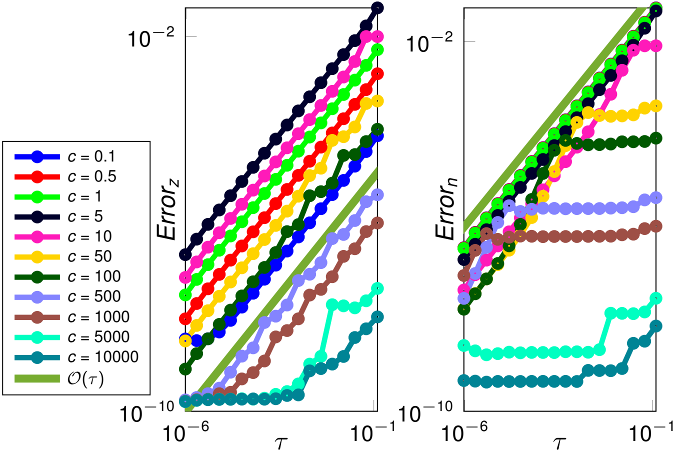
Then, there exists a $T_r > 0$ and $\tau_0 > 0$ such that for $\tau \leq \tau_0$ and $t_n \leq T_r$ we have for all $c > 0$ that

$$\|z(t_n) - z^n\|_r + \|n(t_n) - n^n\|_r \leq \tau K_{r,t_n,M},$$

where the constant $K_{r,t_n,M}$ can be chosen independently of c.

Numerical experiments

- **Order plot: First-order UA scheme:**

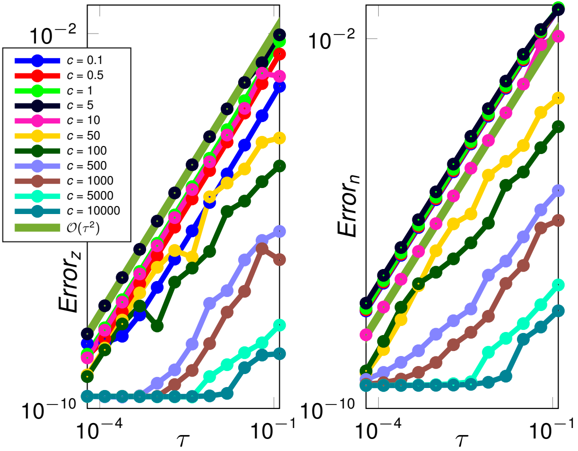


Simulation on $x \in [0, 2\pi]$, $t \in [0, 0.125]$, $\tau_{ref} \approx 10^{-7}$ and $M = 256$.

Numerical experiments

Karlsruhe Institute of Technology

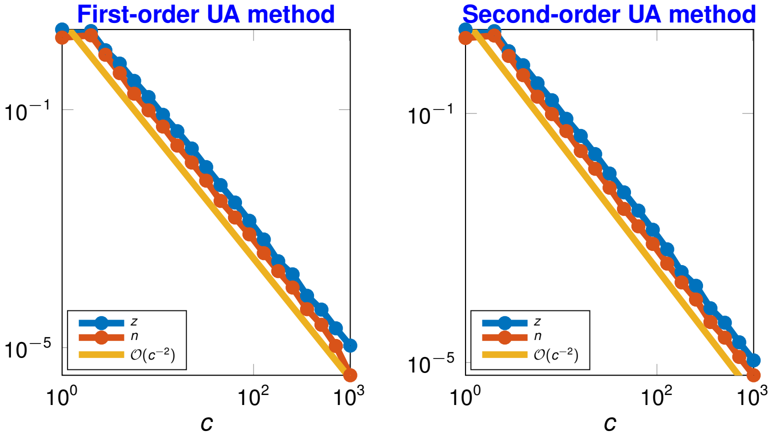
- **Order plot: Second-order UA scheme:**



Simulation on $x \in [0, 2\pi]$, $t \in [0, 0.125]$, $\tau_{ref} \approx 10^{-7}$ and $M = 256$.

Numerical experiments

- **Limit approximation:**



Simulation on $x \in [0, 2\pi]$, $t \in [0, 1]$, $\tau_{ref} \approx 10^{-6}$ and $M = 256$.

Klein-Gordon-Zakharov system

Work in progress

Model problem

Consider the **Klein-Gordon-Zakharov (KGZ) system:**

Model problem

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$$\begin{aligned} c^{-2}\partial_{tt}z - \Delta z + c^2 z &= -nz, \\ \alpha^{-2}\partial_{tt}n - \Delta n &= \Delta|z|^2 \end{aligned}$$

with initial conditions

$$\begin{aligned} z(0) &= z_0, \quad \partial_t z(0) = c^2 z_1, \\ n(0) &= n_0, \quad \partial_t n(0) = \alpha n_1, \end{aligned}$$

in the **non-singular limit regime**, i.e. $\alpha = \gamma c$, $\gamma \in \mathbb{R}_+$.

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$$

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$$
\begin{aligned}
z(0) = z_0, &\quad \partial_t z(0) = c^2 z_1, \\
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\end{aligned}
$$

in the **non-singular limit regime**, i.e. $\boldsymbol{\alpha = \gamma c}$, $\gamma \in \mathbb{R}_+$.

- Derivation: We want to follow the procedure for the KGS system analogously

The twisting

KGZ as **first-order system in time** with $z = \frac{1}{2}(u + \overline{u})$ and $n = \Re(h)$

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$$i\partial_t u = -c\langle\nabla\rangle_c u - \frac{1}{2}c\langle\nabla\rangle_c^{-1}\Re(h)(u + \overline{u}),$$

$$i\partial_t h = -\alpha\langle\nabla\rangle_0 h - \frac{1}{4}\alpha\langle\nabla\rangle_0|u + \overline{u}|^2.$$

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- **Ansatz from MFE:**

$$u_* = \mathrm{e}^{-ic^2 t}u, \qquad h_* = \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t}h$$

The twisting

KGZ as **first-order system in time** with $z = \frac{1}{2}\left(u + \overline{u}\right)$ and $n = \Re\left(h\right)$

$$
\begin{aligned}
i\partial_t u &= -c\langle\nabla\rangle_c u - \frac{1}{2} c\langle\nabla\rangle_c^{-1}\Re\left(h\right)\left(u+\overline{u}\right),\\
i\partial_t h &= -\alpha\langle\nabla\rangle_0 h - \frac{1}{4}\alpha\langle\nabla\rangle_0 |u+\overline{u}|^2.
\end{aligned}
$$

- **Ansatz from MFE:**

$$
u_* = \mathrm{e}^{-ic^2 t} u, \qquad h_* = \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t} h
$$

- First-order system in u_*, h_*

$$
\begin{aligned}
i\partial_t u_* &= \mathcal{A}_c u_* - \frac{1}{2} c\langle\nabla\rangle_c^{-1}\Re\left(\mathrm{e}^{i\alpha\langle\nabla\rangle_0 t} h_*\right)\left(u_* + \mathrm{e}^{-2ic^2 t}\overline{u_*}\right)\\
i\partial_t h_* &= -\frac{\alpha}{4}\langle\nabla\rangle_0 \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t}\left(2|u_*|^2 + \mathrm{e}^{2ic^2 t} u_*^2 + \mathrm{e}^{-2ic^2 t}\overline{u_*}^2\right)
\end{aligned}
$$

The twisting

KGZ as **first-order system in time** with $z = \frac{1}{2}(u + \overline{u})$ and $n = \Re(h)$

$$i\partial_t u = -c\langle\nabla\rangle_c u - \frac{1}{2} c\langle\nabla\rangle_c^{-1} \Re(h)(u + \overline{u}),$$
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- **Ansatz from MFE:**

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- First-order system in u_*, h_*

$$i\partial_t u_* = \mathcal{A}_c u_* - \frac{1}{2} c\langle\nabla\rangle_c^{-1} \Re\left(\mathrm{e}^{i\alpha\langle\nabla\rangle_0 t} h_*\right)\left(u_* + \mathrm{e}^{-2ic^2 t}\overline{u_*}\right)$$
$$i\partial_t h_* = -\frac{\boxed{\alpha}}{4}\langle\nabla\rangle_0 \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t}\left(2|u_*|^2 + \mathrm{e}^{2ic^2 t} u_*^2 + \mathrm{e}^{-2ic^2 t}\overline{u_*}^2\right)$$

Problem

$$i\partial_t h_* = -\frac{\alpha}{4}\langle\nabla\rangle_0 \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t}\left(2|u_*|^2 + \mathrm{e}^{2ic^2t}u_*^2 + \mathrm{e}^{-2ic^2t}\overline{u_*}^2\right)$$

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$$i\partial_t h_* = -\frac{\alpha}{4}\langle\nabla\rangle_0 \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t}\left(2|u_*|^2 + \mathrm{e}^{2ic^2 t}u_*^2 + \mathrm{e}^{-2ic^2 t}\overline{u_*}^2\right)$$

- **Integrating:**

$$h_*(t_n+\tau) = h(t_n) + \frac{i\alpha}{4}\langle\nabla\rangle_0 \int_0^\tau \mathrm{e}^{-i\alpha\langle\nabla\rangle_0(t_n+s)}\left(2|u_*(t_n+s)|^2 + \mathrm{e}^{2ic^2(t_n+s)}u_*^2(t_n+s) + c.c.\right)\mathrm{d}s$$

Problem

$$i\partial_t h_* = -\frac{\alpha}{4}\langle\nabla\rangle_0 \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t}\left(2|u_*|^2 + \mathrm{e}^{2ic^2t}u_*^2 + \mathrm{e}^{-2ic^2t}\overline{u_*}^2\right)$$

- **Integrating:**

$$h_*(t_n+\tau) = h(t_n) + \frac{i\alpha}{4}\langle\nabla\rangle_0\int_0^\tau \mathrm{e}^{-i\alpha\langle\nabla\rangle_0(t_n+s)}\left(2|u_*(t_n+s)|^2 + \mathrm{e}^{2ic^2(t_n+s)}u_*^2(t_n+s) + c.c.\right)\mathrm{d}s$$

- Taylor expansion and integrating the high oscillatory phases exactly

$$h_*^{n+1} = h_*^n + \frac{i\alpha}{4}\langle\nabla\rangle_0\mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t_n}\Big[2\tau\varphi_1\big(-i\alpha\langle\nabla\rangle_0\tau\big)|u_*^n|^2 + \mathrm{e}^{2ic^2t_n}\tau\varphi_1\big(i(-\alpha\langle\nabla\rangle_0 + 2c^2)\tau\big)(u_*^n)^2 + c.c.\Big]$$

Problem

$$h_*^{n+1} = h_*^n + \frac{i\alpha}{2}\langle\nabla\rangle_0 \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t_n}\tau\varphi_1\big(-i\alpha\langle\nabla\rangle_0\tau\big)\,|u_*^n|^2$$

$$+\frac{i\alpha}{4}\langle\nabla\rangle_0 \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t_n}\Big(\mathrm{e}^{2ic^2 t_n}\tau\varphi_1\big(i(-\alpha\langle\nabla\rangle_0 + 2c^2)\tau\big)(u_*^n)^2 + c.c.\Big)$$

Problem

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$$+\frac{i\alpha}{4}\langle\nabla\rangle_0 \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t_n}\Big(\mathrm{e}^{2ic^2 t_n}\tau\varphi_1\big(i(-\alpha\langle\nabla\rangle_0+2c^2)\tau\big)(u_*^n)^2 + c.c.\Big)$$

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$$+\frac{i\alpha}{4}\langle\nabla\rangle_0 \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t_n}\Big(\mathrm{e}^{2ic^2 t_n}\tau\varphi_1\big(i(-\alpha\langle\nabla\rangle_0+2c^2)\tau\big)(u_*^n)^2 + c.c.\Big)$$

- For I_1 we have

$$I_1 = \frac{i\alpha}{2}\langle\nabla\rangle_0 \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t_n}\tau\frac{\mathrm{e}^{-i\alpha\langle\nabla\rangle_0\tau}-1}{-i\alpha\langle\nabla\rangle_0\tau}$$

Problem

$$h_*^{n+1} = h_*^n + \underbrace{\frac{i\alpha}{2}\langle\nabla\rangle_0 \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t_n}\tau\varphi_1\big(-i\alpha\langle\nabla\rangle_0\tau\big)}_{=:I_1}|u_*^n|^2$$
$$+\frac{i\alpha}{4}\langle\nabla\rangle_0 \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t_n}\Big(\mathrm{e}^{2ic^2 t_n}\tau\varphi_1\big(i(-\alpha\langle\nabla\rangle_0+2c^2)\tau\big)(u_*^n)^2 + c.c.\Big)$$

- For I_1 we have

$$I_1 = \frac{i\alpha}{2}\langle\nabla\rangle_0 \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t_n}\tau\frac{\mathrm{e}^{-i\alpha\langle\nabla\rangle_0\tau}-1}{-i\alpha\langle\nabla\rangle_0\tau}$$
$$= \frac{1}{2}\Big(\mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t_n} - \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t_{n+1}}\Big)$$

UA scheme for the KGZ system

First-order uniformly accurate scheme

$$
\begin{aligned}
u_*^{n+1} &= \mathrm{e}^{-i\mathcal{A}_c\tau} u_*(t_n) + \frac{i}{2} c\langle\nabla\rangle_c^{-1} \mathrm{e}^{i\tau\mathcal{A}_c}\left[I_1^n - I_2^n\right],\\
h_*^{n+1} &= h_*^n + \frac{1}{2}\left(\mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t_n} - \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t_{n+1}}\right)|u_*^n|^2\\
&\quad + \frac{1}{4}\langle\nabla\rangle_0 \frac{\mathrm{e}^{-i(\alpha\langle\nabla\rangle_0+2c^2)\tau} - 1}{-(\langle\nabla\rangle_0 + 2c\gamma^{-1})} \mathrm{e}^{-i(\alpha\langle\nabla\rangle_0+2c^2)t_n} \overline{u_*^n}^2\\
&\quad + \frac{1}{4}\langle\nabla\rangle_0 \frac{\mathrm{e}^{i(-\alpha\langle\nabla\rangle_0+2c^2)\tau} - 1}{-\langle\nabla\rangle_0 + 2c\gamma^{-1}} \mathrm{e}^{i(-\alpha\langle\nabla\rangle_0+2c^2)t_n} (u_*^n)^2,
\end{aligned}
$$

with

$$
\begin{aligned}
I_1^n &= \frac{1}{2}\left[\mathrm{e}^{i\alpha\langle\nabla\rangle_0 t_n}\tau\varphi_1\left(i\alpha\langle\nabla\rangle_0\tau\right) + \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t_n}\tau\varphi_1\left(-i\alpha\langle\nabla\rangle_0\tau\right)\right]\Re(h_*^n)\left(u_*^n + \mathrm{e}^{-2ic^2t}\overline{u_*^n}\right),\\
I_2^n &= \frac{1}{2i}\left[\mathrm{e}^{i\alpha\langle\nabla\rangle_0 t_n}\tau\varphi_1\left(i\alpha\langle\nabla\rangle_0\tau\right) - \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t_n}\tau\varphi_1\left(-i\alpha\langle\nabla\rangle_0\tau\right)\right]\Im(h_*^n)\left(u_*^n + \mathrm{e}^{-2ic^2t}\overline{u_*^n}\right).
\end{aligned}
$$

UA scheme for the KGZ system

First-order uniformly accurate scheme

$$u_*^{n+1} = \mathrm{e}^{-i\mathcal{A}_c\tau} u_*(t_n) + \frac{i}{2} c\langle\nabla\rangle_c^{-1} \mathrm{e}^{i\tau\mathcal{A}_c}\left[I_1^n - I_2^n\right],$$

$$\begin{aligned} h_*^{n+1} &= h_*^n + \frac{1}{2}\left(\mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t_n} - \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t_{n+1}}\right)|u_*^n|^2 \\ &\quad + \frac{1}{4}\langle\nabla\rangle_0 \frac{\mathrm{e}^{-i(\alpha\langle\nabla\rangle_0 + 2c^2)\tau} - 1}{-(\langle\nabla\rangle_0 + 2c\gamma^{-1})} \mathrm{e}^{-i(\alpha\langle\nabla\rangle_0 + 2c^2)t_n} \overline{u_*^n}^2 \\ &\quad + \frac{1}{4}\langle\nabla\rangle_0 \frac{\mathrm{e}^{i(-\alpha\langle\nabla\rangle_0 + 2c^2)\tau} - 1}{\boxed{-\langle\nabla\rangle_0 + 2c\gamma^{-1}}} \mathrm{e}^{i(-\alpha\langle\nabla\rangle_0 + 2c^2)t_n} (u_*^n)^2, \end{aligned}$$

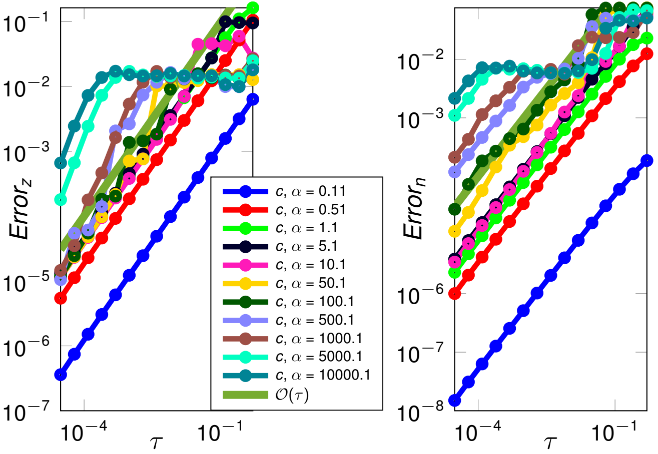
with

$$I_1^n = \frac{1}{2}\left[\mathrm{e}^{i\alpha\langle\nabla\rangle_0 t_n}\tau\varphi_1\left(i\alpha\langle\nabla\rangle_0\tau\right) + \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t_n}\tau\varphi_1\left(-i\alpha\langle\nabla\rangle_0\tau\right)\right]\Re(h_*^n)\left(u_*^n + \mathrm{e}^{-2ic^2 t}\overline{u_*^n}\right),$$

$$I_2^n = \frac{1}{2i}\left[\mathrm{e}^{i\alpha\langle\nabla\rangle_0 t_n}\tau\varphi_1\left(i\alpha\langle\nabla\rangle_0\tau\right) - \mathrm{e}^{-i\alpha\langle\nabla\rangle_0 t_n}\tau\varphi_1\left(-i\alpha\langle\nabla\rangle_0\tau\right)\right]\Im(h_*^n)\left(u_*^n + \mathrm{e}^{-2ic^2 t}\overline{u_*^n}\right).$$

First numerical experiments ($\gamma = 1$)

- **Order plot:**



Simulation on $x \in [0, 2\pi]$, $t \in [0, 1]$, $\tau_{ref} \approx 10^{-6}$ and $M = 256$.

UA scheme for the KGZ system

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Questions:

- Right twisting of h?
- Right calculation of the UA scheme?
- Convergence to the numerical scheme of the limit system?

UA schemes for KGS and KGZ systems

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Remark

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- Derivation of the schemes for the KGZ equation also works for $z \in \mathbb{C}$, i.e. $z = \frac{1}{2}(u + \overline{v})$.

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- Generalization to **higher order schemes**:
  Insert Duhamel's formula for $u_*(t_n + s)$ into $u_*(t_n + \tau)$ and go on analogously to the derivation of the first-order scheme.

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- Generalization to **higher order schemes**:
  Insert Duhamel's formula for $u_*(t_n + s)$ into $u_*(t_n + \tau)$ and go on analogously to the derivation of the first-order scheme.
- For KGS also the second-order schemes converge in the limit to the corresponding second-order numerical method for the limit equation.

Outlook

- Work-precision plots for KGS and KGZ.
- Error analysis of the first-order scheme for the KGZ system.

- Construct higher-order methods for the KGZ system.
- Error analysis of the higher-order methods.

- Can we twist (KG, KGS, KGZ) such that $\partial_{tt} u_{**} = \mathcal{O}(1)$?

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