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ADI splitting for the Maxwell equations

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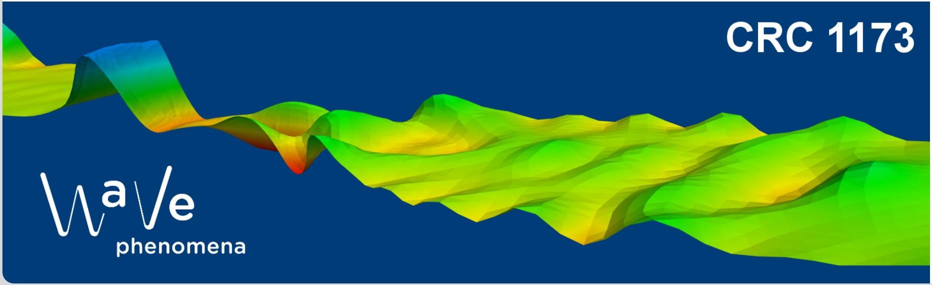

Contents

Maxwell Equations (MEs)

On a cuboid, say $Q := (-1, 1)^3$, we consider for $t \geq 0$

$$\begin{aligned}
\partial_t(\varepsilon \mathbf{E}(t)) &= \operatorname{rot} \mathbf{H}(t) - (\sigma \mathbf{E}(t) + \mathbf{J}_0(t)), \\
\partial_t(\mu \mathbf{H}(t)) &= -\operatorname{rot} \mathbf{E}(t),
\end{aligned}$$

$$\operatorname{div}(\varepsilon \mathbf{E}(t)) = \rho(t), \qquad \operatorname{div}(\mu \mathbf{H}(t)) = 0,$$

$$\mathbf{E}(t) \times \nu = 0, \qquad \mu \mathbf{H}(t) \cdot \nu = 0, \qquad \text{on } \partial Q =: \Gamma,$$

$$\mathbf{E}(0) = \mathbf{E}_0, \qquad \mathbf{H}(0) = \mathbf{H}_0.$$

ν denotes the outer unit normal.

Given: $(\mathbf{E}_0, \mathbf{H}_0)$, $\mathbf{J}_0$ and $\rho(0)$

Coeffcients: $\varepsilon, \mu, \sigma \in W^{1,\infty}(Q, \mathbb{R}) \cap W^{2,3}(Q, \mathbb{R})$ and $\varepsilon, \mu \geq \delta > 0$

We look for solutions $(\mathbf{E}, \mathbf{H})$ in $C^1([0, \infty), L^2(Q)^6)$. Numerical analysis is done on $[0, T]$ for a $T > 0$.

Function spaces

We denote $\Gamma_1 := \{x \in \overline{Q} \mid x_1 = -1 \text{ or } x_1 = 1\}$ and Γ_2, Γ_3 analogously.

We define

$$H(Q, \mathrm{rot}) : = \{u \in L^2(Q)^3 \mid \mathrm{rot}\, u \in L^2(Q)^3\} \qquad \text{and}$$
$$H_0(Q, \mathrm{rot}) : = \{u \in H(Q, \mathrm{rot}) \mid u \times \nu = 0\}.$$

Let $X := L^2(Q)^6$ be equipped with

$$\big((u, v) \mid (\varphi, \psi)\big)_X := \int_Q (\varepsilon u \cdot \varphi + \mu v \cdot \psi)\, \mathrm{d}x$$

for all $(u, v), (\varphi, \psi) \in X$.

Function spaces

$$X = L^2(Q)^6 \qquad \big((u,v) \mid (\varphi,\psi)\big)_X := \int_Q (\varepsilon u \cdot \varphi + \mu v \cdot \psi)\,\mathrm{d}x$$

We define its subspaces

$$X_{\mathrm{div}}^{(0)} := \{(u,v) \in X \mid \mathrm{div}(\mu v) = 0, \mu v \cdot \nu = 0 \text{ on } \Gamma, \mathrm{div}(\varepsilon u) \in L^2(Q)\},$$
$$X_{\mathrm{div}}^{(1)} := \{(u,v) \in X \mid \mathrm{div}(\mu v) = 0, \mu v \cdot \nu = 0 \text{ on } \Gamma, \mathrm{div}(\varepsilon u) \in H^1_{00}(Q)\},$$

being equipped with

$$\|(u,v)\|^2_{X_{\mathrm{div}}^{(0)}} := \|(u,v)\|^2_X + \|\mathrm{div}(\varepsilon u)\|^2_{L^2},$$
$$\|(u,v)\|^2_{X_{\mathrm{div}}^{(1)}} := \|(u,v)\|^2_X + \|\mathrm{div}(\varepsilon u)\|^2_{H^1} + \sum_{\widetilde{\Gamma} \text{ side of } Q} \|\mathrm{div}(\varepsilon u)\|^2_{H_0^{1/2}(\widetilde{\Gamma})},$$

where

$$H_0^{1/2}(\widetilde{\Gamma}) = \big(L^2(\widetilde{\Gamma}), H_0^1(\widetilde{\Gamma})\big)_{1/2,2}.$$

The Maxwell operator

(Compare [HJS, 2014]) We define the Maxwell operator

$$M := \begin{pmatrix} -\frac{\sigma}{\varepsilon} I & \frac{1}{\varepsilon}\operatorname{rot} \\ -\frac{1}{\mu}\operatorname{rot} & 0 \end{pmatrix}$$

with $D(M) := H_0(Q, \operatorname{rot}) \times H(Q, \operatorname{rot})$. Maxwell equations become

$$\partial_t \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} = M \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} + \begin{pmatrix} -\frac{1}{\varepsilon}\mathbf{J}_0(t) \\ 0 \end{pmatrix}$$

in X for $t \geq 0$, where $\rho(t) := \operatorname{div}(\varepsilon \mathbf{E}(t))$. We define the restrictions $M_{\text{div}}^{(0)}$ and $M_{\text{div}}^{(1)}$ to $X_{\text{div}}^{(0)}$ and $X_{\text{div}}^{(1)}$ with

$$D(M_{\text{div}}^{(0)}) := D(M) \cap X_{\text{div}}^{(0)} \qquad \text{and} \qquad D(M_{\text{div}}^{(1)}) := D(M) \cap X_{\text{div}}^{(1)}.$$

Observe that $M_{\text{div}}^{(0)}$ maps into $X_{\text{div}}^{(0)}$ and $M_{\text{div}}^{(1)}$ into $X_{\text{div}}^{(1)}$.

The splitting operators

(Compare [HJS, 2014]) Split rot $= C_1 - C_2$ with

$$C_1 := \begin{pmatrix} 0 & 0 & \partial_2 \\ \partial_3 & 0 & 0 \\ 0 & \partial_1 & 0 \end{pmatrix} \quad \text{and} \quad C_2 := \begin{pmatrix} 0 & \partial_3 & 0 \\ 0 & 0 & \partial_1 \\ \partial_2 & 0 & 0 \end{pmatrix}$$

and define

$$A := \begin{pmatrix} -\frac{\sigma}{2\varepsilon} I & \frac{1}{\varepsilon} C_1 \\ \frac{1}{\mu} C_2 & 0 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} -\frac{\sigma}{2\varepsilon} I & -\frac{1}{\varepsilon} C_2 \\ -\frac{1}{\mu} C_1 & 0 \end{pmatrix}$$

with

$$D(A) : = \{(u, v) \in X \mid (C_1 v, C_2 u) \in X + \text{b.c.}\} \quad \text{and}$$
$$D(B) : = \{(u, v) \in X \mid (C_2 v, C_1 u) \in X + \text{b.c.}\}$$

We have $M = A + B$ on $D(A) \cap D(B) \subseteq D(M)$.

The splitting scheme

Let $\tau > 0$ and $t_n := n\tau$ for $n \in \mathbb{N}$. We introduce (compare [OS, 2012])

$$S^{I}_{\tau,n+1} w := \left(I - \tfrac{\tau}{2} B\right)^{-1} \left(I + \tfrac{\tau}{2} A\right) \cdot$$
$$\cdot \left[\left(I - \tfrac{\tau}{2} A\right)^{-1} \left(I + \tfrac{\tau}{2} B\right) w - \tfrac{\tau}{2\varepsilon} \left(\mathbf{J}_0(t_n) + \mathbf{J}_0(t_{n+1}), 0\right) \right]$$

for $w \in D(B)$ and suitable $\mathbf{J}_0$. Note that with

$$S^{I,(1)}_{\tau,n+1} w_1 := \left(I - \tfrac{\tau}{2} A\right)^{-1} \left(I + \tfrac{\tau}{2} B\right) w_1 \qquad \text{for } w_1 \in D(B) \qquad \text{and}$$
$$S^{I,(2)}_{\tau,n+1} w_2 := \left(I - \tfrac{\tau}{2} B\right)^{-1} \left(I + \tfrac{\tau}{2} A\right) w_2 \qquad \text{for } w_2 \in D(A)$$

we can write

$$S^{I}_{\tau,n+1} w = S^{I,(2)}_{\tau,n+1} \left[S^{I,(1)}_{\tau,n+1} w - \tfrac{\tau}{2\varepsilon} \left(\mathbf{J}_0(t_n) + \mathbf{J}_0(t_{n+1}), 0\right) \right].$$

Solutions to the Maxwell equations in $X_{\mathrm{div}}^{(0)}$

Proposition 1

$M_{\mathrm{div}}^{(0)}$ generates a C_0-semigroup on $X_{\mathrm{div}}^{(0)}$, giving the unique solution of (MEs) in $C^1\big([0,\infty), X_{\mathrm{div}}^{(0)}\big) \cap C\big([0,\infty), D(M_{\mathrm{div}}^{(0)})\big)$ if $(\mathbf{J}_0, 0) \in C\big([0,\infty), D(M_{\mathrm{div}}^{(0)})\big)$ and $(\mathbf{E}_0, \mathbf{H}_0) \in D(M_{\mathrm{div}}^{(0)})$.

Solutions to the Maxwell equations in $X_{\mathrm{div}}^{(0)}$

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Proposition 1

$M_{\mathrm{div}}^{(0)}$ generates a C_0-semigroup on $X_{\mathrm{div}}^{(0)}$, giving the unique solution of (MEs) in $C^1\big([0,\infty), X_{\mathrm{div}}^{(0)}\big) \cap C\big([0,\infty), D(M_{\mathrm{div}}^{(0)})\big)$ if $(\mathbf{J}_0, 0) \in C\big([0,\infty), D(M_{\mathrm{div}}^{(0)})\big)$ and $(\mathbf{E}_0, \mathbf{H}_0) \in D(M_{\mathrm{div}}^{(0)})$.

Proposition 2

$M_{\mathrm{div}}^{(1)}$ generates a C_0-semigroup on $X_{\mathrm{div}}^{(1)}$, giving the unique solution of (MEs) in $C^1\big([0,\infty), X_{\mathrm{div}}^{(1)}\big) \cap C\big([0,\infty), D(M_{\mathrm{div}}^{(1)})\big)$ if $(\mathbf{J}_0, 0) \in C\big([0,\infty), D(M_{\mathrm{div}}^{(1)})\big)$ and $(\mathbf{E}_0, \mathbf{H}_0) \in D(M_{\mathrm{div}}^{(1)})$.

Efficiency of the scheme

From $(\mathbf{E}_{n+1/2}, \mathbf{H}_{n+1/2}) := S^{I,(1)}_{\tau,n+1}(\mathbf{E}_n, \mathbf{H}_n) = (I - \frac{\tau}{2}A)^{-1}(I + \frac{\tau}{2}B)(\mathbf{E}_n, \mathbf{H}_n)$ we get

$$(1 + \tfrac{\sigma\tau}{4\varepsilon})\mathbf{E}_{n+1/2} = (1 - \tfrac{\sigma\tau}{2\varepsilon})\mathbf{E}_n - \tfrac{\tau}{2\varepsilon} C_2 \mathbf{H}_n + \tfrac{\tau}{2\varepsilon} C_1 \color{blue}{\mathbf{H}_{n+1/2}},$$
$$\color{blue}{\mathbf{H}_{n+1/2}} = \mathbf{H}_n - \tfrac{\tau}{2\mu} C_1 \mathbf{E}_n + \color{red}{\tfrac{\tau}{2\mu} C_2 \mathbf{E}_{n+1/2}}.$$

Efficiency of the scheme

From $(\mathbf{E}_{n+1/2}, \mathbf{H}_{n+1/2}) := S^{I,(1)}_{\tau,n+1}(\mathbf{E}_n, \mathbf{H}_n) = (I - \frac{\tau}{2}A)^{-1}(I + \frac{\tau}{2}B)(\mathbf{E}_n, \mathbf{H}_n)$ we get

$$\begin{aligned}(1 + \tfrac{\sigma\tau}{4\varepsilon})\mathbf{E}_{n+1/2} &= (1 - \tfrac{\sigma\tau}{2\varepsilon})\mathbf{E}_n - \tfrac{\tau}{2\varepsilon}C_2\mathbf{H}_n + \tfrac{\tau}{2\varepsilon}C_1\mathbf{H}_{n+1/2},\\ \mathbf{H}_{n+1/2} &= \mathbf{H}_n - \tfrac{\tau}{2\mu}C_1\mathbf{E}_n + \tfrac{\tau}{2\mu}C_2\mathbf{E}_{n+1/2}.\end{aligned}$$

Plugging the second equation into the first one yields

$$\begin{aligned}\Big((1 + \tfrac{\sigma\tau}{4\varepsilon})I - \tfrac{\tau^2}{4\varepsilon}D^{(1)}_\mu\Big)\mathbf{E}_{n+1/2} &= (1 - \tfrac{\sigma\tau}{2\varepsilon})\mathbf{E}_n + \tfrac{\tau}{2\varepsilon}\operatorname{rot}\mathbf{H}_n + \tfrac{\tau}{2\varepsilon}C_1\mathbf{H}_n\\ &\quad - \tfrac{\tau^2}{4\varepsilon}C_1\tfrac{1}{\mu}C_1\mathbf{E}_n,\\ \mathbf{H}_{n+1/2} &= \mathbf{H}_n - \tfrac{\tau}{2\mu}C_1\mathbf{E}_n + \tfrac{\tau}{2\mu}C_2\mathbf{E}_{n+1/2},\end{aligned}$$

with

$$D^{(1)}_\mu := C_1\tfrac{1}{\mu}C_2 = \begin{pmatrix}\partial_2\frac{1}{\mu}\partial_2 & 0 & 0\\ 0 & \partial_3\frac{1}{\mu}\partial_3 & 0\\ 0 & 0 & \partial_1\frac{1}{\mu}\partial_1\end{pmatrix}.$$

Efficiency of the scheme

$$\Big(\big(1+\tfrac{\sigma\tau}{4\varepsilon}\big)I - \tfrac{\tau^2}{4\varepsilon}D_\mu^{(1)}\Big)\mathbf{E}_{n+1/2} = \big(1-\tfrac{\sigma\tau}{2\varepsilon}\big)\mathbf{E}_n + \tfrac{\tau}{2\varepsilon}\,\mathrm{rot}\,\mathbf{H}_n + \tfrac{\tau}{2\varepsilon}C_1\mathbf{H}_n - \tfrac{\tau^2}{4\varepsilon}C_1\tfrac{1}{\mu}C_1\mathbf{E}_n,$$

$$\mathbf{H}_{n+1/2} = \mathbf{H}_n - \tfrac{\tau}{2\mu}C_1\mathbf{E}_n + \tfrac{\tau}{2\mu}C_2\mathbf{E}_{n+1/2},$$

with

$$D_\mu^{(1)} := C_1\tfrac{1}{\mu}C_2 = \begin{pmatrix} \partial_2\frac{1}{\mu}\partial_2 & 0 & 0 \\ 0 & \partial_3\frac{1}{\mu}\partial_3 & 0 \\ 0 & 0 & \partial_1\frac{1}{\mu}\partial_1 \end{pmatrix}.$$

$\Rightarrow$ Implicit first equation decouples into three one-dimensional equations!
(See Namiki '99, Chen/Zhang/Zheng '00 for the case $\sigma = 0$.)

The second half step is treated similarly after adding the inhomogeneity.

(Nearly) preservation of the divergence

Theorem 1 (Preservation of the divergence) (E., Schnaubelt, 2016)

Let $(\mathbf{J}_0, 0) \in C\Big([0,T], D\big((M_{\mathrm{div}}^{(1)})^2\big)\Big) \cap C^2\Big([0,T], D(M_{\mathrm{div}}^{(1)})\Big)$ and $(\mathbf{E}_0, \mathbf{H}_0) \in D\big((M_{\mathrm{div}}^{(1)})^2\big)$. Then there exists a $\tau_0 > 0$ s.t. for all $0 < \tau \le \tau_0$ and $N \in \mathbb{N}$ with $N\tau \le T$ we have

$$\Big\|\Big(\big(\mathrm{div}(\varepsilon \mathbf{E}_N), \mathrm{div}(\mu \mathbf{H}_N)\big) - \big(\mathrm{div}(\varepsilon \mathbf{E}_0), 0\big) + \int_0^{N\tau} \big(\mathrm{div}(\mathbf{J}_0(s)), 0\big)\,\mathrm{d}s$$
$$+ \sum_{k=0}^{N-1} \frac{\tau}{2}\big(\mathrm{div}(\tfrac{\sigma}{2}\mathbf{E}_{k+1} + \sigma \mathbf{E}_{k+1/2} + \tfrac{\sigma}{2}\mathbf{E}_k), 0\big)\Big\|_{H^{-1}}$$
$$\le C\tau$$

for a constant C only depending on $(\mathbf{E}_0, \mathbf{H}_0)$, $\mathbf{J}_0$, ε, μ, σ and T.

Strong convergence of the scheme

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Theorem 2 (Strong convergence) (E., Schnaubelt, 2016)

Let $(\mathbf{J}_0, 0) \in C\Big([0,T], D\big((M_{\text{div}}^{(1)})^2\big)\Big) \cap C^2\Big([0,T], D(M_{\text{div}}^{(1)})\Big)$ and $(\mathbf{E}_0, \mathbf{H}_0) \in D\big((M_{\text{div}}^{(1)})^3\big)$. Then for all $\tau > 0$ and $n \in \mathbb{N}$ with $n\tau \leq T$ we have

$$\Big\| S_{\tau,n}^{I} \cdots S_{\tau,1}^{I} (\mathbf{E}_0, \mathbf{H}_0) - (\mathbf{E}(n\tau), \mathbf{H}(n\tau)) \Big\|_{L^2} \leq C\tau^2$$

with a constant C depending only on $(\mathbf{E}_0, \mathbf{H}_0)$, $\mathbf{J}_0$, ε, μ, σ and T.

Weak convergence of the scheme

Define

$$Y := \Big\{(u,v) \in (H^1(Q))^6 \,|\, u_j = 0 \text{ on } \Gamma \setminus \Gamma_j,\, v_j = 0 \text{ on } \Gamma_j, \text{for all } j \in \{1,2,3\}\Big\}.$$

Theorem 3 (Weak convergence) (E., Schnaubelt, 2016)

Let $(\mathbf{J}_0, 0) \in C\Big([0,T], D\big((M_{\text{div}}^{(0)})\big)\Big) \cap C^2\Big([0,T], X_{\text{div}}^{(0)})\Big)$ and $(\mathbf{E}_0, \mathbf{H}_0) \in D\big((M_{\text{div}}^{(0)})^2\big)$. Then for all $\tau > 0$ and $n \in \mathbb{N}$ with $n\tau \leq T$ we have

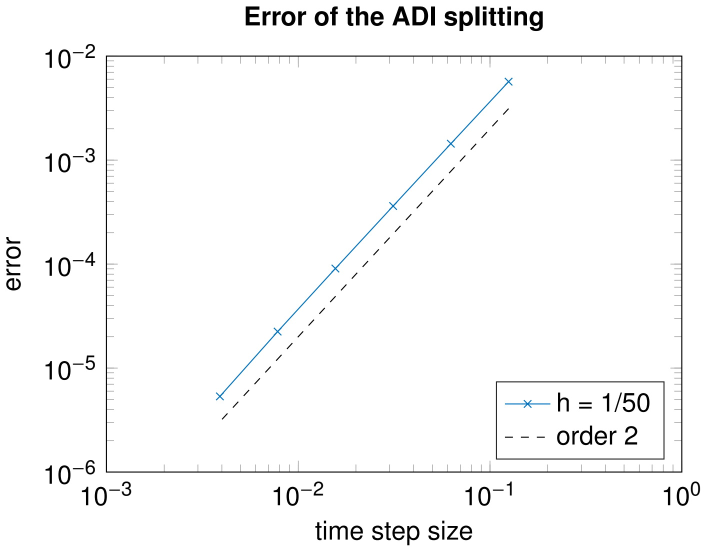
$$\Big| \Big\langle S_{\tau,n}^{I} \cdots S_{\tau,1}^{I} (\mathbf{E}_0, \mathbf{H}_0) - (\mathbf{E}(n\tau), \mathbf{H}(n\tau)), (\varphi, \psi) \Big\rangle_Y \Big| \leq C\tau^2$$

for all $(\varphi, \psi) \in Y$ with a constant C depending only on $(\mathbf{E}_0, \mathbf{H}_0)$, $\mathbf{J}_0$, ε, μ, σ and T.

Numerical experiments (I)

- spatial domain $Q = (0,1)^3$, discretized by the Yee grid with 50 grid points per direction
- time domain $[0,1]$
- time steps size 2^{-k}, $k = 3, \dots, 8$
- reference solution: ADI method with time step size 2^{-10}
- ε, μ, σ smooth (arctan)
- $\mathbf{J}_0$ smooth in space (mollifier) and time (trigonometric)
- $(\mathbf{E}_0, \mathbf{H}_0)$ smooth in space and time (trigonometric)

Numerical experiments (II)

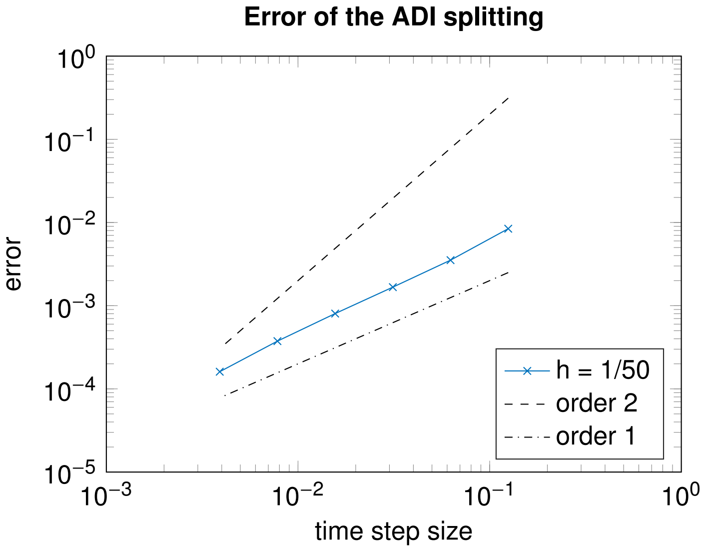


Numerical experiments (III)

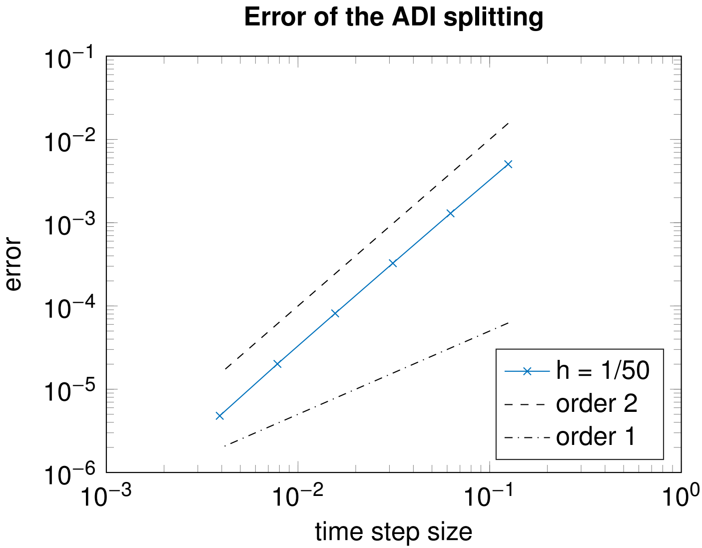
Reduction of the convergence order:

- Choose $\mathbf{J}_0$ smooth of the type arctan in space.
  $\Rightarrow$ Does not fulfill the boundary conditions of $D\big((M_{\text{div}}^{(1)})^2\big)$.
- Choose $(\mathbf{E}_0, \mathbf{H}_0)$ smooth so that it does not fulfill the boundary conditions (in space) of $D\big((M_{\text{div}}^{(1)})^3\big)$.

Numerical experiments (IV)



Numerical experiments (V)



Summary

Results:

- Second order convergence in L^2 and in "H^{-1}"
- First order preservation of the divergence in H^{-1}
- Numerical confirmation of the convergence order
- Order reduction for violated boundary conditions of $\mathbf{J}_0$

Aims of further research:

- Order reduction for violated boundary conditions of $(\mathbf{E}_0, \mathbf{H}_0)$
- Numerical confirmation of the divergence preservation
- Generalizations of the domain to unions of cuboids

- Piecewise smooth coefficients
- Analysis of the space discretisation