Trigonometric time-integrators for the Zakharov system

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&
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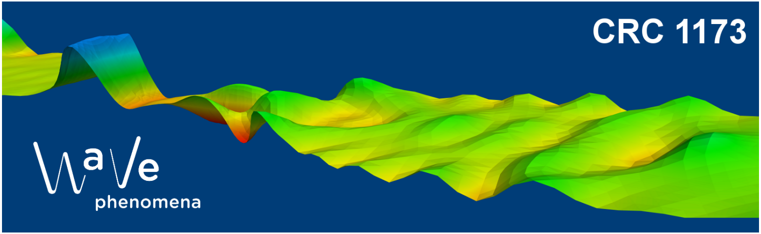

Trifels, Oct. 2015

Joint work with G. Schneider (Stuttgart):

Efficient time-integrators for Klein-Gordon-Zakharov system

$$c^{-2}\partial_{tt}E - \Delta E + c^2 E = -uE, \quad \alpha^{-2}\partial_{tt}u - \Delta u = \Delta|E|^2 \quad \text{(KGZ)}$$

in high plasma-frequency regime $c \gg 1$.

- Problem: Highly-oscillatory solutions!

Figure: $c = 16$

Figure: $c = 32$

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in high plasma-frequency regime $c \gg \infty$.

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- Idea: Filter out high-oscillations & only solve limit-system!

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Approximation:

$$E(t,x) = \mathrm{e}^{ic^2t}E_0(t,x) + \mathrm{e}^{-ic^2t}\overline{E}_0(t,x) + \mathcal{O}(c^{-2})$$
$$u(t,x) = u_0(t,x) + \mathcal{O}(c^{-2})$$

(E_0, u_0) solve **non-oscillatory** limit system ("$c = \infty$" in (KGZ))

$$i\partial_t E_0 + \Delta E_0 = u_0 E_0, \quad \alpha^{-2}\partial_{tt}u_0 - \Delta u_0 = \Delta|E_0|^2 \quad \text{(Zakharov)}$$

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Limit system = Zakharov

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+ Looks very promising (no oscillations!) – numerically!

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But ...

– Difficulty: Structure of nonlinear coupling!

Zakharov

$$i\partial_t E = -\Delta E + uE, \quad \partial_{tt} u = \Delta u + \Delta |E|^2 \quad (x \in \mathbb{T}^d) \qquad \text{(Z)}$$

Mild solutions

$$\begin{aligned} E(t) =& \mathrm{e}^{it\Delta} E(0) - i \int_0^t \mathrm{e}^{i(t-\xi)\Delta} u(\xi) E(\xi) \mathrm{d}\xi, \\ u(t) =& \cos(t|\nabla|) u(0) + \sin(t|\nabla|) |\nabla|^{-1} u'(0) \\ & + \int_0^t \sin((t-\xi)|\nabla|) |\nabla| |E(\xi)|^2 \mathrm{d}\xi. \end{aligned}$$

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"Naive" bounds: ($s > d/2$)

$$\text{(B1)} \quad \|E(t)\|_s \le \|E(0)\|_s + c\int_0^t \|u(\xi)\|_s\|E(\xi)\|_s\mathrm{d}\xi,$$

$$\text{(B2)} \quad \|u(t)\|_s \le \|u(0)\|_s + \|u'(0)\|_{s-1} + c\int_0^t \|E(\xi)\|_{s+1}^2\mathrm{d}\xi$$

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(B2) $\|u(t)\|_s \le \|u(0)\|_s + \|u'(0)\|_{s-1} + c \int_0^t \|E(\xi)\|_{s+1}^2 d\xi$

$\rightarrow$ Standard fix-point argument fails in H^s (loss of derivative)

Zakharov = Schrödinger coupled with wave

Good spaces$^{(*)}$ = Bourgain spaces: $X^S_{s,b}$ $X^W_{s,b}$

$$\|f\|^2_{X^S_{s,b}} = \sum_{n\in\mathbb{Z}} (1+|n|)^{2s} \int_{-\infty}^{\infty} (1+|\tau+n^2|)^{2b} |\hat{f}(\tau,n)|^2 \mathrm{d}\tau$$

$$= \|\mathrm{e}^{-it\Delta} f(t,\cdot)\|_{H^b_t(\mathbb{R}:H^s(\mathbb{T}))}$$

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$(*)$ One shows local-wellposedness of (Z) in $X^S_{s,1/2} \times X^W_{l,1/2}$ with $0 \le s-l \le 1$, $1/2 \le l+1/2 \le 2s$
[Bourgain (94'), Takaoka (97'), Kishimoto (14')]

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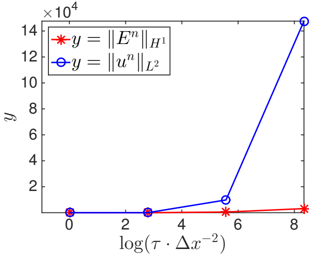
But: How to adapt "well" to numerical analysis?
(... still unclear to me!)

Q: Time integration of Zakharov system?

$$i\partial_t E = -\Delta E + uE, \quad \partial_{tt} u = \Delta u + \Delta |E|^2 \quad (x \in \mathbb{T}^d) \qquad \text{(Z)}$$

Up to now numerical methods[1] require

$$\mathcal{C}^\infty\text{-solutions + CFL condition } (\tau \cdot \Delta x^{-2} < 1) \qquad (*)$$



(CFL $(*)$ means: time-step $\times$ discretized Δ $=$ bounded)

Can we construct a method without $(*)$?

[1] Glassey (92'), Chang (95') (FD-methods); Payne et al (83'), Markowich et al (04'), Bao et al ('05), . . .

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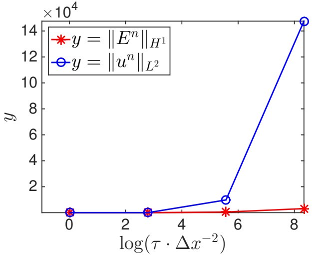

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[Slide from Oberwolfach workshop March 2014]

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Idea: [Ozawa & Tsutsumi[2]] Set $F := \partial_t E$ & reformulate (Z):

$$i\partial_t F = -\Delta F + uF + (\partial_t u)\left(E(0) + \int_0^t F(\xi)\mathrm{d}\xi\right), \; F(0) = \partial_t E(0),$$

$$\partial_{tt} u = \Delta u + \Delta|E|^2, \qquad \text{(Z}^*\text{)}$$

$$(-\Delta + 1)E = iF - (u-1)\left(E(0) + \int_0^t F(\xi)\mathrm{d}\xi\right).$$

[2] Existence and Smoothing Effect of Solutions for the Zakharov Eq.s (92')

$$i\partial_t E = -\Delta E + uE, \quad \partial_{tt} u = \Delta u + \Delta |E|^2 \quad (x \in \mathbb{T}^d) \tag{Z}$$

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NO loss of derivative! $\rightarrow$ Instead of (Z) solve (Z*) numerically!

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Herr, S. (15'): Trigonometric time-integration scheme for (Z$^{(*)}$): Convergence of order τ^γ $(0 < \gamma \leq 1)$ in

$$H^s \times H^{s+2} \times H^{s+1} \times H^s \ni \text{error}(F, E, u, u'), \quad s > d/2$$

for $t \in [0, T]$ if

$$H^{s+2+2\gamma} \times H^{s+1+2\gamma} \times H^{s+2\gamma} \ni (E(t), u(t), u'(t)).$$

No CFL nor $\mathcal{C}^\infty$-data required!

[2] Existence and Smoothing Effect of Solutions for the Zakharov Eq.s (92')

Numerical Experiment: [Trigonometric time-integrator for (Z$^{(*)}$)]

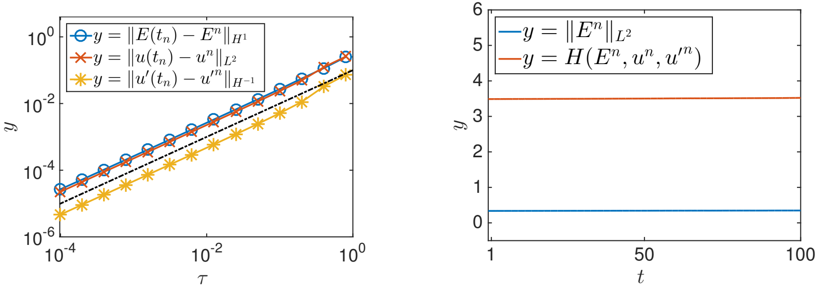

Figure: Left: Orderplot (double logarithmic; dashed line: slope one) $(\Delta x = 0.0245)$. Right: Energy conservation (CFL ≈ 6.5).

Energy $(\hat{u'}_0(0) = 0)$

$$H(E, u, u')$$
$$= \int_{\mathbb{T}^d} |\nabla E(t,x)|^2 + u(t,x)|E(t,x)| + \frac{1}{2}||\nabla|^{-1}u'(t,x)|^2 + \frac{1}{2}|u(t,x)|^2 \mathrm{d}x.$$

Convergence Theorem:

Let $s > d/2$, $0 < \gamma \leq 1$ and T such that

$$m_{s+2+2\gamma}(T) := \sup_{t \leq T} \{\|E(t)\|_{s+2+2\gamma} + \|u(t)\|_{s+1+2\gamma} + \|u'(t)\|_{s+2\gamma}\} < \infty, \tag{R}$$

where $\|\cdot\|_s := \|\cdot\|_{H^s(\mathbb{T}^d)}$. Then: $\exists \tau_0 > 0$ s.t. $\forall \tau \leq \tau_0$, $t_n \leq T$

$$\|E(t_n) - E^n\|_{s+2} + \|u(t_n) - u^n\|_{s+1} + \|u'(t_n) - u'^n\|_s \leq C(T)\tau^\gamma.$$

○ LWP on $\mathbb{T}^d$ [Bourgain (94'), Takaoka (97'), Kishimoto (13')]: $T > 0$ in (R)

○ Energy space: If $1 \leq d \leq 3$ and for some $\varepsilon, \delta > 0$

$$m_{\max(d/2+\varepsilon,1)+2+2\delta}(T) < \infty \quad \text{and} \quad \tau \leq \tau_0(\delta),\ t_n \leq T \tag{R*}$$

$$\|E(t_n) - E^n\|_1 + \|u(t_n) - u^n\|_0 + \|u'(t_n) - u'^n\|_{-1} \leq C(T)\tau.$$

(Quantum) Zakharov

$$i\partial_t E_\vartheta - \Omega_\vartheta^2 E_\vartheta = E_\vartheta u_\vartheta, \quad \partial_{tt} u_\vartheta + \Omega_\vartheta^2 u_\vartheta = \Delta |E_\vartheta|^2. \qquad \text{(QZ)}$$

$\Omega_\vartheta^2 := -\Delta + \vartheta \Delta^2$ ($\vartheta = 0$ classical)

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$+$ $\exists C > 0$ s.t.

$$\forall \vartheta > 0 \, : \, \left\| \Delta \Omega_\vartheta^{-1} f \right\|_s \leq C \inf\left\{ \vartheta^{-1/2} \|f\|_s, \|f\|_{s+1} \right\}. \qquad (*)$$

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$-$ Using $(*)$ leads to error constant $\mathrm{e}^{\vartheta^{-1/2} T}$!

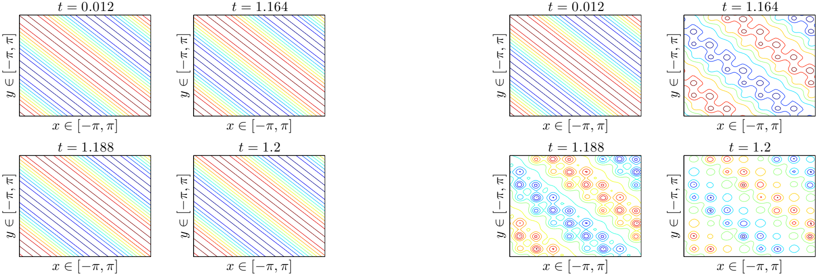

Figure: Splitting method for (QZ): Soliton simulation. Left: Strongly quantum regime $\vartheta = 1$. Right: "Classical regime" $\vartheta = 0.01$.

Idea: Consider (QZ*) system in $(E_\vartheta, \partial_t E_\vartheta =: F_\vartheta, u_\vartheta, \partial_t u_\vartheta)$!

S. (15'): Trigonometric time-integration scheme for (QZ$^{(*)}$):

$$F_\vartheta^{n+1} = \mathrm{e}^{-i\tau\Omega_\vartheta^2} F_\vartheta^n + i\tau \frac{1-\mathrm{e}^{-i\tau\Omega_\vartheta^2}}{-i\tau\Omega_\vartheta^2} \left(u_\vartheta^n F_\vartheta^n + u_\vartheta'^n E_\vartheta^0 + u_\vartheta'^n \big(\tau \sum_{k=0}^{n} F_\vartheta^k\big) \right),$$

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$[\Omega_\vartheta^2 := -\Delta + \vartheta \Delta^2]$

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$[\Omega_\vartheta^2 := -\Delta + \vartheta\Delta^2]$

Uniformly accurate in

+ τ w.r.t. to ϑ [$E_\vartheta(t_n) - E_\vartheta^n = \mathcal{O}(\tau)$]

+ ϑ w.r.t. to τ [Quantum to classical Approx. $E_0^n - E_\vartheta^n = \mathcal{O}(\vartheta)$]

(no $\Delta x-$ nor $\vartheta-$ CFL!)

(Quantum) Zakharov

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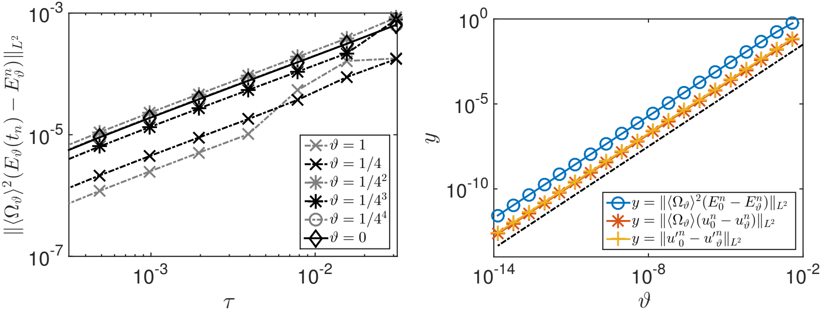

Figure: Orderplot (double logarithmic). Left: convergence in time (several ϑ). Right: convergence in ϑ ($\tau = 10^{-2}$).

Ongoing projects in this ”direction”:

- With S. Herr & F. Rousset: Trigonometric integrators for the Zakharov system in the subsonic regime

$$i\partial_t E + \Delta E = uE,\ \alpha^{-2}\partial_{tt}u - \Delta u = \Delta|E|^2 \qquad (\alpha \gg 1)$$

- With L. Gauckler: Trigonometric integrators for the quadratic quasilinear wave equation

$$\partial_{tt}u - \partial_{xx}u + u = \frac{1}{2}(u^2)_{xx}$$

Preprints:

(Zakharov)

- S. Herr, K. Schratz (15'): Trigonometric integrators for the Zakharov system
- K. Schratz (15'): Splitting methods for the quantum Zakharov system
- K. Schratz (15'): Uniformly accurate trigonometric integrators for the quantum Zakharov system

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(Highly-oscillatory Klein-Gordon-type eq.s)

- P. Krämer, K. Schratz (15'): Efficient time-integration of the Maxwell-Klein-Gordon system in the non-relativistic regime
- M. Daub, G. Schneider, K. Schratz (15'): From the Klein-Gordon-Zakharov to the Klein-Gordon equation