Karlsruhe Institute of Technology

On Lawson methods and trees

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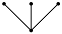
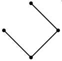
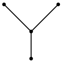
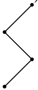
$$y^{(4)}(0) = \big(f'''[f,f,f]\big)(y_0) + 3\big(f''[f'f,f]\big)(y_0)$$

$$+ \big(f'f''[f,f]\big)(y_0) + \big(f'f'f'f\big)(y_0)$$

Outline

1. Lawson methods

2. Outdoor excursion

3. Order and convergence

4. Example: Linear problems

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Problem

consider semilinear stiff problem

$$u'(t) = Au(t) + g\big(u(t)\big), \qquad u(0) = u_0$$

- A matrix of large norm or A differential operator (unbounded) s.t.

  $$\left\| \mathrm{e}^{tA} \right\| \le C, \qquad t \ge 0$$

  w.l.o.g. $C = 1$ (for $C > 1$ use $\|v\|_\star = \sup_{t\ge 0} \left\|\mathrm{e}^{tA} v\right\|$)
- g “nice”

applications: (discretizations of) pdes

- heat equation, convection diffusion equation, etc
- (nonlinear) Schrödinger equation, Maxwell equations

Lawson methods, 1967: key idea

$$u'(t) = Au(t) + g\big(u(t)\big), \qquad u(0) = u_0$$

- transformation of variables

$$w(t) = \mathrm{e}^{-tA}u(t)$$

- differentiation yields (hopefully) nonstiff ode for w

$$w'(t) = \mathrm{e}^{-tA}(-Au + u') = \mathrm{e}^{-tA}g(u) = \mathrm{e}^{-tA}g\big(\mathrm{e}^{tA}w\big)$$

Lawson method:

- solve ode for w with explicit Runge–Kutta method
- transform back to original u variables

Lawson methods

s-stage Runge–Kutta method given by a_{ij}, b_i, c_i

$$U_i = \mathrm{e}^{c_i hA} u_0 + h \sum_{j=1}^{i-1} a_{ij} \mathrm{e}^{(c_i - c_j)hA} g(U_j), \qquad i = 1, \dots, s$$

$$u_1 = \mathrm{e}^{hA} u_0 + h \sum_{i=1}^{s} b_i \mathrm{e}^{(1-c_i)hA} g(U_i)$$

example: Lawson-Euler method:

$$u_1 = \mathrm{e}^{hA} u_0 + h\mathrm{e}^{hA} g(u_0)$$

Lawson methods

s-stage Runge–Kutta method given by a_{ij}, b_i, c_i

$$U_i = \mathrm{e}^{c_i hA} u_0 + h \sum_{j=1}^{i-1} a_{ij} \mathrm{e}^{(c_i - c_j)hA} g(U_j), \qquad i = 1, \ldots, s$$

$$u_1 = \mathrm{e}^{hA} u_0 + h \sum_{i=1}^{s} b_i \mathrm{e}^{(1-c_i)hA} g(U_i)$$

discussion:

- if $c_1 \leq \ldots \leq c_s$, then scheme is suited for parabolic and hyperbolic problems (excludes Dopri, etc.)
- otherwise, we need $\left\| \mathrm{e}^{tA} \right\| \leq 1$ for all $t \in \mathbb{R}$
- requires evaluation or approximation of $\mathrm{e}^{hA} v$
- special case of exponential integrator (using only exponentials)

Failure of Lawson methods

consider scalar ivp

$$u'(t) = Au(t) + 1, \qquad u(0) = u_0 = -A^{-1}, \qquad A < 0$$

with solution $u(t) = u(0) = -A^{-1}$

- exponential Euler method is exact:

$$u_1 = \mathrm{e}^{hA} u_0 + h\varphi_1(hA) = u_0, \qquad \varphi_1(z) = \frac{e^z - 1}{z}$$

- Lawson Euler method

$$u_1 = \mathrm{e}^{hA} u_0 + h\mathrm{e}^{hA} = \mathrm{e}^{hA}(-A^{-1} + h)$$

  gives reasonable results only in nonstiff case $hA \to 0$

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convergence analysis: H., Ostermann, 2005

Success of Lawson methods

in

- Kassam, Trefethen, 2005 (integrated factor method):
  KdV, Burgers, Kuramoto-Sivashinsky, Allen-Cahn, periodic b.c.
- Cano, Gonzáles-Pachón, 2014:
  nonlinear Schrödinger equation, periodic b.c.
- Balac, Fernandez, Mahé, Méhats, Texier-Picar, 2014:
  generalized nonlinear Schrödinger equation in optics

full order of convergence is observed numerically

aim of this talk:

- explain this behavior theoretically:

  If the solution is sufficiently regular, then the Lawson method converges with the same order as the underlying Runge–Kutta method.

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Classical order of RK methods

reminder: how to prove error estimates for RK methods

consider autonomous ivp

$$y' = f(y), \qquad y(0) = y_0,$$

with f sufficiently smooth

Taylor's theorem

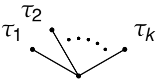
$$y(h) = \sum_{k=0}^{p} y^{(k)}(0)\frac{h^k}{k!} + O(h^{p+1}), \qquad h \to 0$$

higher derivatives of y obtained by repeated differentiation of the ode

Trees

Taylor expansion of exact solution

- order $\varrho(\tau)$ = number of nodes of τ
- elementary differential $D(\tau)$ defined recursively by
  - $D(\bullet)(y) = f(y)$,
  - $D(\tau)(y) = f^{(k)}(y)\Big[D(\tau_1)(y), \ldots, D(\tau_k)(y)\Big]$ for $\tau =$



Theorem (Butcher, 1963; Hairer, Wanner, 1974; ...)

The solution of $y' = f(y)$, $y(0) = y_0$ satisfies

$$y^{(k)}(0) = \sum_{\substack{\tau \in \mathcal{T} \\ \varrho(\tau)=k}} \alpha(\tau) D(\tau)(y_0), \qquad k = 1, 2, 3, \ldots$$

for certain coefficients $\alpha(\tau)$, which are independent of the ode.

Taylor expansion of numerical solution

for $y' = f(y)$

$$y_1 = y_0 + h \sum_{i=1}^{s} b_i f(Y_i), \qquad Y_i = y_0 + h \sum_{j=1}^{s} a_{ij} f(Y_j)$$

Theorem (Hairer, Wanner, 1974, ...)

The numerical solution $y_1 \approx y(h)$ satisfies

$$y_1^{(k)}(0) = \sum_{\substack{\tau \in \mathcal{T} \\ \varrho(\tau) = k}} \phi(\tau) \alpha(\tau) D(\tau)(y_0), \qquad k = 1, 2, 3, \ldots$$

with the same coefficients $\alpha(\tau)$ as for the exact solution.

conclusion: RK method is of order p if $\phi(\tau) = 1$ for all τ with $\varrho(\tau) \leq p$

RK methods vs Lawson methods

Runge–Kutta method

for $y' = f(y)$, $y(0) = y_0$

$$y_1 = y_0 + h \sum_{i=1}^{s} b_i f(Y_i),$$

$$Y_i = y_0 + h \sum_{j=1}^{i-1} a_{ij} f(Y_j)$$

exact solution

$$y(h) = y_0 + \int_0^h f\big(y(\sigma)\big) \mathrm{d}\sigma$$

Lawson method

for $u' = Au + g(u)$, $u(0) = u_0$

$$u_1 = \mathrm{e}^{hA} u_0 + h \sum_{i=1}^{s} b_i \mathrm{e}^{(1-c_i)hA} g(U_i),$$

$$U_i = \mathrm{e}^{c_i hA} u_0 + h \sum_{j=1}^{i-1} a_{ij} \mathrm{e}^{(c_i-c_j)hA} g(U_j)$$

exact solution

$$u(h) = \mathrm{e}^{hA} u_0 + \int_0^h \mathrm{e}^{(h-\sigma)A} g\big(u(\sigma)\big) \mathrm{d}\sigma$$

Iterated v.o.c. formula

notation:

$$g_\eta = g\big(\mathrm{e}^{\eta hA}u_0\big), \qquad g_\eta^{(k)} = g^{(k)}\big(\mathrm{e}^{\eta hA}u_0\big), \qquad k \geq 1$$

$$\begin{aligned}
u(h) &= \mathrm{e}^{hA}u_0 + h\int_0^1 \mathrm{e}^{(1-\sigma)hA} g\big(u(\sigma h)\big)\mathrm{d}\sigma \\
&= \mathrm{e}^{hA}u_0 + h\int_0^1 \mathrm{e}^{(1-\sigma)hA} g\Big(\mathrm{e}^{\sigma hA}u_0 + h\int_0^\sigma \mathrm{e}^{(\sigma-\eta)hA} g\big(u(\eta h)\big)\mathrm{d}\eta\Big)\mathrm{d}\sigma \\
&= \mathrm{e}^{hA}u_0 + h\int_0^1 \mathrm{e}^{(1-\sigma)hA} g_\sigma \mathrm{d}\sigma \\
&\quad + h^2 \int_0^1 \mathrm{e}^{(1-\sigma)hA} g'_\sigma \int_0^\sigma \mathrm{e}^{(\sigma-\eta)hA} g_\eta \mathrm{d}\eta\mathrm{d}\sigma \\
&\quad + \mathcal{O}(h^3)
\end{aligned}$$

Lubich, Jahnke, 2000; Thalhammer, 2008; Lubich, 2008

Iterated v.o.c. formula, cont'd

notation:

$$g_\eta = g\big(\mathrm{e}^{\eta hA}u_0\big), \qquad g_\eta^{(k)} = g^{(k)}\big(\mathrm{e}^{\eta hA}u_0\big), \qquad k \ge 1$$

up to order four:

$$\begin{aligned}
u(h) = \mathrm{e}^{hA}u_0 &+ h\int_0^1 \mathrm{e}^{(1-\sigma)hA} g_\sigma \,\mathrm{d}\sigma \\
&+ h^2 \int_0^1 \mathrm{e}^{(1-\sigma)hA} g'_\sigma \int_0^\sigma \mathrm{e}^{(\sigma-\eta)hA} g_\eta \,\mathrm{d}\eta\,\mathrm{d}\sigma \\
&+ h^3 \int_0^1 \mathrm{e}^{(1-\sigma)hA} g'_\sigma \int_0^\sigma \mathrm{e}^{(\sigma-\eta)hA} g'_\eta \int_0^\eta \mathrm{e}^{(\eta-\xi)hA} g_\xi \,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\sigma \\
&+ \tfrac{1}{2}h^3 \int_0^1 \mathrm{e}^{(1-\sigma)hA} g''_\sigma \Big[\int_0^\sigma \mathrm{e}^{(\sigma-\eta)hA} g_\eta \,\mathrm{d}\eta, \int_0^\sigma \mathrm{e}^{(\sigma-\xi)hA} g_\xi \,\mathrm{d}\xi\Big] \mathrm{d}\sigma \\
&+ \mathcal{O}(h^4)
\end{aligned}$$

Iterated v.o.c. formula, cont'd

notation:

$$g_\eta = g\big(\mathrm{e}^{\eta hA}u_0\big), \qquad g_\eta^{(k)} = g^{(k)}\big(\mathrm{e}^{\eta hA}u_0\big), \qquad k \geq 1$$

up to order four:

$$\begin{aligned} u(h) = \mathrm{e}^{hA}u_0 &+ h\int_0^1 \mathrm{e}^{(1-\sigma)hA} g_\sigma \,\mathrm{d}\sigma \\ &+ h^2 \int_0^1 \mathrm{e}^{(1-\sigma)hA} g'_\sigma \int_0^\sigma \mathrm{e}^{(\sigma-\eta)hA} g_\eta \,\mathrm{d}\eta\,\mathrm{d}\sigma \\ &+ h^3 \int_0^1 \mathrm{e}^{(1-\sigma)hA} g'_\sigma \int_0^\sigma \mathrm{e}^{(\sigma-\eta)hA} g'_\eta \int_0^\eta \mathrm{e}^{(\eta-\xi)hA} g_\xi \,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\sigma \\ &+ \tfrac{1}{2}h^3 \int_0^1 \mathrm{e}^{(1-\sigma)hA} g''_\sigma \Big[\int_0^\sigma \mathrm{e}^{(\sigma-\eta)hA} g_\eta \,\mathrm{d}\eta, \int_0^\sigma \mathrm{e}^{(\sigma-\xi)hA} g_\xi \,\mathrm{d}\xi\Big]\mathrm{d}\sigma \\ &+ \mathcal{O}(h^4) \end{aligned}$$

Iterated v.o.c. formula, cont'd

notation:

$$g_\eta = g\big(\mathrm{e}^{\eta hA} u_0\big), \qquad g_\eta^{(k)} = g^{(k)}\big(\mathrm{e}^{\eta hA} u_0\big), \qquad k \geq 1$$

up to order four:

$$\begin{aligned} u(h) = \mathrm{e}^{hA} u_0 &+ h \int_0^1 \mathrm{e}^{(1-\sigma)hA} g_\sigma \,\mathrm{d}\sigma \\ &+ h^2 \int_0^1 \mathrm{e}^{(1-\sigma)hA} g'_\sigma \int_0^\sigma \mathrm{e}^{(\sigma-\eta)hA} g_\eta \,\mathrm{d}\eta \,\mathrm{d}\sigma \\ &+ h^3 \int_0^1 \mathrm{e}^{(1-\sigma)hA} g'_\sigma \int_0^\sigma \mathrm{e}^{(\sigma-\eta)hA} g'_\eta \int_0^\eta \mathrm{e}^{(\eta-\xi)hA} g_\xi \,\mathrm{d}\xi \,\mathrm{d}\eta \,\mathrm{d}\sigma \\ &+ \tfrac{1}{2} h^3 \int_0^1 \mathrm{e}^{(1-\sigma)hA} g''_\sigma \Big[\int_0^\sigma \mathrm{e}^{(\sigma-\eta)hA} g_\eta \,\mathrm{d}\eta, \int_0^\sigma \mathrm{e}^{(\sigma-\xi)hA} g_\xi \,\mathrm{d}\xi \Big] \mathrm{d}\sigma \\ &+ \mathcal{O}(h^4) \end{aligned}$$

Iterated v.o.c. formula, cont'd

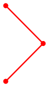
notation:

$$g_\eta = g\big(\mathrm{e}^{\eta hA}u_0\big), \qquad g_\eta^{(k)} = g^{(k)}\big(\mathrm{e}^{\eta hA}u_0\big), \qquad k \geq 1$$

up to order four:

$$u(h) = \mathrm{e}^{hA}u_0 + h\int_0^1 \mathrm{e}^{(1-\sigma)hA} g_\sigma \,\mathrm{d}\sigma$$

$$+ h^2 \int_0^1 \mathrm{e}^{(1-\sigma)hA} g_\sigma' \int_0^\sigma \mathrm{e}^{(\sigma-\eta)hA} g_\eta \,\mathrm{d}\eta\,\mathrm{d}\sigma$$



$$+ h^3 \int_0^1 \mathrm{e}^{(1-\sigma)hA} g_\sigma' \int_0^\sigma \mathrm{e}^{(\sigma-\eta)hA} g_\eta' \int_0^\eta \mathrm{e}^{(\eta-\xi)hA} g_\xi \,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\sigma$$

$$+ \tfrac{1}{2}h^3 \int_0^1 \mathrm{e}^{(1-\sigma)hA} g_\sigma'' \Big[\int_0^\sigma \mathrm{e}^{(\sigma-\eta)hA} g_\eta \,\mathrm{d}\eta, \int_0^\sigma \mathrm{e}^{(\sigma-\xi)hA} g_\xi \,\mathrm{d}\xi\Big]\,\mathrm{d}\sigma$$

$$+ \mathcal{O}(h^4)$$

Iterated v.o.c. formula, cont'd

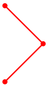
notation:

$$g_\eta = g\big(\mathrm{e}^{\eta hA}u_0\big), \qquad g_\eta^{(k)} = g^{(k)}\big(\mathrm{e}^{\eta hA}u_0\big), \qquad k \geq 1$$

up to order four:

$$u(h) = \mathrm{e}^{hA}u_0 + h\int_0^1 \mathrm{e}^{(1-\sigma)hA}{\color{red}g_\sigma}\,\mathrm{d}\sigma$$

$$+\,h^2\int_0^1 \mathrm{e}^{(1-\sigma)hA}{\color{red}g'_\sigma}\int_0^\sigma \mathrm{e}^{(\sigma-\eta)hA}{\color{red}g_\eta}\,\mathrm{d}\eta\,\mathrm{d}\sigma$$



$$+\,h^3\int_0^1 \mathrm{e}^{(1-\sigma)hA}{\color{red}g'_\sigma}\int_0^\sigma \mathrm{e}^{(\sigma-\eta)hA}{\color{red}g'_\eta}\int_0^\eta \mathrm{e}^{(\eta-\xi)hA}{\color{red}g_\xi}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\sigma$$

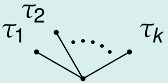
$$+\,\tfrac{1}{2}h^3\int_0^1 \mathrm{e}^{(1-\sigma)hA}{\color{red}g''_\sigma}\Big[\int_0^\sigma \mathrm{e}^{(\sigma-\eta)hA}{\color{red}g_\eta}\,\mathrm{d}\eta,\int_0^\sigma \mathrm{e}^{(\sigma-\xi)hA}{\color{red}g_\xi}\,\mathrm{d}\xi\Big]\mathrm{d}\sigma$$

$$+\,\mathcal{O}(h^4)$$

Elementary integrals

Definition

For $\tau \in \mathcal{T}$ and $0 \leq \zeta \leq 1$ we define $G_\zeta(\tau)$ recursively as:

- $G_\zeta(\bullet)(v) = \int_0^\zeta \mathrm{e}^{(\zeta-\sigma)hA} g(\mathrm{e}^{\sigma hA} v)\,\mathrm{d}\sigma$
- for $\tau =$ 

set

$$G_\zeta(\tau)(v) = \int_0^\zeta \mathrm{e}^{(\zeta-\sigma)hA} g^{(k)}(\mathrm{e}^{\sigma hA} v)\left[G_\sigma(\tau_1)(v), \ldots, G_\sigma(\tau_k)(v)\right]\mathrm{d}\sigma.$$

- $G(\tau) = G_1(\tau)$; $F(\tau)$ denotes integrand of $G(\tau)$.

$G(\tau)$ is $\varrho(\tau)$-fold multivariate integral, $F(\tau)$ is function of $\varrho(\tau)$ variables.

Expansion of exact solution

Theorem

The solution of $u' = Au + g(u)$, $u(0) = u_0$ *satisfies*

$$u(h) = \mathrm{e}^{hA}u_0 + \sum_{\tau\in\mathcal{T}} h^{\varrho(\tau)}\gamma(\tau)G(\tau)(u_0)$$

with certain coefficients $\gamma(\tau)$ *which are independent of the differential equation.*

Proof. Isomorphism $\tau \simeq D(\tau) \simeq G(\tau)$

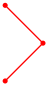
Expansion of numerical solution

$$
\begin{aligned}
u_1 = \mathrm{e}^{hA} u_0 &+ h \sum_{i=1}^{s} b_i \mathrm{e}^{(1-c_i)hA} g_{c_i} \\
&+ h^2 \sum_{i=1}^{s} b_i \mathrm{e}^{(1-c_i)hA} g'_{c_i} \sum_{j=1}^{i-1} a_{ij} \mathrm{e}^{(c_i-c_j)hA} g_{c_j} \\
&+ h^3 \sum_{i=1}^{s} b_i \mathrm{e}^{(1-c_i)hA} g'_{c_i} \sum_{j=1}^{i-1} a_{ij} \mathrm{e}^{(c_i-c_j)hA} g'_{c_j} \sum_{k=1}^{j-1} a_{jk} \mathrm{e}^{(c_j-c_k)hA} g_{c_k} \\
&+ \tfrac{1}{2} h^3 \sum_{i=1}^{s} b_i \mathrm{e}^{(1-c_i)hA} g''_{c_i} \Big[\sum_{j=1}^{i-1} a_{ij} \mathrm{e}^{(c_i-c_j)hA} g_{c_j}, \sum_{k=1}^{i-1} a_{ik} \mathrm{e}^{(c_i-c_k)hA} g_{c_k} \Big] \\
&+ \mathcal{O}(h^4)
\end{aligned}
$$

Expansion of numerical solution

$$u_1 = \mathrm{e}^{hA} u_0 + h \sum_{i=1}^{s} b_i \mathrm{e}^{(1-c_i)hA} g_{c_i}$$

$$+ h^2 \sum_{i=1}^{s} b_i \mathrm{e}^{(1-c_i)hA} g'_{c_i} \sum_{j=1}^{i-1} a_{ij} \mathrm{e}^{(c_i-c_j)hA} g_{c_j}$$



$$+ h^3 \sum_{i=1}^{s} b_i \mathrm{e}^{(1-c_i)hA} g'_{c_i} \sum_{j=1}^{i-1} a_{ij} \mathrm{e}^{(c_i-c_j)hA} g'_{c_j} \sum_{k=1}^{j-1} a_{jk} \mathrm{e}^{(c_j-c_k)hA} g_{c_k}$$

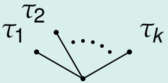
$$+ \frac{1}{2} h^3 \sum_{i=1}^{s} b_i \mathrm{e}^{(1-c_i)hA} g''_{c_i} \Big[\sum_{j=1}^{i-1} a_{ij} \mathrm{e}^{(c_i-c_j)hA} g_{c_j}, \sum_{k=1}^{i-1} a_{ik} \mathrm{e}^{(c_i-c_k)hA} g_{c_k} \Big]$$

$$+ \mathcal{O}(h^4)$$

Elementary quadrature formulas

Definition

For $\tau \in \mathcal{T}$ we define $\widehat{G}(\tau)$ recursively as:

- $\widehat{G}(\bullet)(v) = \sum_{i=1}^{s} b_i \mathrm{e}^{(1-c_i)hA} g_{c_i}, \qquad \widehat{G}_i(\bullet)(v) = \sum_{j=1}^{i-1} a_{ij} \mathrm{e}^{(c_i-c_j)hA} g_{c_j}$
- for $\tau =$ 

we set

$$\widehat{G}(\tau)(v) = \sum_{i=1}^{s} b_i \mathrm{e}^{(1-c_i)hA} g_{c_i}^{(k)} \big[\widehat{G}_i(\tau_1)(v), \dots, \widehat{G}_i(\tau_k)(v)\big]$$

$$\widehat{G}_i(\tau)(v) = \sum_{j=1}^{i-1} a_{ij} \mathrm{e}^{(c_i-c_j)hA} g_{c_j}^{(k)} \big[\widehat{G}_j(\tau_1)(v), \dots, \widehat{G}_j(\tau_k)(v)\big]$$

Expansion of numerical solution

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Theorem

The Lawson approximation satisfies

$$u_1 = \mathrm{e}^{hA} u_0 + \sum_{\tau \in \mathcal{T}} h^{\varrho(\tau)} \gamma(\tau) \widehat{G}(\tau)(u_0)$$

with the same coefficients $\gamma(\tau)$ as for the exact solution.

Proof. Isomorphism $\tau \simeq D(\tau) \simeq G(\tau) \simeq \widehat{G}(\tau)$

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Example: methods of order one

$$u_1 = \mathrm{e}^{hA} u_0 + h \sum_{i=1}^{s} b_i \mathrm{e}^{(1-c_i)hA} g(\mathrm{e}^{c_i hA} u_0) + \mathcal{O}(h^2)$$

$$u(h) = \mathrm{e}^{hA} u_0 + h \int_0^1 \mathrm{e}^{(1-\sigma)hA} g(\mathrm{e}^{\sigma hA} u_0) \mathrm{d}\sigma + \mathcal{O}(h^2)$$

hence $u_1 - u(h) = \mathcal{O}(h^2)$ if

$$\widehat{G}(\bullet)(u_0) - G(\bullet)(u_0) = \mathcal{O}(h)$$

well known:

$$\widehat{G}(\bullet)(u_0) - G(\bullet)(u_0) = h \int_0^1 \kappa_p(\sigma) F'(\bullet)(\sigma) \mathrm{d}\sigma$$

order only depends on bounds on $F'(\bullet)$

Convergence

Theorem

A Lawson method is of order p if

$$\widehat{G}(\tau)(u_0) - G(\tau)(u_0) = \mathcal{O}(h^{p+1-\varrho(\tau)}), \qquad \text{for all } \tau \in \mathcal{T}, \quad \varrho(\tau) \le p$$

Proof. Follows directly from expansion of exact and numerical solution.

Theorem

If

$$F(\tau) \in C^{p+1-\varrho(\tau)} \qquad \text{for all } \tau \in \mathcal{T}, \quad \varrho(\tau) \le p$$

and if the underlying the Runge–Kutta method is of order p, then the Lawson method is of order p.

Main result

Theorem

If

- $F(\tau) \in C^{p+1-\varrho(\tau)}$ *for all* $\tau \in \mathcal{T}$ *with* $\varrho(\tau) \leq p$ *and*
- *underlying Runge–Kutta method is of order* p
- $0 < h \leq h_0$

then the Lawson method converges with order p*, i.e.*

$$\|u(t_n) - u_n\| \leq Ch^p, \qquad t_n = nh \leq T,$$

where C *and* h_0 *are independent of* n*,* h*, and* A*.*

Sketch of proof

- expand $F(\tau)(\sigma_1, \ldots, \sigma_{\varrho(\tau)})$ into a Taylor polynomial of degree $p + 1 - \varrho(\tau)$
- underlying RK method is of order k, hence multivariate quadrature formula is exact for all polynomials of degree $p + 1 - \varrho(\tau)$
- stability
- Lady Windermere's fan

Outline

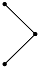
Linear problems

$$u' = Au + Bu, \qquad u(0) = u_0, \qquad B \text{ bounded}$$

integrands of elementary integrals

$$F(\bullet)(v)(\sigma) = \mathrm{e}^{(1-\sigma)hA} B \mathrm{e}^{\sigma hA} v$$

$F($  $)(v)(\sigma) = \mathrm{e}^{(1-\sigma_1)hA} B \mathrm{e}^{(\sigma_1-\sigma_2)hA} B \mathrm{e}^{\sigma_2 hA} v$

$F($  $)(v)(\sigma) = \mathrm{e}^{(1-\sigma_1)hA} B \mathrm{e}^{(\sigma_1-\sigma_2)hA} B \mathrm{e}^{(\sigma_2-\sigma_3)hA} B \mathrm{e}^{\sigma_3 hA} v$

- order one: $F(\bullet) \in C^1$

$$F'(\bullet)(v) = \mathrm{e}^{(1-\sigma)hA}[B, A]\mathrm{e}^{\sigma hA} v, \qquad [B, A] = BA - AB$$

- same regularity condition as for splitting methods (Jahnke, Lubich, 2000; Thalhammer, 2008; Lubich, 2008)

Linear problems – order two

- order two: $F(\tau) \in C^{3-\varrho(\tau)}$, $\varrho(\tau) \leq 2$

$$F''(\bullet)(v)(\sigma) = \mathrm{e}^{(1-\sigma)hA}[A,[A,B]]\mathrm{e}^{\sigma hA}v$$

$$F'({}_{\bullet}\!\diagup^{\bullet})(v)(\sigma) = \Big[\mathrm{e}^{(1-\sigma_1)hA}[B,A]\mathrm{e}^{(\sigma_1-\sigma_2)hA}B\mathrm{e}^{\sigma_2 hA}v,$$

$$\mathrm{e}^{(1-\sigma_1)hA}B\mathrm{e}^{(\sigma_1-\sigma_2)hA}[B,A]\mathrm{e}^{\sigma_2 hA}v\Big]$$

Summary

- Lawson methods for $u' = Au + g(u)$
- expansion of exact and numerical solution based on iterated v.o.c. formula
- interprete Lawson methods as multivariate quadrature formulas
- convergence result showing exactly the required regularity assumptions
- for linear problems: same assumptions as for splitting methods
- for semilinear problems: additional terms, not just commutators
- generalization to nonlinear v.o.c. formula possible