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A note on the Gautschi-type method for oscillatory second-order differential equations

Volker Grimm

Heinrich-Heine-Universität, Mathematisches Institut, Lehrstuhl für Angewandte Mathematik, Universitätsstraße 1, 40225 Düsseldorf, Germany.
e-mail: grimm@am.uni-duesseldorf.de

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Summary The Gautschi-type method has been proposed by Hochbruck and Lubich for oscillatory second-order differential equations. They conjecture that this method allows for a uniform error bound independent of the size of the system. The conjecture is proved in this note.

1 Introduction

In [5], Hochbruck and Lubich consider the Gautschi-type method for the solution of systems of oscillatory second-order differential equations

$$y'' = -Ay + g(y), \qquad y(0) = y_0,\, y'(0) = y_0', \tag{1}$$

where A is a symmetric and positive semi-definite real matrix of arbitrarily large norm. The aim is to use step sizes that are not restricted by the large norm of A or, in more physical terms, by the frequencies of A, which are the eigenvalues of $\Omega := \sqrt{A}$. The Gautschi-type method, which is based on the requirement that it solves exactly linear problems with constant inhomogeneity g, is given by

$$y_{n+1} - 2\cos(h\Omega)\, y_n + y_{n-1} = h^2 \operatorname{sinc}^2\left(\frac{h\Omega}{2}\right) g(\phi(h\Omega)y_n),$$

with the *filter function* ϕ whose purpose is to filter out resonant frequencies at integer multiples of π.

The following error bound for the Gautschi-type method is proved in [5].

Theorem 1 *If the solution of system (1) satisfies the finite-energy condition*

$$\frac{1}{2}y'(t)^T y'(t) + \frac{1}{2}y(t)^T Ay(t) \leq \frac{1}{2}K^2$$

for $0 \leq t \leq T$*, then the error of the Gautschi-type method for* $0 \leq t_n = nh \leq T$ *is bounded by,*

$$\|y(t_n) - y_n\| \leq h^2 C\,\ell(n, N),$$

where C *depends on* $\|y(t_0)\|, T,$ $\|g\|, \|g_y\|, \|g_{yy}\|,$ ϕ *and* K*. The term* $\ell(n, N) := \min\{\log(n+1)\ \log(N+1), \sqrt{N}\}$ *is slowly growing with the number* n *of steps taken and the dimension* N *of system (1).*

The conjecture is that the term $\ell(n, N)$ can be dropped. This question is of obvious interest if the system (1) arises from a semi-discretisation of wave equations.

The conjecture is proved in Section 2 by using an alternative technique of estimating the componentwise product $E_n \bullet G$ of the Jacobian G of g with a matrix E_n describing all possible resonances between the frequencies.

2 Proof of the uniform error bound

To prove the conjecture of Hochbruck and Lubich in [5], one has to bound $\|E_n \bullet G\|$ in the norm induced by the Euclidean norm, where $E_n = (\epsilon_n(\alpha_j, \alpha_k))_{j,k=1}^N$ and

$$\epsilon_n(\alpha, \beta) = \frac{1}{n}\frac{1}{\beta} S_n(\alpha, \beta) I(\alpha, \beta),$$

with

$$S_n(\alpha, \beta) = 2\sum_{j=0}^{n-1} \frac{\sin(j+1)\alpha}{\sin\alpha} e^{-ij\beta}$$

and

$$I(\alpha, \beta) = -\left(\frac{\cos\beta - \cos\alpha}{\beta^2 - \alpha^2} + \frac{1}{2} \cdot \frac{\sin^2\frac{\alpha}{2}}{\left(\frac{\alpha}{2}\right)^2} \cdot \phi(\beta) \right).$$

A uniform bound is stated in Lemma 1 and proved for the filter function

$$\phi(\beta) = \frac{\sin\beta}{\beta}. \tag{2}$$

This filter function is chosen for the sake of a simpler proof, but the technique of the proof can be used to derive bounds of the same type for many more filter functions.

Lemma 1 *Let E_n be the matrix $(\epsilon_n(\alpha_j,\beta_k))_{j,k=1}^N$, where α_j, β_k are arbitrary real numbers and the filter function (2) is used. In the matrix norm induced by the Euclidean norm the componentwise product of E_n with an arbitrary $N\times N$ matrix G is bounded by*

$$\|E_n \bullet G\| \le \frac{1}{6}\|G\|.$$

Note: The bound is independent of n and of the size N of the matrices involved.

Proof The crux is to use the two-dimensional Fourier transform. If it can be shown that $\epsilon_n(\alpha,\beta)$ is the Fourier transform of an L^1-function $\check{\epsilon}_n(x,y)$, that is

$$\epsilon_n(\alpha,\beta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \check{\epsilon}_n(x,y)e^{-i\alpha x - i\beta y}\,dxdy =: \mathcal{F}(\check{\epsilon}_n(x,y)), \tag{3}$$

then it holds that

$$E_n \bullet G = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \check{\epsilon}_n(x,y)\,D_\alpha(x)\ \cdot G \cdot\ D_\beta(y)\,dxdy$$

with

$$\begin{aligned} D_\alpha(x) &= \operatorname{diag}(e^{-i\alpha_1 x},\cdots,e^{-i\alpha_N x}),\\ D_\beta(y) &= \operatorname{diag}(e^{-i\beta_1 y},\cdots,e^{-i\beta_N y}).\end{aligned}$$

Hence it follows

$$\|E_n \bullet G\| \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\check{\epsilon}_n(x,y)|\|G\|dxdy = \|\check{\epsilon}_n\|_1\|G\|.$$

Standard Fourier techniques can be used to find $\check{\epsilon}_n$. An indicator function for the set A is designated by $1_A(x,y)$ in the following. The equations

$$\frac{\cos\beta - \cos\alpha}{\beta^2-\alpha^2} = \mathcal{F}\left(-\frac{1}{4}1_{\{|x|+|y|\le 1\}}(x,y)\right)$$

and

$$\frac{1}{2}\cdot\frac{\sin^2\frac{\alpha}{2}}{\left(\frac{\alpha}{2}\right)^2}\cdot\underbrace{\frac{\sin\beta}{\beta}}_{=\phi(\beta)} = \mathcal{F}\left(\frac{1}{4}(1-|x|)1_{[-1,1]^2}(x,y)\right)$$

are readily justified. The second equation follows by two one-dimensional Fourier transformations and this is where the filter function $\phi(\beta)$ enters the proof. The one-dimensional inverse Fourier transform of the filter function $\phi(\beta)$ needs to be known. If one defines

$$h(x,y) := -\frac{1}{4}1_{\{|x|+|y|\le 1\}}(x,y) + \frac{1}{4}(1-|x|)1_{[-1,1]^2}(x,y)$$

then

$$-I(\alpha,\beta) = \mathcal{F}\left(h(x,y)\right),$$

and therefore

$$\frac{i}{\beta} I(\alpha,\beta) = \mathcal{F}(f(x,y)), \quad \text{with} \quad f(x,y) := \int_{-\infty}^{y} h(x,v)\,dv, \tag{4}$$

where $f \in L^1$ has bounded support. A straightforward calculation using (4) now shows that $\epsilon_n(\alpha,\beta)$ (with filter (2)) is the Fourier transform of the function

$$\check{\epsilon}_n(x,y) = -\frac{2i}{n}\sum_{j=0}^{n-1}\sum_{l=0}^{j} f(x-j+2l, y-j), \tag{5}$$

where

$$f(x,y) = \begin{cases} 0 & \text{for } |x| \geq 1 \text{ or } |y| \geq 1 \\ \frac{1}{4}(1-|x|)(y+1) & \text{for } |x| < 1, -1 < y \leq -1+|x| \\ -\frac{1}{4}|x|y & \text{for } |x| < 1, -1+|x| < y < 1-|x| \\ \frac{1}{4}(1-|x|)(y-1) & \text{for } |x| < 1, 1-|x| \leq y < 1. \end{cases}$$

Since f has bounded support, the same is true for $\check{\epsilon}_n$ for fixed n, and hence it is in L^1. The bound $\|\check{\epsilon}_n\|_1 = 1/6$ for all n is derived by calculating the L^1-norm of $\check{\epsilon}_n$ explicitly. The support of $\check{\epsilon}_n$ is a triangle composed of squares. Fortunately, the shifted versions of function f in (5) cancel out each other in the interior of the triangle and $\check{\epsilon}_n$ is only non-zero on the boundary consisting of $4n$ unit squares where the function $\check{\epsilon}_n$ has L^1-norm $(2/n)\cdot\|f\|_1/4 = 1/(24n)$ (cf. Fig. 1). Hence $\|\check{\epsilon}_n\|_1 = 4n/(24n) = 1/6$. The pictorial calculation of $\|\check{\epsilon}_n\|_1$ can be conducted rigorously by using the function

$$b(x,y) = \begin{cases} -\frac{1}{4}xy & \text{for } y < 1-x \\ \frac{1}{4}(1-x)(y-1) & \text{for } 1-x \leq y \end{cases}$$

to write f as

$$\begin{aligned} f(x,y) = {} & b(x,y)\cdot 1_{[0,1]^2}(x,y) + b(-x,y)\cdot 1_{[0,1]^2}(x+1,y) \\ & - b(-x,-y)\cdot 1_{[0,1]^2}(x+1,y+1) - b(x,-y)\cdot 1_{[0,1]^2}(x,y+1). \end{aligned}$$

The function b has the property

$$b(x,y) = b(1-x,1-y), \qquad \text{for all} \quad x,y. \tag{6}$$

Fig. 1. $-\frac{1}{2i}\check{\epsilon}_1(x,y) = f(x,y)$ and $-\frac{3}{2i}\check{\epsilon}_3(x,y)$ with filter (2)

A straightforward but tedious calculation using (6) shows that $\check{\epsilon}_n$ is given by

$$\begin{aligned}\check{\epsilon}_n(x,y) = \frac{-2i}{n}\sum_{j=-1}^{n-2} -b(-x-j-1,-y+j+1)\cdot 1_{[0,1]^2}(x+j+2,y-j)\\ -\frac{2i}{n}\sum_{j=0}^{n-1} -b(x-j,-y+j)\cdot 1_{[0,1]^2}(x-j,y-j+1)\\ -\frac{2i}{n}\sum_{l=0}^{n-1} b(x-n+2l+1,y-n+1)\cdot 1_{[0,1]^2}(x-n+2l+1,y-n+1)\\ -\frac{2i}{n}\sum_{l=1}^{n} b(-x+n-2l+1,y-n+1)\cdot 1_{[0,1]^2}(x-n+2l,y-n+1).\end{aligned}$$

Since the interior of the support of the functions in this sum is disjoint, the L^1-norm of $\check{\epsilon}_n(x,y)$ can be calculated by calculating the L^1-norm of each function in the sum. By using simple integral transformations, it can be seen that all functions have the same L^1-norm, namely $\|(2/n)\cdot b\cdot 1_{[0,1]^2}\|_1 = (2/n)\cdot\|f\|_1/4$, hence

$$\|\check{\epsilon}_n\|_1 = 4n\cdot\frac{2}{n}\cdot\int_0^1\int_0^1 |b(x,y)|dx\,dy = \frac{1}{6} \qquad (= 2\|f\|_1).$$

□

3 Concluding remarks

The Gautschi-type method allows for a uniform error bound with respect to frequencies and to the dimension of the system solved. A bound uniform in frequencies has already been proved in [5], and the independence of the dimension is shown in this note by proving the conjecture. Lemma 1 is a replacement for the Lemmas 4 and 5 in

[5] and not only gives improved error bounds but also simplifies the original proof considerably.

The new technique to estimate the componentwise product of matrices has been inspired by a similar technique of Hochbruck and Lubich in [4] where they used a one-dimensional Fourier transform for a matrix that depends only on differences of frequencies. The generalisation for any two-dimensional function evaluated at the frequencies using a two-dimensional Fourier transform is not obvious and only possible due to the oscillatory nature of the filter functions. It is hoped that the presentation of this new technique is helpful for other scientists working with oscillatory differential equations. A survey of this current research area can be found in [3].

The removal of the logarithmic term in [5] implies a uniform error bound for more general equations examined in [1] and [2]. This can be seen easily since the proof of the error bound for the more general equations is based on Theorem 1.

Lemma 1 improves the original bound for all dimensions since $1/6 \leq \min\{\log(n+1)\,\log(N+1), \sqrt{N}\}$ for $N, n \geq 1$. But the most important consequence of Lemma 1 is that the accuracy of the Gautschi-type method in time is independent of the grid chosen in space and the relevance this has for semi-discretisations of partial differential equations.

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