IMPLICIT RUNGE–KUTTA METHODS AND DISCONTINUOUS GALERKIN DISCRETIZATIONS FOR LINEAR MAXWELL'S EQUATIONS

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Abstract. In this paper we consider implicit Runge–Kutta methods for the time-integration of linear Maxwell's equations. We first present error bounds for the abstract Cauchy problem which respect the unboundedness of the differential operators using energy techniques. The error bounds hold for algebraically stable and coercive methods such as Gauß and Radau collocation methods. The results for the abstract evolution equation are then combined with a discontinuous Galerkin discretization in space using upwind fluxes. For the case that permeability and permittivity are piecewise constant functions, we show error bounds for the full discretization, where the constants do not deteriorate if the spatial mesh width tends to zero.

Key words. implicit Runge–Kutta methods, time integration, discontinuous Galerkin finite elements, error analysis, evolution equations, Maxwell's equations

AMS subject classifications. Primary: 65M12, 65M15. Secondary: 65M60, 65J10.

1. Introduction.

In recent years, there has been a great interest in solving the Maxwell's equations numerically using discontinuous Galerkin (dG) finite element methods for the spatial discretization, see the recent textbooks [6, 16]. This approach is particularly attractive if one is interested in the simulation of wave propagation in composite materials, where the electric permittivity and the magnetic permeability are discontinuous. For the full discretization, discontinuous Galerkin methods have to be supplemented with suitable time integration schemes. Explicit time integrators can exploit the block diagonal structure of the mass matrix of discontinuous Galerkin schemes and thus lead to fully explicit schemes. The RKDG methods of [5] achieve high-order convergence both in space and time by using a strong stability preserving Runge-Kutta schemes in time. We refer to [10] for more details on SSP-RK methods.

On the other hand, explicit methods suffer from step size restrictions due to stability requirements (CFL condition). Note that the CFL condition restricts the time step size no matter how many small elements appear in the grid, i.e., for uniformly refined grids as well as for locally refined grids. However, for the latter, local time stepping methods are a very attractive alternative, see, e.g., [11, 19, 27, 28].

In contrast, implicit methods can be used with large time steps at the cost of solving linear or, in general, even nonlinear systems. Since this additional computational effort only pays off if the time step size can be chosen significantly larger than for explicit methods, the advantage becomes more significant if the spatial grid is sufficiently fine. However, the efficiency of implicit methods strongly depends on the availability of fast linear solvers. For elliptic problems, it is well known that multigrid preconditioning allows to solve the linear systems in linear complexity. Such results are not yet available for the linear systems arising in hyperbolic problems. Nevertheless, we showed in [17] that even with a standard parallel implementation which was not optimized for the particular application, implicit time integration of linear Maxwell's equations outperforms explicit methods on a rather large test problem. For details on the implementation we refer to [17, Sec. 5] and for details on the numerical experiments we refer to [17, Sec. 6].

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Note that in the case that only a small part of the mesh contains small elements, then fast linear solvers are available by using precondioners which essentially solve the problem on the fine grid only. We are thus convinced that implicit methods have a great potential for solving wave type problems numerically.

It is the aim of our paper to analyze the full discretization error of linear Maxwell's equations with discontinuous Galerkin methods with upwind fluxes in space and implicit Runge–Kutta methods in time. For details on an extension to central fluxes we refer to [24]. Moreover, our analysis easily carries over to $H(\mathrm{curl})$-conforming finite element discretizations using Nédélec elements, cf. [22]. In fact, the situation for conforming methods is much simpler than for discontinuous Galerkin methods, since the discrete and the continuous bilinear forms coincide.

In a different framework and for more general first-order systems, error bounds for explicit Runge–Kutta methods of order two and three have been proven in [4]. For constant permittivity and permeability, an optimal convergence rate of $k+1/2$ in space, where k denotes the degree of polynomials in the dG method, and s in time, where s denotes the number of stages of the explicit Runge-Kutta method, was shown. For dG methods with central fluxes and the leap-frog method, error bounds of order k in space and order two in time have been shown in [9]. Also in the context of dG methods with central fluxes, a locally-implicit time integration method was suggested in [27] and analyzed in [7] and [23].

Implicit time integration for linear, abstract initial value problems

$$\partial_t u(t) + Au(t) = f(t), \quad u(0) = u_0, \tag{1.1}$$

where the operator A is a generator of a bounded C_0 semigroup, cf. [8, 25], has been studied in [1] generalizing earlier work in [2, 3] for the homogeneous case $f \equiv 0$. The elegant proofs are based on a Hille-Phillips operational calculus using Laplace transformations. The results shown in these papers can be applied to our situation and then yield convergence of order $s+1$ in time for s-stage collocation methods, in particular for Gauß and Radau methods. The error estimates for the full discretization can also be generalized to the application considered here. Using an elliptic projection yields the same order of convergence as our analysis but requires more regularity of the solution while using an L^2-projection leads to an order reduction in space.

To the best of our knowledge, the analysis used in [1, 2, 3] does not generalize to nonlinear problems. Therefore, in this paper, we follow a different approach using energy techniques, which were motivated by the analysis presented in [20] for quasilinear parabolic problems and L-stable Runge–Kutta methods and [21] for Gauß–Runge–Kutta time discretizations of wave equations on evolving surfaces. The assumptions in [20] exclude Gauß collocation methods, which are particularly interesting for hyperbolic problems. Our analysis applies to implicit Runge–Kutta methods which are algebraically stable and coercive (see, [14, Sections IV.12 and IV.14]) and Section 3.1 below). The analysis presented here for linear Maxwell's equations will be a first step to prove error bounds for more general problems such as quasilinear Maxwell's equations, acoustic and elastic wave equations, etc. These results will be reported elsewhere.

Our paper is organized as follows: In Section 2 we provide the analytical framework for Maxwell's equations for linear isotropic materials with perfectly conducting boundary conditions.

In Section 3, we present error estimates for the time discretization of the abstract initial value problem by s-stage implicit Runge–Kutta methods using an energy technique motivated by [20]. We show full temporal order of convergence for the implicit

Euler method and for the implicit midpoint rule while in general, the temporal order will suffer from order reduction to order $s+1$. It is well known that full temporal order will not be achieved for stiff problems without additional regularity assumptions, which are often unnatural in the context of time-dependent partial differential equations, cf. [1]. Our error bounds depend in an explicit form on the regularity of the solution. We also discuss the relation to the earlier work [1] showing the same result with a different technique.

Section 4 deals with the discontinuous Galerkin approximation of Maxwell's equations using upwind fluxes. It is known that the optimal convergence rate of dG methods applied to first order hyperbolic equations is $k+1/2$ when polynomials of degree k are used [18, 26]. Upwind fluxes for Maxwell's equation are considered in [15] (proof of order k) and [29] (proof of order $k+1/2$ for dispersive media). For Maxwell's equations with smooth coefficients written as second order PDE system and interior penalty dG method, order $k+1$ was proved [12, 13]. We present a convergence result for the error of the semidiscretization in a form required for the analysis of the full discretization.

Section 5 contains error bounds for the full discretization for the implicit Euler scheme and for general higher order implicit Runge–Kutta methods by generalizing the results from Section 3 to the discrete Maxwell operator. For the sake of readability, some technical details are postponed to the appendix. We also show that for the homogeneous Maxwell equations, the divergence is conserved in a weak sense, see also [9] for a related result for explicit Runge–Kutta methods.

2. Analytic framework.

Let Ω be an open, bounded, Lipschitz domain in $\mathbb{R}^d$, and let $T>0$ be a finite time. We consider Maxwell's equations for linear isotropic materials with perfectly conducting boundary conditons:

$$\begin{aligned}
&\mu\partial_t \mathbf{H} + \nabla\times \mathbf{E} = 0, && \Omega\times(0,T),\\
&\epsilon\partial_t \mathbf{E} - \nabla\times \mathbf{H} = 0, && \Omega\times(0,T),\\
&\nabla\cdot(\epsilon\mathbf{E}) = 0, \quad \nabla\cdot(\mu\mathbf{H}) = 0, && \Omega\times(0,T),\\
&n\times\mathbf{E} = 0, \quad n\cdot(\mu\mathbf{H}) = 0, && \partial\Omega\times(0,T),\\
&\mathbf{E}(0) = \mathbf{E}^0, \quad \mathbf{H}(0) = \mathbf{H}^0, && \Omega.
\end{aligned} \tag{2.1}$$

The differential operators and boundary conditions are understood in the sense of distributions and traces, respectively. For the coefficients we assume that

$$\mu,\epsilon \in L^\infty(\Omega), \quad \epsilon,\mu \geq \delta > 0. \tag{2.2}$$

The Hilbert space $V := L^2(\Omega)^3 \times L^2(\Omega)^3$ will be equipped with the weighted scalar product

$$\left(\begin{pmatrix}\mathbf{H}_1\\ \mathbf{E}_1\end{pmatrix}, \begin{pmatrix}\mathbf{H}_2\\ \mathbf{E}_2\end{pmatrix}\right)_V = \int_\Omega \mu\mathbf{H}_1\cdot\mathbf{H}_2 + \epsilon\mathbf{E}_1\cdot\mathbf{E}_2. \tag{2.3}$$

We define

$$V_0 := \left\{(\mathbf{H},\mathbf{E}) \in L^2(\Omega)^6 \,|\, \nabla\cdot(\epsilon\mathbf{E}) = 0,\ \nabla\cdot(\mu\mathbf{H}) = 0,\ n\cdot(\mu\mathbf{H}) = 0\right\} \subset V.$$

The subspace V_0 is closed in V because of the closedness of $\nabla\cdot$ and the continuity of the normal trace since the differential operators are defined in distributional sense. The Maxwell operator

$$A\begin{pmatrix}\mathbf{H}\\ \mathbf{E}\end{pmatrix} := \begin{pmatrix}\mu^{-1}\,\nabla\times\mathbf{E}\\ -\epsilon^{-1}\,\nabla\times\mathbf{H}\end{pmatrix} \tag{2.4}$$

is defined on its domain $D(A) = H(\mathrm{curl}, \Omega) \times H_0(\mathrm{curl}, \Omega)$ and the graph norm is denoted by

$$\|v\|_{D(A)} := (\|v\|_V^2 + \|Av\|_V^2)^{1/2}. \tag{2.5}$$

With respect to the weighted inner product (2.3), the Maxwell operator satisfies the following more general assumption for V and $(\cdot,\cdot)_V$ given above:

Assumption 2.1. *Let V be a Hilbert space with inner product $(\cdot,\cdot)_V$. We assume that the linear operator $A : D(A) \to V$ is skew adjoint, in particular*

$$(Av, w)_V = -(v, Aw)_V, \qquad \textit{for all } v, w \in D(A). \tag{2.6}$$

In the following, we consider the abstract initial value problem

$$\partial_t u(t) + Au(t) = 0, \quad u(0) = \begin{pmatrix} \mathbf{H}^0 \\ \mathbf{E}^0 \end{pmatrix} \tag{2.7}$$

for an operator A satisfying Assumption 2.1. Since A generates a unitary C_0-group, $\|u(t)\|_V = \|u(0)\|_V$. For the Maxwell operator, $\|\cdot\|_V$ corresponds to the electromagnetic energy, which is thus preserved.

Moreover, by Stone's theorem [25, Theorem 1.10.8.], for $(\mathbf{H}^0, \mathbf{E}^0)^T \in D(A) \cap V_0$ a unique solution $u \in C^1(\mathbb{R}; V_0) \cap C(\mathbb{R}; D(A) \cap V_0)$ of (2.7) exists. Therefore it is sufficient that the initial values $\mathbf{E}^0$ and $\mathbf{H}^0$ satisfy the divergence condition and that $\mathbf{H}^0$ satisfies the boundary condition to ensure that these conditions are valid for $\mathbf{E}(t)$ and $\mathbf{H}(t)$ for all $t \in [0, T]$.

More generally, we consider the inhomogeneous problem

$$\partial_t u(t) + Au(t) = f(t), \quad u(0) = \begin{pmatrix} \mathbf{H}^0 \\ \mathbf{E}^0 \end{pmatrix} \tag{2.8}$$

for $f \in C(0, T; V)$, see also [4].

3. Time discretization by implicit Runge–Kutta methods.

We start by discretizing the abstract Cauchy problem (2.8) in time using implicit s-stage Runge–Kutta methods. This yields approximations $U^{ni} \approx u(t_n + c_i\tau)$ and $u_{n+1} \approx u(t_{n+1})$ defined by

$$\begin{aligned} &\dot{U}^{ni} + AU^{ni} = f^{ni}, \qquad f^{ni} = f(t_n + c_i\tau), \qquad i = 1, \ldots s, \\ &U^{ni} = u^n + \tau \sum_{j=1}^{s} a_{ij} \dot{U}^{nj}, \quad i = 1, \ldots s, \\ &u^{n+1} = u^n + \tau \sum_{i=1}^{s} b_i \dot{U}^{ni}. \end{aligned} \tag{3.1}$$

3.1. Algebraically stable and coercive Runge–Kutta methods.

Recall that a Runge–Kutta method with s distinct nodes $0 \le c_i \le 1$ and weights $\mathcal{Q} = (a_{ij})_{i,j=1}^s$ and $b = (b_i)_{i=1}^s$ is called *algebraically stable* [14, Definition IV.12.5], if $b_i \ge 0$ for $i = 1, \ldots, s$ and

$$\mathcal{M} = (m_{ij})_{i,j=1}^s, \qquad \text{with} \qquad m_{ij} = b_i a_{ij} + b_j a_{ji} - b_i b_j \tag{3.2}$$

is positive semidefinite. It is well known that Gauß, Radau IA ($c_1 = 0$), and Radau IIA ($c_s = 1$) collocation methods are algebraically stable, cf. [14, Theorem IV.12.9].

For our analysis, we also need the coercivity condition that there exists a diagonal, positive definite matrix $\mathcal{D} \in \mathbb{R}^{s,s}$ and a positive scalar α such that

$$u^T \mathcal{D} \mathcal{Q}^{-1} u \geq \alpha u^T \mathcal{D} u \qquad \text{for all} \qquad u \in \mathbb{R}^s. \tag{3.3}$$

This condition also plays an important role in proving the existence of Runge–Kutta approximations, cf. [14, Section IV.14]. For Gauß collocation methods, (3.3) is satisfied for $\mathcal{D} = \mathcal{B}(\mathcal{C}^{-1} - I)$, where $\mathcal{B} := \operatorname{diag}(b_1, \ldots, b_s))$ and $\mathcal{C} := \operatorname{diag}(c_1, \ldots, c_s)$. For Radau IA and Radau IIA collocation methods it is satisfied for $\mathcal{D} = \mathcal{B}(I - \mathcal{C})$ and $\mathcal{D} = \mathcal{B}\mathcal{C}^{-1}$, respectively. For Gauß and Radau methods, the constant α is given explicitly in terms of the nodes c_i, $i = 1, \ldots, s$, see [14, Theorem IV.14.5].

REMARK 3.1. *For A-stable collocation methods such as Gauß- and Radau methods, a unique solution of the linear system defining the interior Runge-Kutta approximations U^{ni}, $i = 1, \ldots, s$ exists because A is skew-adjoint. This can be easily seen using the coercivity condition* (3.3).

3.2. Error bounds. We will present error bounds for algebraically stable Runge–Kutta methods satisfying the coercivity condition (3.3).

Defects. We start by inserting the exact solution of (2.8) into the numerical scheme using the notation

$$\tilde{u}^n = u(t_n), \qquad \tilde{U}^{ni} = u(t_n + c_i\tau), \qquad \dot{\tilde{U}}^{ni} = u'(t_n + c_i\tau).$$

This yields

$$\begin{aligned} \dot{\tilde{U}}^{ni} + A\tilde{U}^{ni} &= f^{ni}, \\ \tilde{U}^{ni} &= \tilde{u}^n + \tau \sum_{j=1}^{s} a_{ij} \dot{\tilde{U}}^{nj} + \Delta^{ni}, \\ \tilde{u}^{n+1} &= \tilde{u}^n + \tau \sum_{i=1}^{s} b_i \dot{\tilde{U}}^{ni} + \delta^{n+1}. \end{aligned} \tag{3.4}$$

For Gauss collocation methods with $s \geq 1$, Radau collocation methods with $s \geq 2$ and $u^{(s+1)} \in L^2(0, T; D(A))$, $u^{(s+2)} \in L^2(0, T; V)$ the defects are given by

$$\Delta^{ni} = \tau^s \int_{t_n}^{t_{n+1}} u^{(s+1)}(t) \kappa_i \left(\frac{t - t_n}{\tau} \right) dt = \mathcal{O}(\tau^{s+1}), \tag{3.5a}$$

$$\delta^{n+1} = \tau^{s+1} \int_{t_n}^{t_{n+1}} u^{(s+2)}(t) \kappa \left(\frac{t - t_n}{\tau} \right) dt = \mathcal{O}(\tau^{s+2}). \tag{3.5b}$$

Here κ_i and κ denote the Peano kernels corresponding to the quadrature rules defining the Runge–Kutta method. They are uniformly bounded with constants depending on the Runge–Kutta coefficients only. Hence we have

$$\tau \sum_{n=1}^{N} \left(\sum_{i=1}^{s} \left\| \Delta^{ni} \right\|_{D(A)}^2 + \left\| \frac{1}{\tau} \delta^{n+1} \right\|_V^2 \right) \leq C \tau^{2(s+1)} B(u, s, T), \tag{3.6}$$

where

$$B(u,s,T) = \int_0^T \left\| u^{(s+1)}(t) \right\|_{D(A)}^2 dt + \int_0^T \left\| u^{(s+2)}(t) \right\|_V^2 dt. \tag{3.7}$$

REMARK 3.2. *The order of the defect δ^{n+1} is not sharp, if the solution is more regular. More precisely, for Gauß collocation methods and $u^{(2s+1)} \in L^2(0,T,V)$ we have $\delta^{n+1} = \mathcal{O}(\tau^{2s+1})$, while for Radau collocation methods and $u^{(2s)} \in L^2(0,T,V)$ we have $\delta^{n+1} = \mathcal{O}(\tau^{2s})$.*

However, we cannot exploit this in our convergence analysis, since the global order is determined by the stage order, which is s for all collocation methods.

By subtracting (3.4) from (3.1), the time integration errors

$$e^n := u^n - \tilde{u}^n, \quad E^{ni} := U^{ni} - \tilde{U}^{ni}$$

satisfy

$$\dot{E}^{ni} + AE^{ni} = 0, \quad i = 1, \dots s, \tag{3.8a}$$

$$E^{ni} = e^n + \tau \sum_{j=1}^{s} a_{ij} \dot{E}^{nj} - \Delta^{ni}, \quad i = 1, \dots s, \tag{3.8b}$$

$$e^{n+1} = e^n + \tau \sum_{i=1}^{s} b_i \dot{E}^{ni} - \delta^{n+1}. \tag{3.8c}$$

Let $\Delta^n = \left(\Delta^{n1} \dots \Delta^{ns}\right)^T$, $E^n = \left(E^{n1} \dots E^{ns}\right)^T$, $\dot{E}^n = \left(\dot{E}^{n1} \dots \dot{E}^{ns}\right)^T$. Then, (3.8) can be written in a more compact form as

$$\begin{aligned} &\dot{E}^n + (I \otimes A)E^n = 0, \\ &E^n = \mathbb{1} \otimes e^n + \tau(\mathcal{O} \otimes I)\dot{E}^n - \Delta^n, \\ &e^{n+1} = e^n + \tau(b^T \otimes I)\dot{E}^n - \delta^{n+1}, \end{aligned} \tag{3.9}$$

where $\mathbb{1} = [1 \, \dots \, 1]^T$.

Energy techniques. The following analysis uses an energy technique motivated by [20].

LEMMA 3.3. *The error $e^n = u^n - u(t_n)$ satisfies*

$$\begin{aligned} \left\| e^{n+1} \right\|_V^2 - \left\| e^n \right\|_V^2 \le \frac{C}{T+1} \tau \left(\left\| e^n \right\|_V^2 + \sum_{i=1}^{s} \left\| E^{ni} \right\|_V^2 \right) \\ + C(T+1)\tau \left(\sum_{i=1}^{s} \left\| \Delta^{ni} \right\|_{D(A)}^2 + \left\| \frac{1}{\tau} \delta^{n+1} \right\|_V^2 \right). \end{aligned} \tag{3.10}$$

Here, the constant C only depends on $\mathcal{O}$, b, and s.

Proof. Taking the inner product of (3.8c) with itself we obtain

$$\left\| e^{n+1} \right\|_V^2 = \left\| e^n + \tau \sum_{i=1}^{s} b_i \dot{E}^{ni} \right\|_V^2 - 2(\delta^{n+1}, e^n + \tau \sum_{i=1}^{s} b_i \dot{E}^{ni})_V + \left\| \delta^{n+1} \right\|_V^2. \tag{3.11}$$

We estimate each of these three terms separately. For the first term we have

$$\left\| e^n + \tau \sum_{i=1}^{s} b_i \dot{E}^{ni} \right\|_V^2 = \|e^n\|_V^2 + 2\tau \sum_{i=1}^{s} b_i (e^n, \dot{E}^{ni})_V + \tau^2 \sum_{i,j=1}^{s} b_i b_j (\dot{E}^{ni}, \dot{E}^{nj})_V \quad (3.12)$$

Using (3.8b), we write e^n as

$$e^n = E^{ni} - \tau \sum_{j=1}^{s} a_{ij} \dot{E}^{nj} + \Delta^{ni}.$$

Inserting this identity into the second term of (3.12) we obtain

$$\begin{aligned} \left\| e^n + \tau \sum_{i=1}^{s} b_i \dot{E}^{ni} \right\|_V^2 &= \|e^n\|_V^2 + 2\tau \sum_{i=1}^{s} b_i (E^{ni} + \Delta^{ni}, \dot{E}^{ni})_V \\ &\quad + \tau^2 \sum_{i,j=1}^{s} (b_i b_j - b_i a_{ij} + b_j a_{ji})(\dot{E}^{ni}, \dot{E}^{nj})_V. \end{aligned}$$

Since the method is algebraically stable, the last term is not positive and we end up with

$$\left\| e^n + \tau \sum_{i=1}^{s} b_i \dot{E}^{ni} \right\|_V^2 \le \|e^n\|_V^2 + 2\tau \sum_{i=1}^{s} b_i (\dot{E}^{ni}, E^{ni} + \Delta^{ni})_V. \quad (3.13)$$

The skew-symmetry (2.6) of the operator A implies

$$\begin{aligned} (\dot{E}^{ni}, E^{ni})_V &= -(AE^{ni}, E^{ni})_V = 0, \\ (\dot{E}^{ni}, \Delta^{ni})_V &= -(AE^{ni}, \Delta^{ni})_V = (E^{ni}, A\Delta^{ni})_V \le \left\|E^{ni}\right\|_V \left\|A\Delta^{ni}\right\|_V. \end{aligned}$$

$A\Delta^{ni}$ is well defined because of $u^{(s+1)} \in L^2(0,T;D(A))$. For arbitrary $\gamma > 0$, Young's inequality gives

$$\left\| e^n + \tau \sum_{i=1}^{s} b_i \dot{E}^{ni} \right\|_V^2 \le \|e^n\|_V^2 + \tau \sum_{i=1}^{s} b_i \left(\frac{1}{\gamma} \left\|E^{ni}\right\|_V^2 + \gamma \left\|A\Delta^{ni}\right\|_V \right). \quad (3.14)$$

To bound the second term in (3.11) first observe that

$$(\delta^{n+1}, e^n)_V \le \tau \left\| \frac{1}{\tau}\delta^{n+1} \right\|_V \|e^n\|_V \le \frac{1}{2\gamma}\tau \|e^n\|_V^2 + \frac{\gamma}{2}\tau \left\| \frac{1}{\tau}\delta^{n+1} \right\|_V^2. \quad (3.15)$$

In order to bound $(\delta^{n+1}, \dot{E}^{ni})_V$ we use the compact form (3.9). From the second equation of (3.9) we have

$$\dot{E}^n = \frac{1}{\tau}(\mathcal{O}^{-1} \otimes I)(E^n + \Delta^n - \mathbb{1} \otimes e^n).$$

If we denote the inverse of the Runge–Kutta matrix by

$$\mathcal{O}^{-1} = (\omega_{ij})_{i,j}, \quad (3.16)$$

then

$$\dot{E}^{ni} = \frac{1}{\tau}\sum_{j=1}^{s} \omega_{ij}(E^{nj} + \Delta^{nj} - e^n). \tag{3.17}$$

Hence we have

$$\begin{aligned}\tau(\delta^{n+1}, \dot{E}^{ni})_V &\le \tau \left\| \frac{1}{\tau}\delta^{n+1} \right\|_V \sum_{j=1}^{s} |\omega_{ij}| \left(\|e^n\|_V + \left\|E^{nj}\right\|_V + \left\|\Delta^{nj}\right\|_V \right) \\ &\le \frac{\gamma}{2}\tau \left\| \frac{1}{\tau}\delta^{n+1} \right\|_V^2 + \frac{C}{\gamma}\tau\Big(\|e^n\|_V^2 + \sum_{j=1}^{s} \left\|E^{nj}\right\|_V^2 + \sum_{j=1}^{s} \left\|\Delta^{nj}\right\|_V^2 \Big),\end{aligned}$$

where $C = C(\mathcal{O}, b, s)$. Since the right-hand side does not depend on i and $\sum_i b_i = 1$, $b_i \ge 0$, we conclude from (3.15) that

$$\begin{aligned}(\delta^{n+1}, e^n + \tau \sum_{i=1}^{s} b_i \dot{E}^{ni})_V \le \frac{C}{\gamma}\tau\Big(\|e^n\|_V^2 + \sum_{j=1}^{s} \left\|E^{nj}\right\|_V^2 + \sum_{j=1}^{s} \left\|\Delta^{nj}\right\|_V^2 \Big) \\ + \gamma\tau \left\| \frac{1}{\tau}\delta^{n+1} \right\|_V^2 .\end{aligned} \tag{3.18}$$

Inserting (3.14) and (3.18) for $\gamma = 1 + T$ into (3.11), and writing $\left\|\delta^{n+1}\right\|_V^2 = \tau^2 \left\|\delta^{n+1}/\tau\right\|_V^2$ finally proves the Lemma. ◻

In order to use the Gronwall inequality, we have to bound $\left\|E^{ni}\right\|_V$ in terms of $\|e^n\|_V$ and the defects.

Lemma 3.4. *The error of the inner stages satisfies*

$$\sum_{i=1}^{s} \left\|E^{ni}\right\|_V^2 \le C \left(\|e^n\|_V^2 + \sum_{i=1}^{s} \left\|\Delta^{ni}\right\|_V^2 \right), \tag{3.19}$$

where, $C = C(\mathcal{O}, s, \mathcal{D}, \alpha)$.

Proof. From (3.9) we conclude

$$E^n = \mathbb{1} \otimes e^n - \tau(\mathcal{O} \otimes A)E^n - \Delta^n.$$

Multiplying by $\mathcal{D}\mathcal{O}^{-1} \otimes I$, where $\mathcal{D} = \operatorname{diag}(d_1, \dots, d_s)$ is the diagonal matrix arising in the coercivity property (3.3), and taking the inner product with E^n yields

$$(E^n, (\mathcal{D}\mathcal{O}^{-1} \otimes I)E^n)_{V^s} = -\tau(E^n, (\mathcal{D} \otimes A)E^n)_{V^s} + (E^n, (\mathcal{D}\mathcal{O}^{-1} \otimes I)(\mathbb{1} \otimes e^n - \Delta^n))_{V^s}.$$

The coercivity implies the lower bound

$$(E^n, (\mathcal{D}\mathcal{O}^{-1} \otimes I)E^n)_{V^s} \ge \alpha \sum_{i=1}^{s} d_i \left\|E^{ni}\right\|_V^2 .$$

Note that

$$(E^n, (\mathcal{D} \otimes A)E^n)_{V^s} = \sum_{i=1}^{s} d_i (E^{ni}, AE^{ni})_V = 0$$

since A is skew-symmetric, cf. (2.6). Using the notation (3.16) for the entries of $\mathcal{Q}^{-1}$ we obtain

$$
\begin{aligned}
(E^n, (\mathcal{D}\mathcal{Q}^{-1} \otimes I)(\mathbb{1} \otimes e^n - \Delta^n))_{V^s} &= \sum_{i,j=1}^{s} d_i \omega_{ij} (E^{ni}, e^n - \Delta^{nj})_V \\
&\leq \gamma \sum_{i=1}^{s} d_i \left\| E^{ni} \right\|_V^2 + \frac{C}{\gamma} \left(\left\| e^n \right\|_V^2 + \sum_{i=1}^{s} \left\| \Delta^{ni} \right\|_V^2 \right).
\end{aligned} \tag{3.20}
$$

Choosing $\gamma = \frac{\alpha}{2}$ completes the proof. □

Main result (time discretization error). From the bounds on the defects and the energy technique we are now able to state and prove our main result for the error of the time integration scheme.

THEOREM 3.5. *Let u be the solution of (2.8) and assume that $u^{(s+1)} \in L^2(0,T;D(A))$ and $u^{(s+2)} \in L^2(0,T;V)$. Then the error of an s-stage algebraically stable and coercive Runge–Kutta method of order at least two satisfies*

$$
\left\| e_N \right\|_V \leq C(T+1)^{1/2} \tau^{s+1} B(u,s,T)^{1/2},
$$

where B is defined in (3.7). The constant $C = C(\mathcal{Q}, b)$ is independent of u.

Proof. Inserting (3.19) into (3.10) we get

$$
\left\| e^{n+1} \right\|_V^2 - \left\| e^n \right\|_V^2 \leq \frac{C\tau}{T+1} \left\| e^n \right\|_V^2 + C(T+1)\tau \left(\sum_{i=1}^{s} \left\| \Delta^{ni} \right\|_{D(A)}^2 + \left\| \frac{1}{\tau} \delta^{n+1} \right\|_V^2 \right).
$$

Applying a discrete Gronwall lemma in differential form, we have for τ sufficiently small

$$
\left\| e_N \right\|_V^2 \leq C e^{N \frac{C}{T+1} \tau} (T+1) \tau \sum_{n=1}^{N} \left(\sum_{i=1}^{s} \left\| \Delta^{ni} \right\|_{D(A)}^2 + \left\| \frac{1}{\tau} \delta^{n+1} \right\|_V^2 \right)
$$

The assumptions on the regularity of u and the bound (3.6) for the defects prove the desired result. □

REMARK 3.6. *In [1, Theorem 1] (with order and strict order $s+1$), the following closely related result is proved in a completely different way:*

$$
\left\| e_N \right\|_V \leq C \tau^{s+1} \left(\int_0^T \left\| u^{(s+1)}(t) \right\|_{D(A)} dt + \int_0^T \left\| u^{(s+2)}(t) \right\|_V dt \right). \tag{3.21}
$$

We nevertheless presented our proof here since it will provide the basis for our analysis of the fully discrete problem.

REMARK 3.7. *For the implicit Euler method (Radau IIA for $s=1$), the following convergence result can be shown with a simple proof*

$$
\left\| e^N \right\|_V \leq C(T+1)^{1/2} \tau \Big(\int_0^T \left\| u''(t) \right\|_V^2 dt \Big)^{1/2}.
$$

Details of the proof can be found in Section 5.1 for the fully discrete case.

REMARK 3.8. *For the Maxwell operator (2.4) and $f \equiv 0$, the Runge–Kutta solution preserves the divergence, i.e,*

$$
\nabla \cdot (\epsilon \mathbf{E}^n) = \nabla \cdot (\epsilon \mathbf{E}^0), \qquad \nabla \cdot (\mu \mathbf{H}^n) = \nabla \cdot (\mu \mathbf{H}^0), \qquad n = 1, 2, \ldots.
$$

4. Spatial discretization of Maxwell's equations. We discretize (2.8) in space using a discontinuous Galerkin (dG) method, see, e.g., [6, 16].

4.1. Discontinous Galerkin approximations. For the purpose of presentation, we restrict ourselves to simplicial meshes. However, all our results hold for more general meshes as well.

We use the following notation: by $\mathbb{P}_k$ we denote the set of all multivariate polynomials of degree at most k. $\mathcal{T}_h = \{K\}$ is a simplicial mesh of a polyhedron $\Omega \subset \mathbb{R}^d$ consisting of elements K, i.e., $\Omega = \bigcup K$. h_K denotes the diameter of K and r_K denotes the radius of the largest ball inscribed in K. The index h refers to the maximum diameter of all elements of $\mathcal{T}_h$. The set of faces is denoted by $\mathcal{F}_h = \mathcal{F}_h^{\text{int}} \cup \mathcal{F}_h^{\text{ext}}$, where $\mathcal{F}_h^{\text{int}}$ and $\mathcal{F}_h^{\text{ext}}$ consist of all interior and all exterior faces, respectively. If $F \in \mathcal{F}_h^{\text{int}}$ is an interior face, then there exists a neighboring element K_F of K with $\partial K \cap \partial K_F = F$. By n_F we denote the unit normal of a face F, where the orientation of n_F is fixed once and forever for each inner face. For a boundary face F, n_F is an outward normal vector. With $v_K := v_h|_K$ we denote the restriction of a function v_h to an element K. Jumps of v_h on an interior face F with normal vector n_F pointing from K to K_F are defined as

$$[\![v_h]\!]_F := (v_{K_F})|_F - (v_K)|_F.$$

Note that the sign of the jump on face F is fixed by the direction of the normal vector n_F. The discontinuous Galerkin space w.r.t. the mesh $\mathcal{T}_h$ is defined as the space of piecewise polynomials with fixed polynomial degree k:

$$V_h = \left\{ v_h \in L^2(\Omega) \mid v_h|_K \in \mathbb{P}_k(K) \right\}^6. \tag{4.1}$$

In general, $V_h \not\subset D(A)$, so that the method is nonconforming.

As in [6, Def. 1.38], the following properties of the mesh are required:

Assumption 4.1. *We suppose shape and contact regularity of the mesh, which means that there exists a constant $\rho > 0$, such that all elements K and their neighboring elements K_F satisfy*

$$\frac{h_{K_F}}{h_K} \geq \rho, \qquad \frac{r_K}{h_K} \geq \rho. \tag{4.2}$$

Assumption 4.2. *We suppose that $\mu_K := \mu|_K$ and $\epsilon_K := \epsilon|_K$ are constant for each $K \in \mathcal{T}_h$.*

By $\pi_h : V \to V_h$ we denote the L^2-orthogonal projection on V_h which satisfies

$$(v_h, u - \pi_h u)_{0,\Omega} = 0 \qquad \text{for all} \qquad u \in V,\ v_h \in V_h. \tag{4.3}$$

Note that for piecewise constant coefficients, we have

$$(v_h, u - \pi_h u)_V = 0 \qquad \text{for all} \qquad u \in V,\ v_h \in V_h. \tag{4.4}$$

Additionally, we use broken Sobolev spaces

$$H^q(\mathcal{T}_h) := \left\{ v \in L^2(\Omega) \mid v|_K \in H^q(K),\, \forall K \in \mathcal{T}_h \right\}, \quad q \in \mathbb{N}, \tag{4.5}$$

which are Hilbert spaces with seminorm and norm defined via

$$|u|^2_{H^q(\mathcal{T}_h)} := \sum_{K \in \mathcal{T}_h} |u|^2_{H^q(K)}, \qquad \|u\|^2_{H^q(\mathcal{T}_h)} := \sum_{j=0}^{q} |u|^2_{H^q(\mathcal{T}_h)},$$

respectively.

After discretization by a discontinuous Galerkin method with upwind fluxes, we end up with the space semidiscrete problem

$$\partial_t u_h + A_h u_h = f_h, \quad u_h(0) = \pi_h u_0, \tag{4.6}$$

where $u_h \in C^1(0,T;V_h)$ is the semidiscrete solution and $f_h = \pi_h f$. Given $u_h = (\mathbf{H}_h, \mathbf{E}_h)$ and $w_h = (\phi_h, \psi_h) \in V_h$, the dG operator $A_h : V_h \to V_h$ is given as

$$\begin{aligned}(A_h u_h, w_h)_V := &\sum_K \Big((\nabla\times \mathbf{E}_h, \phi_h)_{0,K} - (\nabla\times \mathbf{H}_h, \psi_h)_{0,K} \Big) \\ &+ \sum_{F\in\mathcal{F}_h^{\mathrm{int}}} \Big((n_F \times [\![\mathbf{E}_h]\!]_F, \alpha_K \phi_K + \alpha_{K_F}\phi_{K_F})_{0,F} \\ &\qquad - (n_F \times [\![\mathbf{H}_h]\!]_F, \beta_K \psi_K + \beta_{K_F}\psi_{K_F})_{0,F} \\ &\qquad + \gamma_F (n_F \times [\![\mathbf{E}_h]\!]_F, n_F \times [\![\psi_h]\!]_F)_{0,F} \\ &\qquad + \delta_F (n_F \times [\![\mathbf{H}_h]\!]_F, n_F \times [\![\phi_h]\!]_F)_{0,F} \Big) \\ &+ \sum_{F\in\mathcal{F}_h^{\mathrm{ext}}} \Big(-(n\times \mathbf{E}_h, \phi_h)_{0,F} + 2\gamma_F (n\times\mathbf{E}_h, n\times\psi_h)_{0,F} \Big).\end{aligned} \tag{4.7}$$

If $c_K = (\epsilon_K \mu_K)^{-1/2}$ denotes the speed of light in the media of element K, then

$$\begin{aligned}\alpha_{K,F} &= \frac{c_{K_F}\epsilon_{K_F}}{c_{K_F}\epsilon_{K_F} + c_K\epsilon_K} = \frac{1}{1 + \left(\frac{\epsilon_K \mu_{K_F}}{\mu_K \epsilon_{K_F}}\right)^{1/2}}, \\ \beta_{K,F} &= \frac{c_{K_F}\mu_{K_F}}{c_{K_F}\mu_{K_F} + c_K\mu_K} = \frac{1}{1 + \left(\frac{\mu_K \epsilon_{K_F}}{\epsilon_K \mu_{K_F}}\right)^{1/2}}, \\ \gamma_F &= \frac{1}{c_{K_F}\mu_{K_F} + c_K\mu_K}, \qquad \delta_F = \frac{1}{c_{K_F}\epsilon_{K_F} + c_K\epsilon_K}.\end{aligned}$$

Note that

$$\alpha_{K,F} + \alpha_{K_F,F} = 1, \qquad \beta_{K,F} + \beta_{K_F,F} = 1, \qquad \alpha_{K,F} = \beta_{K_F,F}. \tag{4.8}$$

By (4.7), A_h is also well defined as an operator from $V_h + \big(D(A) \cap H^1(\mathcal{T}_h)^6\big)$ to V_h. This allows to show the following consistency property.

Lemma 4.3. *For $u \in D(A) \cap H^1(\mathcal{T}_h)^6$ we have*

$$A_h u = \pi_h A u. \tag{4.9}$$

Proof. For $u = (\mathbf{H}, \mathbf{E}) \in D(A) \cap H^1(\mathcal{T}_h)^6$ we have $n_F \times [\![\mathbf{E}]\!]_F = n_F \times [\![\mathbf{H}]\!]_F = 0$ for $F \in \mathcal{F}_h^{\mathrm{int}}$ and $n_F \times \mathbf{E} = 0$ for $F \in \mathcal{F}_h^{\mathrm{ext}} \subset \partial\Omega$. Hence the sum over the faces vanishes and for all $w_h = (\psi_h, \phi_h) \in V_h$ we have

$$\begin{aligned}(A_h u, w_h)_V &= \sum_K \Big((\nabla\times \mathbf{E}, \phi_h)_{0,K} - (\nabla\times \mathbf{H}, \psi_h)_{0,K} \Big) \\ &= (\nabla\times \mathbf{E}, \phi_h)_{0,\Omega} - (\nabla\times \mathbf{H}, \psi_h)_{0,\Omega} = (Au, w_h)_V.\end{aligned}$$

This is equivalent to (4.9). □

LEMMA 4.4. *For all $u_h = (\mathbf{H}_h, \mathbf{E}_h) \in V_h$ we have*

$$(A_h u_h, u_h)_V = \sum_{F\in\mathcal{F}_h^{int}} \Big(\gamma_F \left\|n_F \times [\![\mathbf{E}_h]\!]_F\right\|_{0,F}^2 + \delta_F \left\|n_F \times [\![\mathbf{H}_h]\!]_F\right\|_{0,F}^2\Big) + \sum_{F\in\mathcal{F}_h^{ext}} 2\gamma_F \left\|n_F \times \mathbf{E}_h\right\|_{0,F}^2 \geq 0. \tag{4.10}$$

In particular, $-A_h$ is dissipative on V_h.

Proof. Integration by parts yields

$$\sum_K \Big((\nabla\times \mathbf{E}_h, \mathbf{H}_h)_{0,K} - (\nabla\times \mathbf{H}_h, \mathbf{E}_h)_{0,K}\Big) = \sum_{F\in\mathcal{F}_h^{\text{int}}} \Big((n_F \times \mathbf{E}_K, \mathbf{H}_K)_{0,F} + (n_{K_F} \times \mathbf{E}_{K_F}, \mathbf{H}_{K_F})_{0,F}\Big) + \sum_{F\in\mathcal{F}_h^{\text{ext}}} (n_F \times \mathbf{E}_h, \mathbf{H}_h)_{0,F}.$$

Inserting this into the definition of A_h and using $\alpha_K + \beta_K = 1$, we obtain

$$\begin{aligned}(A_h u_h, u_h)_V = \sum_{F\in\mathcal{F}_h^{\text{int}}} \Big(&\alpha_K (n_F \times \mathbf{E}_{K_F}, \mathbf{H}_K)_{0,F} - \alpha_{K_F}(n_F \times \mathbf{E}_K, \mathbf{H}_{K_F})_{0,F} \\ &- \beta_K (n_F \times \mathbf{H}_{K_F}, \mathbf{E}_K)_{0,F} + \beta_{K_F}(n_F \times \mathbf{H}_K, \mathbf{E}_{K_F})_{0,F} \\ &+ \gamma_F \left\|n_F \times [\![\mathbf{E}_h]\!]_F\right\|_{0,F}^2 + \delta_F \left\|n_F \times [\![\mathbf{H}_h]\!]_F\right\|_{0,F}^2 \Big) \\ &+ \sum_{F\in\mathcal{F}_h^{\text{ext}}} 2\gamma_F \left\|n_F \times \mathbf{E}_h\right\|_{0,F}^2 .\end{aligned}$$

By (4.8), the first and the fourth term sum to zero, and so do the second and the third term. This shows (4.10). □

The previous lemma implies that the electromagnetic energy is non-increasing if $f = 0$:

$$\partial_t \left\|u_h\right\|_V^2 \leq 0 \tag{4.11}$$

The following integration by parts formula will be used frequently later. It allows to move all derivatives and tangential jumps to the test functions.

LEMMA 4.5. *For $u = (\mathbf{H}, \mathbf{E}) \in V_h + \big(D(A) \cap H^1(\mathcal{T}_h)^6\big)$ and $w_h = (\phi_h, \psi_h) \in V_h$ the following relation holds:*

$$\begin{aligned}(A_h u, w_h)_V = &\sum_K \Big((\mathbf{E}, \nabla\times \phi_h)_{0,K} - (\mathbf{H}, \nabla\times \psi_h)_{0,K}\Big) \\ &+ \sum_{F\in\mathcal{F}_h^{int}} \Big((\alpha_K \mathbf{E}_{K_F} + \alpha_{K_F}\mathbf{E}_K + \delta_F n_F \times [\![\mathbf{H}]\!]_F, n_F \times [\![\phi_h]\!]_F)_{0,F} \\ &- (\beta_K \mathbf{H}_{K_F} + \beta_{K_F}\mathbf{H}_K - \gamma_F n_F \times [\![\mathbf{E}]\!]_F, n_F \times [\![\psi_h]\!]_F)_{0,F}\Big) \\ &- \sum_{F\in\mathcal{F}_h^{ext}} (\mathbf{H}, n_F \times \psi_h)_{0,F} - 2\gamma_F (n_F \times \mathbf{E}, n_F \times \psi_h)_{0,F}.\end{aligned} \tag{4.12}$$

Proof. Integration by parts and (4.8). □

4.2. Error of spatial discretization. Applying the L^2-projection π_h to the continuous problem (2.8) and using the consistency property of Lemma 4.3, the exact solution satisfies

$$\partial_t \pi_h u + A_h u = f_h, \quad \pi_h u(0) = \pi_h u_0. \tag{4.13}$$

We define errors

$$e(t) = u_h(t) - u(t) = e_h(t) - e_\pi(t), \tag{4.14a}$$

where

$$e_h(t) = u_h(t) - \pi_h u(t), \qquad e_\pi(t) := u(t) - \pi_h u(t). \tag{4.14b}$$

For the projection error we have the following result:

LEMMA 4.6. *For $u \in H^{k+1}(K)^6$, the following error bounds hold:*

$$\|e_\pi\|_{0,K} \le C h_K^{k+1} |u|_{H^{k+1}(K)^6} \tag{4.15}$$

and

$$\|e_{\pi,K}\|_{0,F} \le C h_K^{k+1/2} |u|_{H^{k+1}(K)^6}. \tag{4.16}$$

where C is independent of K (and thus also of h_K).

Proof. See [6, Lemma 1.58 and Lemma 1.59]. □

The error e_h satisfies the following error bound:

THEOREM 4.7. *Let $u \in C^0\big(0,T; D(A) \cap H^{k+1}(\mathcal{T}_h)^6\big) \cap C^1(0,T;V)$ be a solution of (2.8) and $u_h \in C^1(0,T;V_h)$ be a solution of the semidiscrete problem (4.6) Then, the error $e_h = u_h - u$ satisfies*

$$\|e_h(T)\|_V^2 + \int_0^T (A_h e_h(t), e_h(t))_V \, dt \le C h^{2k+1} \int_0^T |u(t)|^2_{H^{k+1}(\mathcal{T}_h)^6} \, dt, \tag{4.17}$$

where C is independent of u and h.

Proof. Subtracting (4.13) from (4.6), we obtain

$$\partial_t e_h(t) + A_h e_h(t) = A_h e_\pi(t). \tag{4.18}$$

Taking the V-inner product with e_h and integrating from 0 to T we obtain

$$\frac{1}{2}\int_0^T \frac{d}{dt}\|e_h(t)\|_V^2 \, dt + \int_0^T (A_h e_h(t), e_h(t))_V \, dt = \int_0^T (A_h e_\pi(t), e_h(t))_V \, dt.$$

For the right-hand side we use Lemma 4.5. Since e_π is the projection error we have

$$(e_\pi^{\mathbf{E}}, \nabla\times e_h^{\mathbf{H}})_{0,K} = 0, \quad (e_\pi^{\mathbf{H}}, \nabla\times e_h^{\mathbf{E}})_{0,K} = 0.$$

Hence, the first sum in (4.12) vanishes and we obtain

$$(A_h e_\pi, e_h)_V = \sum_{F\in\mathcal{F}_h^{\text{int}}} \Big((\alpha_K e^{\mathbf{E}}_{\pi,K_F} + \alpha_{K_F} e^{\mathbf{E}}_{\pi,K} + \delta_F n_F \times [\![e_\pi^{\mathbf{H}}]\!]_F, n_F \times [\![e_h^{\mathbf{H}}]\!]_F)_{0,F}$$

$$
\begin{aligned}
&- (\beta_K e^{\mathbf{H}}_{\pi,K_F} + \beta_{K_F} e^{\mathbf{H}}_{\pi,K} - \gamma_F n_F \times [\![e^{\mathbf{E}}_\pi]\!]_F, n_F \times [\![e^{\mathbf{E}}_h]\!]_F)_{0,F}\Big) \\
&- \sum_{F \in \mathcal{F}_h^{\mathrm{ext}}} (e^{\mathbf{H}}_\pi, n_F \times e^{\mathbf{E}}_h)_{0,F} - 2\gamma_F (n_F \times e^{\mathbf{E}}_\pi, n_F \times e^{\mathbf{E}}_h)_{0,F}.
\end{aligned}
$$

Using Cauchy-Schwarz and Young's inequalities and then Lemmas 4.4 and 4.6, we get

$$
(A_h e_\pi, e_h)_V \le \frac{1}{2}(A_h e_h, e_h)_V + Ch^{2k+1} \left| u \right|^2_{H^{k+1}(\mathcal{T}_h)^6}. \tag{4.19}
$$

Since $e_h(0) = 0$ this proves (4.17). ∎

COROLLARY 4.8. *If the assumptions of Theorem 4.7 are satisfied, then the semidiscrete error $e = e_h - e_\pi$ is bounded by*

$$
\|e(T)\|_V^2 + \int_0^T (Ae(t), e(t))_V \, dt \le Ch^{2k+1} \Big(\int_0^T |u(t)|^2_{H^{k+1}(\mathcal{T}_h)^6} \, dt + h \, |u(T)|^2_{H^{k+1}(\mathcal{T}_h)^6} \Big),
$$

where C is independent of u and h.

Proof. We have $(A_h v, e_\pi)_V = 0$ for all $v \in V_h + D(A)$ by (4.4). This yields the identity

$$
(A_h(e_h - e_\pi), e_h - e_\pi)_V = (A_h e_h, e_h)_V - (A_h e_\pi, e_h)_V.
$$

The estimate now follows directly from Lemma 4.6 and Theorem 4.7. ∎

5. Full discretization of Maxwell's equations.

5.1. Error of full discretization for the implicit Euler method. In this section we consider the implicit Euler method for the time integration of (4.6):

$$
u_h^{n+1} = u_h^n + \tau(-A_h u_h^{n+1} + f_h^{n+1}). \tag{5.1}
$$

We treat this scheme separately because its analysis simplifies considerably compared to general higher order methods. Moreover, since stage order and order of the implicit Euler scheme coincide, it does not fit into the assumptions of Theorem 5.4 below.

THEOREM 5.1. *Let $u \in C\big(0, T; D(A) \cap H^{k+1}(\mathcal{T}_h)^6\big)$ denote the solution of (2.8) and assume that $u'' \in L^2(0, T, V)$. Then, for $\tau \le 3/4(T+1)$, the error of the implicit Euler method is bounded by*

$$
\begin{aligned}
&\big\| e_h^N \big\|_V^2 + \tau \sum_{n=0}^{N} (A_h e_h^{n+1}, e_h^{n+1})_V \\
&\quad \le C(T+1)\Big(\tau^2 \int_0^T \|u''(t)\|_V^2 \, dt + h^{2k+1} \max_{t \in [0,T]} |u(t)|^2_{H^{k+1}(\mathcal{T}_h)^6} \Big),
\end{aligned}
$$

where the constant $C = C(\mathcal{Q}, b, k, \rho)$ is independent of h and u.

Proof. The exact solution satisfies

$$
\tilde{u}^{n+1} = \tilde{u}^n + \tau(-A\tilde{u}^{n+1} + f^{n+1}) + \delta^{n+1},
$$

where δ^{n+1} is given in (3.5b) for $s = 0$. Projecting onto L^2 and subtracting from (5.1) yields the error recursion

$$
e_h^{n+1} = e_h^n - \tau A_h (e_h^{n+1} - e_\pi^{n+1}) - \pi_h \delta^{n+1}. \tag{5.2}
$$

Taking the V-inner product with e_h^{n+1} and using (4.19) we obtain

$$
\begin{aligned}
(e_h^{n+1} - e_h^n, e_h^{n+1})_V &+ \frac{1}{2}\tau(A_h e_h^{n+1}, e_h^{n+1})_V \\
&\leq C\tau h^{2k+1} \left|u^{n+1}\right|^2_{H^{k+1}(\mathcal{T}_h)^6} - (\pi_h \delta^{n+1}, e_h^{n+1})_V .
\end{aligned}
\tag{5.3}
$$

We sum from 0 to $N-1$ and use the following representation of the left-hand side

$$
\begin{aligned}
\sum_{n=0}^{N-1} (e_h^{n+1} - e_h^n, e_h^{n+1})_V = \frac{1}{2}\left\|e_h^N\right\|_V^2 &+ \frac{1}{2}\left\|e_h^N\right\|_V^2 - (e_h^{N-1}, e_h^N)_V + \frac{1}{2}\left\|e_h^{N-1}\right\|_V^2 \\
&+ \frac{1}{2}\left\|e_h^{N-1}\right\|_V^2 - (e_h^{N-2}, e_h^{N-1})_V + \frac{1}{2}\left\|e_h^{N-2}\right\|_V^2 + \dots \\
&+ \frac{1}{2}\left\|e_h^1\right\|_V^2 - (e_h^0, e_h^1)_V + \frac{1}{2}\left\|e_h^0\right\|_V^2 - \frac{1}{2}\left\|e_h^0\right\|_V^2 \\
&\geq \frac{1}{2}\left\|e_h^N\right\|_V^2 - \frac{1}{2}\left\|e_h^0\right\|_V^2 .
\end{aligned}
$$

For the right-hand side of (5.3) we have

$$
\sum_{n=0}^{N-1} (\pi_h \delta^{n+1}, e_h^{n+1})_V \leq \frac{\tau}{2} \sum_{n=0}^{N-1} \left((T+1) \left\| \frac{\delta^{n+1}}{\tau} \right\|_V^2 + \frac{1}{T+1} \left\|e_h^{n+1}\right\|_V^2 \right).
$$

For $\tau \leq 3/4(T+1)$, the result follows by a discrete Gronwall inequality. ☐

5.2. Error of full discretization for higher order Runge–Kutta methods. An implicit s-stage Runge–Kutta method applied to (4.6) yields the approximations

$$
\begin{aligned}
\dot{U}_h^{ni} + A_h U_h^{ni} &= f_h^{ni}, \\
U_h^{ni} &= u_h^n + \tau \sum_{j=1}^{s} a_{ij} \dot{U}_h^{nj}, \\
u_h^{n+1} &= u_h^n + \tau \sum_{i=1}^{s} b_i \dot{U}_h^{ni}.
\end{aligned}
\tag{5.4}
$$

For A-stable collocation methods such as Gauß- and Radau methods, a unique solution of the linear system defining the interior Runge-Kutta approximations U_h^{ni}, $i = 1, \dots, s$ exists because A_h is dissipative.

Defects. Inserting the exact solution $\tilde{u}^n = u(t_n)$, $\tilde{U}^{ni} = u(t_n + c_i\tau)$, and $\dot{\tilde{U}}^{ni} = u'(t_n + c_i\tau)$ into the numerical scheme yields

$$
\begin{aligned}
\pi_h \dot{\tilde{U}}^{ni} + A_h \tilde{U}^{ni} &= f_h^{ni}, \\
\tilde{U}^{ni} &= \tilde{u}^n + \tau \sum_{j=1}^{s} a_{ij} \dot{\tilde{U}}^{nj} + \Delta^{ni}, \\
\tilde{u}^{n+1} &= \tilde{u}^n + \tau \sum_{i=1}^{s} b_i \dot{\tilde{U}}^{ni} + \delta^{n+1},
\end{aligned}
\tag{5.5}
$$

where the defects are given in (3.5). We define errors as

$$
e_h^n := u_h^n - \pi_h \tilde{u}^n, \qquad\qquad e_\pi^n := \tilde{u}^n - \pi_h \tilde{u}^n,
$$

$$E_h^{ni} := U_h^{ni} - \pi_h \tilde{U}^{ni}, \qquad E_\pi^{ni} := \tilde{U}^{ni} - \pi_h \tilde{U}^{ni}, \tag{5.6}$$
$$\dot{E}_h^{ni} := \dot{U}_h^{ni} - \pi_h \dot{\tilde{U}}^{ni}.$$

Subtracting (5.5) from (5.4) yields

$$\dot{E}_h^{ni} + A_h E_h^{ni} = A_h E_\pi^{ni}, \tag{5.7a}$$

$$E_h^{ni} = e_h^n + \tau \sum_{j=1}^s a_{ij} \dot{E}_h^{nj} - \pi_h \Delta^{ni}, \tag{5.7b}$$

$$e_h^{n+1} = e_h^n + \tau \sum_{i=1}^s b_i \dot{E}_h^{ni} - \pi_h \delta^{n+1}. \tag{5.7c}$$

For

$$\Delta^n = \begin{pmatrix} \Delta^{n1} \\ \vdots \\ \Delta^{ns} \end{pmatrix}, \quad E_h^n = \begin{pmatrix} E_h^{n1} \\ \vdots \\ E_h^{ns} \end{pmatrix}, \quad E_\pi^n = \begin{pmatrix} E_\pi^{n1} \\ \vdots \\ E_\pi^{ns}, \end{pmatrix}, \quad \dot{E}_h^n = \begin{pmatrix} \dot{E}_h^{n1} \\ \vdots \\ \dot{E}_h^{ns} \end{pmatrix},$$

we can write (5.7) in compact form as

$$\dot{E}_h^n + (I \otimes A_h) E_h^n = (I \otimes A_h) E_\pi^n \tag{5.8a}$$

$$E_h^n = \mathbb{1} \otimes e_h^n + \tau (\mathcal{O}\!\!\!\!\mathcal{A} \otimes I) \dot{E}_h^n - \pi_h \Delta^n \tag{5.8b}$$

$$e_h^{n+1} = e_h^n + \tau (b^T \otimes I) \dot{E}_h^n - \pi_h \delta^{n+1}. \tag{5.8c}$$

Energy technique. To analyze the error we use the same technique as in the continuous case. Taking the inner product of e_h^{n+1} with itself using (5.7c) yields

$$\left\| e_h^{n+1} \right\|_V^2 = \left\| e_h^n + \tau \sum_{i=1}^s b_i \dot{E}_h^{ni} \right\|_V^2 - 2(\pi_h \delta^{n+1}, e_h^n + \tau \sum_{i=1}^s b_i \dot{E}_h^{ni})_V + \left\| \pi_h \delta^{n+1} \right\|_V^2. \tag{5.9}$$

THEOREM 5.2. *The errors* (5.6) *of the Runge–Kutta method* (5.4) *applied to* (4.6) *satisfy*

$$\begin{aligned} &\left\| e_h^{n+1} \right\|_V^2 - \| e_h^n \|_V^2 + \tau \sum_{i=1}^s b_i (A_h E_h^{ni}, E_h^{ni})_V \\ &\quad \le \frac{C}{T+1} \tau \left(\| e_h^n \|_V^2 + \sum_{i=1}^s \left\| E_h^{ni} \right\|_V^2 \right) + C \tau h^{2k+1} \sum_{i=1}^s b_i \left| u(t_n + c_i \tau) \right|_{H^{k+1}(\mathcal{T}_h)^6}^2 \\ &\qquad + C(T+1) \tau \left(\sum_{i=1}^s \left(\left\| \pi_h \Delta^{ni} \right\|_V^2 + \left| \Delta^{ni} \right|_{H^1(\mathcal{T}_h)^6}^2 \right) + \left\| \frac{1}{\tau} \pi_h \delta^{n+1} \right\|_V^2 \right). \end{aligned}$$

Here the constant $C = C(\mathcal{O}\!\!\!\!\mathcal{A}, b, k, \rho)$ is independent of h and u.

Proof. We estimate each of the three terms in (5.9) separately. The second and the third term can be handled completely analogously to the continuous case. For the second term, (3.18) now reads

$$\begin{aligned} (\pi_h \delta^{n+1}, e_h^n + \tau \sum_{i=1}^s b_i \dot{E}_h^{ni})_V \le & \frac{C}{\gamma} \tau \Big(\| e_h^n \|_V^2 + \sum_{j=1}^s \left\| E_h^{nj} \right\|_V^2 + \sum_{j=1}^s \left\| \pi_h \Delta^{nj} \right\|_V^2 \Big) \\ & + \gamma \tau \left\| \frac{1}{\tau} \pi_h \delta^{n+1} \right\|_V^2 . \end{aligned} \tag{5.10}$$

For the first term of (5.9) we have to work harder. The reason is the additional term in (5.7a) and the fact that A_h is not skew adjoint. However, the derivation of the estimate (3.13) remains true, so that we can start from

$$\left\| e_h^n + \tau \sum_{i=1}^{s} b_i \dot{E}_h^{ni} \right\|_V^2 \leq \|e_h^n\|_V^2 + 2\tau \sum_{i=1}^{s} b_i (\dot{E}_h^{ni}, E_h^{ni} + \pi_h \Delta^{ni})_V.$$

Eliminating $\dot{E}_h^{ni}$ by (5.7a) yields

$$\begin{aligned}(\dot{E}_h^{ni}, E_h^{ni} + \pi_h \Delta^{ni})_V =& (A_h E_\pi^{ni}, E_h^{ni})_V - (A_h E_h^{ni}, E_h^{ni})_V \\ &+ (A_h E_\pi^{ni}, \pi_h \Delta^{ni})_V - (A_h E_h^{ni}, \pi_h \Delta^{ni})_V.\end{aligned}$$

For the first two terms we have by (4.19)

$$\begin{aligned}&(A_h E_\pi^{ni}, E_h^{ni})_V - (A_h E_h^{ni}, E_h^{ni})_V \\ &\qquad \leq -\frac{1}{2}(A_h E_h^{ni}, E_h^{ni})_V + Ch^{2k+1} \left| \tilde{U}^{ni} \right|_{H^{k+1}(\mathcal{T}_h)^6}^2.\end{aligned} \tag{5.11}$$

The bounds for the last two terms are more involved and their proof is postponed to Lemma A.4 below. This lemma yields for $\gamma = T + 1$

$$(A_h E_h^{ni}, \pi_h \Delta^{ni})_V \leq \frac{C}{T+1} \left\| E_h^{ni} \right\|_V^2 + C(T+1) \left| \Delta^{ni} \right|_{H^1(\mathcal{T}_h)^6}^2,$$

and for $\gamma = 1$ using (4.15)

$$(A_h E_\pi^{ni}, \pi_h \Delta^{ni})_V \leq Ch^{2k+2} \left| \tilde{U}^{ni} \right|_{H^{k+1}(\mathcal{T}_h)^6}^2 + C \left| \Delta^{ni} \right|_{H^1(\mathcal{T}_h)^6}^2.$$

This finally gives

$$\begin{aligned}\left\| e_h^n + \tau \sum_{i=1}^{s} b_i \dot{E}_h^{ni} \right\|_V^2 \leq& \|e_h^n\|_V^2 + 2\tau \sum_{i=1}^{s} b_i \Big(-\frac{1}{2}(A_h E_h^{ni}, E_h^{ni})_V \\ &+ Ch^{2k+1} \left| \tilde{U}^{ni} \right|_{H^{k+1}(\mathcal{T}_h)^6}^2 + \frac{C}{T+1} \left\| E_h^{ni} \right\|_V^2 + C(T+1) \left| \Delta^{ni} \right|_{H^1(\mathcal{T}_h)^6}^2 \Big),\end{aligned}$$

which shows the desired bound. ∎

Bound on the inner stages. As in the continuous case, we need to bound the error of the inner stages E_h^{ni} in order to apply a Gronwall lemma.

LEMMA 5.3. *The error of the inner stages satisfies*

$$\begin{aligned}&\sum_{i=1}^{s} \left(\left\| E_h^{ni} \right\|_V^2 + \tau (A_h E_h^{ni}, E_h^{ni})_V \right) \\ &\qquad \leq C \left(\|e_h^n\|_V^2 + \sum_i \left\| \pi_h \Delta^{ni} \right\|_V^2 + \tau h^{2k+1} \sum_i |u(t_n + c_i \tau)|_{H^{k+1}(\mathcal{T}_h)^6}^2 \right),\end{aligned} \tag{5.12}$$

where the constant $C = C(\mathcal{O}, b, k, \rho)$ is independent of h and u.

Proof. We start from (5.8) and write

$$E_h^n = \mathbb{1} \otimes e_h^n + \tau (\mathcal{O} \otimes A_h) \left(E_\pi^n - E_h^n \right) - \pi_h \Delta^n.$$

Multiplying by $\mathcal{D}\mathcal{Q}^{-1} \otimes I$ and taking the inner product with E_h^n gives

$$
\begin{aligned}
(E_h^n, (\mathcal{D}\mathcal{Q}^{-1} \otimes I)E_h^n)_{V^s} &= \tau\, (E_h^n, (\mathcal{D} \otimes A_h)(E_\pi^n - E_h^n))_{V^s} \\
&\quad + (E_h^n, (\mathcal{D}\mathcal{Q}^{-1} \otimes I)(\mathbb{1} \otimes e_h^n - \pi_h \Delta^n))_{V^s}.
\end{aligned}
\tag{5.13}
$$

From the coercivity condition (3.3) we conclude

$$
(E_h^n, (\mathcal{D}\mathcal{Q}^{-1} \otimes I)E_h^n)_{V^s} \geq \alpha \sum_{i=1}^{s} d_i \left\| E_h^{ni} \right\|_V^2 .
$$

Since $\mathcal{D}$ is diagonal we have by (5.11)

$$
(E_h^n, (\mathcal{D} \otimes A_h)(E_\pi^n - E_h^n))_{V^s} \leq \sum_i \left(-\frac{d_i}{2}(A_h E_h^{ni}, E_h^{ni})_V + C h^{2k+1} \left| \tilde{U}^{ni} \right|^2_{H^{k+1}(\mathcal{T}_h)^6} \right).
$$

Treating the last term as in (3.20) for the continuous case and choosing $\gamma = \frac{\alpha}{2}$ shows the result. ◻

Main result (full discretization error). Our main result is contained in the following theorem.

THEOREM 5.4. *Let $u \in C\big(0, T; D(A) \cap H^{k+1}(\mathcal{T}_h)^6\big)$ be the solution of (2.8) and assume that $u^{(s+1)} \in L^2(0, T; D(A) \cap H^1(\mathcal{T}_h)^6)$ and $u^{(s+2)} \in L^2(0, T; V)$. Then the error of an s-stage algebraically stable and coercive Runge–Kutta method of order at least two satisfies*

$$
\begin{aligned}
&\left\| e_h^N \right\|_V + \left(\tau \sum_{n=1}^{N} \sum_{i=1}^{s} b_i (A_h E_h^{ni}, E_h^{ni})_V \right)^{1/2} \\
&\qquad \leq\; C(T+1)^{1/2} \Big(\tau^{s+1} B_h(u, s, T)^{1/2} + h^{k+1/2} \max_{t \in [0,T]} |u(t)|_{H^{k+1}(\mathcal{T}_h)^6} \Big),
\end{aligned}
$$

where

$$
B_h(u, s, T) = \int_0^T \left\| u^{(s+1)}(t) \right\|^2_{H^1(\mathcal{T}_h)^6} dt + \int_0^T \left\| u^{(s+2)}(t) \right\|^2_V dt.
$$

The constant $C = C(\mathcal{Q}, b, k, \rho)$ is independent of h and u.

Proof. The orthogonal projection is stable, i.e.,

$$
\left\| \pi_h \Delta^{ni} \right\|_V \leq \left\| \Delta^{ni} \right\|_V , \quad \left\| \pi_h \delta^{n+1} \right\|_V \leq \left\| \delta^{n+1} \right\|_V .
$$

By Lemma 5.3 we obtain

$$
\begin{aligned}
&\left\| e_h^{n+1} \right\|_V^2 - \| e_h^n \|_V^2 + \tau \sum_{i=1}^{s} b_i (A_h E_h^{ni}, E_h^{ni})_V \\
&\qquad \leq \frac{C}{T+1} \tau \| e_h^n \|_V^2 + C \tau h^{2k+1} \sum_{i=1}^{s} \left| \tilde{U}^{ni} \right|^2_{H^{k+1}(\mathcal{T}_h)^6} \\
&\qquad\quad + C(T+1)\tau \left(\sum_{i=1}^{s} \left\| \Delta^{ni} \right\|^2_{H^1(\mathcal{T}_h)^6} + \left\| \frac{1}{\tau} \delta^{n+1} \right\|^2_V \right).
\end{aligned}
$$

The regularity assumptions on u and (3.5) imply

$$\tau \sum_{n=1}^{N} \left(\sum_{i=1}^{s} \left\| \Delta^{ni} \right\|_{H^1(\mathcal{T}_h)^6}^2 + \left\| \frac{1}{\tau} \delta^{n+1} \right\|_V^2 \right) \leq C \tau^{2(s+1)} B_h(u, s, T).$$

Applying a discrete Gronwall lemma in differential form we end up with

$$\left\| e_h^N \right\|_V^2 + \tau \sum_{n=1}^{N} \sum_{i=1}^{s} b_i (A_h E_h^{ni}, E_h^{ni})_V \leq C(T+1) \tau^{2s+1} B_h(u, s, T) + CTh^{2k+1} \max_{t \in [0,T]} |u(t)|_{H^{k+1}(\mathcal{T}_h)^6}^2 ,$$

from which the result is easily obtained. □

REMARK 5.5. *The convergence results for the fully discrete scheme can be also obtained by applying results from [1] but they require more regularity or give a lower order of convergence.*

COROLLARY 5.6. *Under the assumptions of Theorem 5.4, the error is also bounded by*

$$\sum_{n=1}^{N} \tau \left\| e_h^n \right\|_V^2 \leq CT(T+1) \big(\tau^{2s+2} B_h(u, s, T) + h^{2k+1} \max_{t \in [0,T]} |u(t)|_{H^{k+1}(\mathcal{T}_h)^6}^2 \big).$$

REMARK 5.7. *A similar result can be proven if we use central fluxes instead of the upwind fluxes (4.19). The resulting dG operator A_h^{cf} then satisfies*

$$(A_h^{\mathrm{cf}} e_h, e_h)_V = 0$$
$$(A_h^{\mathrm{cf}} e_\pi, e_h)_V \leq C \left\| e_h \right\|_V^2 + Ch^{2k} |u|_{H^{k+1}(\mathcal{T}_h)^6}^2 .$$

In [9] it was shown that the spatial discretization error is in $\mathcal{O}(h^k)$ for central fluxes. Under the assumptions of Theorem 5.4, the full discretization error is bounded by

$$\left\| e_h^N \right\|_V \leq \; C(T+1)^{1/2} \Big(\tau^{s+1} B_h(u, s, T)^{1/2} + T^{1/2} h^k \max_{t \in [0,T]} |u(t)|_{H^{k+1}(\mathcal{T}_h)^6} \Big).$$

We refer to [24] for details.

Divergence error. We next study the divergence error of the numerical approximation. From the inverse inequality [6, Lemma 1.44] and Theorem 5.4 we immediately obtain

$$\left\| \nabla \cdot e_h^N \right\|_V \leq \; C(T+1)^{1/2} \Big(h^{-1} \tau^{s+1} B_h(u, s, T)^{1/2} + h^{k-1/2} \max_{t \in [0,T]} |u(t)|_{H^{k+1}(\mathcal{T}_h)^6} \Big).$$

In a weak sense, we can even prove that the discrete divergence is preserved exactly if $f = 0$. As in [9] we define a test space $X_h \subset H_0^1(\Omega)$ as the space of continuous, elementwise polynomial functions:

$$X_h = \big\{ v \in C^0(\bar{\Omega}) \mid \; v|_K \in \mathbb{P}_{k+1}(K)^6, \; K \in \mathcal{T}_h \big\} \cap H_0^1(\Omega).$$

By $\langle \cdot, \cdot \rangle_{-1}$ we denote the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, in which

$$\langle \nabla \cdot u, \psi \rangle_{-1} = -(u, \nabla \psi)_{0,\Omega} \qquad \text{for all} \qquad u \in L^2(\Omega)^3, \; \psi \in H_0^1(\Omega).$$

THEOREM 5.8. *Let $f \equiv 0$. Then the Runge–Kutta solution (5.4) satisfies*

$$\begin{cases} \langle \nabla\cdot \epsilon \mathbf{E}_h^{n+1}, \psi\rangle_{-1} = \langle \nabla\cdot \epsilon \mathbf{E}_h^{n}, \psi\rangle_{-1}, \\ \langle \nabla\cdot \mu \mathbf{H}_h^{n+1}, \psi\rangle_{-1} = \langle \nabla\cdot \mu \mathbf{H}_h^{n}, \psi\rangle_{-1}, \end{cases} \qquad \textit{for all} \qquad \psi \in X_h.$$

Moreover, if the initial data is divergence free, then

$$\langle \nabla\cdot \epsilon \mathbf{E}_h^{n}, \psi\rangle_{-1} = \langle \nabla\cdot \mu \mathbf{H}_h^{n}, \psi\rangle_{-1} = 0, \qquad n = 0, 1, 2, \dots.$$

Proof. For $\psi \in X_h \subset H_0^1(\Omega)$, integration by parts shows

$$\langle \nabla\cdot\big(\epsilon(\mathbf{E}_h^{n+1} - \mathbf{E}_h^{n})\big), \psi\rangle_{-1} = -(\epsilon(\mathbf{E}_h^{n+1} - \mathbf{E}_h^{n}), \nabla\psi)_{0,\Omega} = -(u_h^{n+1} - u_h^{n}, \begin{pmatrix} 0 \\ \nabla\psi \end{pmatrix})_V.$$

Using (5.4) and Lemma 4.5 we obtain

$$\langle \nabla\cdot\big(\epsilon(\mathbf{E}_h^{n+1} - \mathbf{E}_h^{n})\big), \psi\rangle_{-1} = \tau \sum_{i=1}^{s} b_i (A_h U_h^{ni}, \begin{pmatrix} 0 \\ \nabla\psi \end{pmatrix})_V = 0,$$

since for functions in X_h we have $\nabla\times \nabla\psi = 0$, $n_F \times [\![\nabla\psi_h]\!]_F = 0$ for $F \in \mathcal{F}_h^{\mathrm{int}}$ and $n \times \nabla\psi_h = 0$ on $\partial\Omega$. The result for $\mathbf{H}$ is proved analogously.

The second part follows from

$$\langle \nabla\cdot \epsilon(\mathbf{E}_h^{0}), \psi\rangle_{-1} = -\int_\Omega \pi_h \mathbf{E}^0 \cdot \epsilon\nabla\psi = -\int_\Omega \epsilon\mathbf{E}^0 \cdot \nabla\psi = \int_\Omega \nabla\cdot(\epsilon\mathbf{E}^0)\psi = 0$$

and similar for $\mathbf{H}$. ∎

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Appendix A. Auxiliary results.

For the proof of Lemma A.4 below we need some auxiliary results:

LEMMA A.1. *If Assumption 4.1 is satisfied, then for $v \in H^1(\mathcal{T}_h)^3$, $w \in L^2(\Omega)^3$ and arbitrary $\gamma > 0$ we have*

$$\sum_K |(w, \nabla\times \pi_h v)_{0,K}| \le \frac{1}{2\gamma} \|w\|_{0,\Omega}^2 + C\gamma \sum_K |v|_{1,K}^2,$$

where $C = C(\rho, k)$ is independent of K.

Proof. The Cauchy-Schwarz inequality and Young's inequality yield

$$|(w, \nabla\times \pi_h v)_{0,K}| \le \frac{1}{2\gamma} \|w\|_{0,K}^2 + \frac{\gamma}{2} \|\nabla\times \pi_h v\|_{0,K}^2. \tag{A.1}$$

From [6, Lemma 1.58] we have

$$\begin{aligned} \|\nabla\times \pi_h v\|_{0,K} &= \|\nabla\times(v + \pi_h v - v)\|_{0,K} \\ &\le \|\nabla\times v\|_{0,K} + \|\nabla\times(\pi_h v - v)\|_{0,K} \\ &\le C\,|v|_{1,K} \end{aligned}$$

for $v \in H^1(K)^3$. Inserting this bound into (A.1) and summing over all elements K shows the desired bound. ∎

LEMMA A.2. *Let Assumption 4.1 be satisfied and let $\gamma > 0$ be arbitrarily chosen. If F is an interior face connecting the elements K and K_F and n_F is the unit normal vector pointing from K to K_F, then for $v \in H(\mathrm{curl}, K \cup K_F) \cap H^1(K)^3 \cap H^1(K_F)^3$, $w \in H^1(K)^3$, we have*

$$|(w, n_F \times [\![\pi_h v]\!]_F)_{0,F}| \leq \frac{1}{\gamma}\Big(h_K^2 \,|w|_{1,K}^2 + 3\,\|w\|_{0,K}^2\Big) + C\gamma\Big(|v|_{1,K}^2 + |v|_{1,K_F}^2\Big). \quad \text{(A.2)}$$

If $F \subset \partial K$ is an exterior face with outward normal vector n_F, $w \in H^1(K)^3$, and $v \in H_0(\mathrm{curl}, \Omega) \cap H^1(K)^3$ then

$$|(w, n_F \times \pi_h v)_{0,F}| \leq \frac{1}{2\gamma}\Big(h_K^2 \,|w|_{1,K}^2 + 3\,\|w\|_{0,K}^2\Big) + C\gamma\,|v|_{1,K}^2\,. \quad \text{(A.3)}$$

In both estimates $C = C(\rho, k)$ is independent of K and K_F.

Proof. By assumption, we have $n_F \times [\![v]\!]_F = 0$. Thus we can write

$$(w, n_F \times [\![\pi_h v]\!]_F)_{0,F} = (w, n_F \times (\pi_h v_{K_F} - v_{K_F}))_{0,F} - (w, n_F \times (\pi_h v_K - v_K))_{0,F}.$$

Using the continuous trace inequality [6, Lemma 1.49] and the polynomial approximation properties on the faces [6, Lemma 1.59] we have

$$\begin{aligned}|(w, n_F \times (\pi_h v_K - v_K))_{0,F}| &\leq \|w\|_{0,F}\,\|n_F \times (\pi_h v_K - v_K)\|_{0,F}\\ &\leq C\Big(|w|_{1,K}\,\|w\|_{0,K} + h_K^{-1}\,\|w\|_{0,K}^2\Big)^{1/2} h_K^{1/2}\,|v|_{1,K}\\ &\leq \frac{1}{2\gamma}\Big(h_K^2\,|w|_{1,K}^2 + 3\,\|w\|_{0,K}^2\Big) + C\gamma\,|v|_{1,K}^2\,.\end{aligned}$$

Analogously, using (4.2), we obtain

$$|(w_K, n_F \times (\pi_h v_{K_F} - v_{K_F}))_{0,F}| \leq \frac{1}{2\gamma}\Big(h_K^2\,|w|_{1,K}^2 + 3\,\|w\|_{0,K}^2\Big) + C\gamma\,|v|_{1,K_F}^2\,.$$

This proves (A.2). To prove (A.3), note that $v \in H_0(\mathrm{curl}, \Omega)$ implies $n_F \times v = 0$ on the exterior face F, so that $n_F \times \pi_h v = n_F \times (\pi_h v - v)$. ∎

LEMMA A.3. *Let Assumption 4.1 be satisfied and let $w \in \mathbb{P}_k(K)^3$ and $\gamma > 0$ be arbitrarily chosen. If F is an interior face connecting the elements K and K_F and n_F is the unit normal vector pointing from K to K_F, then for $v \in H(\mathrm{curl}, K \cup K_F) \cap H^1(K)^3 \cap H^1(K_F)^3$, we have*

$$|(w, n_F \times [\![\pi_h v]\!]_F)_{0,F}| \leq \frac{1}{\gamma}\,\|w\|_{0,K}^2 + C\gamma\Big(|v|_{1,K}^2 + |v|_{1,K_F}^2\Big). \quad \text{(A.4)}$$

If F is an exterior face of K with outward normal vector n_F and $v \in H_0(\mathrm{curl}, \Omega) \cap H^1(K)$, then

$$|(w, n_F \times \pi_h v)_{0,F}| \leq \frac{1}{\gamma}\,\|w\|_{0,K}^2 + C\gamma\,|v|_{1,K}^2\,. \quad \text{(A.5)}$$

In both estimates $C = C(\rho, k)$ independent of K and K_F.

Proof. Analogously to the previous lemma using the discrete trace inequality [6, Lemma 1.52]. ∎

LEMMA A.4. *Suppose that $\mathcal{T}_h$ satisfies Assumption 4.1. Let $v \in \mathcal{D}(A) \cap H^1(\mathcal{T}_h)^6$ and $\gamma > 0$ be arbitrarily chosen. Then, for $u \in H^1(\mathcal{T}_h)^6$ we have*

$$|(A_h u, \pi_h v)_V| \leq \frac{C}{\gamma}\Big(\|u\|_V^2 + \sum_K h_K^2 \,|u|_{1,K}^2\Big) + C\gamma \sum_K |v|_{1,K}^2 \,. \tag{A.6}$$

For $u_h \in V_h$, we have

$$|(A_h u_h, \pi_h v)_V| \leq \frac{C}{\gamma}\,\|u_h\|_V^2 + C\gamma \sum_K |v|_{1,K}^2 \,. \tag{A.7}$$

The constants $C = C(\rho, k)$ are independent h and K.

Proof. We use (4.12) and bound the terms separately. The bound on the sum over the elements follows from Lemma A.1 and the bounds on the sums over the interior and exterior faces from Lemmas A.2 and A.3, respectively. ∎

REFERENCES

[1] P. Brenner, M. Crouzeix, and V. Thomée, *Single step methods for inhomogeneous linear differential equations in Banach space*, RAIRO - Analyse numérique, 12 (1982), pp. 5–26.

[2] P. Brenner and V. Thomée, *On rational approximations of semigroups*, SIAM J. Numer. Anal., 16 (1979), pp. 683–694.

[3] ———, *On rational approximations of groups of operators*, SIAM J. Numer. Anal., 17 (1980), pp. 119–125.

[4] E. Burman, A. Ern, and M. A. Fernández, *Explicit Runge-Kutta schemes and finite elements with symmetric stabilization for first-order linear PDE systems*, SIAM J. Numer. Anal., 48 (2010), pp. 2019–2042.

[5] M.-H. Chen, B. Cockburn, and F. Reitich, *High-order RKDG methods for computational electromagnetics*, J. Sci. Comput., 22/23 (2005), pp. 205–226.

[6] D. A. Di Pietro and A. Ern, *Mathematical aspects of discontinuous Galerkin methods*, vol. 69 of Mathématiques & Applications (Berlin) [Mathematics & Applications], Springer, Heidelberg, 2012.

[7] V. Dolean, H. Fahs, L. Fezoui, and S. Lanteri, *Locally implicit discontinuous Galerkin method for time domain electromagnetics*, J. Comput. Phys., 229 (2010), pp. 512–526.

[8] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, vol. 194 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[9] L. Fezoui, S. Lanteri, S. Lohrengel, and S. Piperno, *Convergence and stability of a discontinuous Galerkin time-domain method for the 3D heterogeneous Maxwell equations on unstructured meshes*, M2AN Math. Model. Numer. Anal., 39 (2005), pp. 1149–1176.

[10] S. Gottlieb, C.-W. Shu, and E. Tadmor, *Strong stability-preserving high-order time discretization methods*, SIAM Review, 43 (2001), pp. 89–112.

[11] M. J. Grote and T. Mitkova, *Explicit local time-stepping methods for Maxwells equations*, Journal Comput. and Appl. Math., 234 (2010), pp. 3283–3302.

[12] M. J. Grote, A. Schneebeli, and D. Schötzau, *Interior penalty discontinuous Galerkin method for Maxwell's equations: energy norm error estimates*, J. Comput. Appl. Math., 204 (2007), pp. 375–386.

[13] ———, *Interior penalty discontinuous Galerkin method for Maxwell's equations: optimal L^2-norm error estimates*, IMA J. Numer. Anal., 28 (2008), pp. 440–468.

[14] E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II. Stiff and Differential-Algebraic Problems*, vol. 14 of Springer Series in Computational Mathematics, Springer, Berlin, Heidelberg, 2nd ed., 1996.

[15] J. S. Hesthaven and T. Warburton, *Nodal high-order methods on unstructured grids. I. Time-domain solution of Maxwell's equations*, J. Comput. Phys., 181 (2002), pp. 186–221.

[16] ———, *Nodal discontinuous Galerkin methods*, vol. 54 of Texts in Applied Mathematics, Springer, New York, 2008. Algorithms, analysis, and applications.

[17] M. Hochbruck, T. Pažur, A. Schulz, E. Thawinan, and C. Wieners, *Efficient time integration for discontinuous Galerkin approximations of linear wave equations*, ZAMM, online first (2014). doi 10.1002/zamm.201300306.
[18] C. Johnson and J. Pitkäranta, *An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation*, Math. Comp., 46 (1986), pp. 1–26.
[19] L. Krivodonova, *An efficient local time-stepping scheme for solution of nonlinear conservation laws*, Journal Comput. Phys., 229 (2010), pp. 8537–8551.
[20] C. Lubich and A. Ostermann, *Runge-Kutta approximation of quasi-linear parabolic equations*, Math. Comp., 64 (1995), pp. 601–627.
[21] D. Mansour, *Gauß–Runge–Kutta time discretization of wave equations on evolving surfaces*, Numer. Math., online first (2014).
[22] P. Monk, *An analysis of Nédélec's method for the spatial discretization of Maxwell's equations*, Journal Comput. Appl. Math., 47 (1993), pp. 101 – 121.
[23] L. Moya, *Temporal convergence of a locally implicit discontinuous Galerkin method for Maxwell's equations*, ESAIM Math. Model. Numer. Anal., 46 (2012), pp. 1225–1246.
[24] T. Pažur, *Error analysis of implicit and exponential time integration of linear Maxwell's equations*, PhD thesis, Karlsruhe Institute of Technology, 2013.
[25] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, vol. 44 of Applied Mathematical Sciences, Springer, New York, 1983.
[26] T. E. Peterson, *A note on the convergence of the discontinuous Galerkin method for a scalar hyperbolic equation*, SIAM J. Numer. Anal., 28 (1991), pp. 133–140.
[27] S. Piperno, *Symplectic local time-stepping in non-dissipative DGTD methods applied to wave propagation problems*, M2AN Math. Model. Numer. Anal., 40 (2006), pp. 815–841 (2007).
[28] A. Taube, M. Dumbser, C.-D. Munz, and R. Schneider, *A high-order discontinuous Galerkin method with time-accurate local time stepping for the Maxwell equations*, International Journal of Numerical Modelling: Electronic Networks, Devices and Fields, 22 (2009), pp. 77–103.
[29] B. Wang, Z. Xie, and Z. Zhang, *Error analysis of a discontinuous Galerkin method for Maxwell equations in dispersive media*, J. Comput. Phys., 229 (2010), pp. 8552–8563.