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Bounding the Error for Approximate Solutions of Almost Linear Complementarity Problems Using Feasible Vectors

Götz Alefeld[1,2]

Institute of Applied and Numerical Mathematics
University of Karlsruhe (Karlsruhe Institute of Technology KIT)

Zhengyu Wang[3]

Department of Mathematics, Nanjing University

Abstract: In this paper we use the concept of a feasible vector in order to bound a solution x^* of an almost linear complementarity problem in a certain set. This set delivers also componentwise error bounds for an approximation $\hat{x}$ to a solution x^*. In the special case that the problem is defined by a so-called H-matrix, we prove that the error bounds are more accurate than the corresponding bounds recently obtained. By numerical examples it will be demonstrated that the new bounds can be better by several orders of magnitude.

Keywords: Complementarity problem, H-matrix, error bound.

Mathematics Subject Classification: 90C33, 65G30, 65K10.

1 Introduction

Let $F : R^n \to R^n$ be a given function. The nonlinear complementarity problem, denoted by $NCP(F)$, is to find a vector $x^* \in R^n$ such that

$$x^* \geq 0, \qquad F(x^*) \geq 0, \qquad (x^*)^T F(x^*) = 0. \tag{1.1}$$

The inequalities are meant componentwise. In this article we consider the problem (1.1) with a so-called *almost linear function*

$$F(x) = Mx + \varphi(x), \tag{1.2}$$

where $M \in R^{n\times n}$ is a given matrix, and $\varphi(x) : R^n \to R^n$ is a given monotonically increasing diagonal function. This means that, the i-th component of $\varphi(x)$ is a function of the i-th variable x_i only, $\varphi_i(x) = \varphi_i(x_i)$, and $\varphi_i(x_i)$ is monotonically increasing, $i = 1, \ldots, n$. If $\varphi(x) \equiv q \in R^n$, the problem reduces to a so-called linear complementarity problem, which we denote by $LCP(M, q)$.

[1]Corresponding author.

[2]Institute of Applied and Numerical Mathematics, University of Karlsruhe (Karlsruhe Institute of Technology KIT), 76128 Karlsruhe, Germany. Email: goetz.alefeld@math.uni-karlsruhe.de.

[3]Department of Mathematics, Nanjing University, Nanjing 210093, China. Email: zywang@nju.edu.cn.

Complementarity problems with functions of the form (1.2) have many real world applications, which has been demonstrated in [5, 6].

Let $\hat{x} \in R^n$ be a given approximation of a solution x^* of the problem $NCP(F)$ with the function F given by (1.2). In this paper we present an new approach of computing a componentwise bound of the error $e(\hat{x}) := \hat{x} - x^*$.

Denote by D and $-B$ the diagonal and off-diagonal part of M, respectively. We define $\mathcal{I} := \{1, \ldots, n\}$, and

$$\alpha := \{i \in \mathcal{I} |\ \hat{x}_i \leq (M\hat{x})_i + \varphi_i(\hat{x}_i)\} \tag{1.3}$$
$$\beta := \{i \in \mathcal{I} |\ \hat{x}_i > (M\hat{x})_i + \varphi_i(\hat{x}_i)\}. \tag{1.4}$$

(Note that $\alpha \cup \beta = \mathcal{I}$.)

$$\tilde{x} := (\tilde{x}_i) \quad \text{with} \quad \tilde{x}_i = \begin{cases} |\hat{x}_i| & \text{if } i \in \alpha \\ 0 & \text{if } i \in \beta \end{cases} \tag{1.5}$$
$$\tilde{y} := (\tilde{y}_i) \quad \text{with} \quad \tilde{y}_i = \begin{cases} (M\hat{x})_i + \varphi_i(\hat{x}_i) & \text{if } i \in \alpha \\ -|(M\hat{x})_i + \varphi_i(\hat{x}_i)| & \text{if } i \in \beta \end{cases} \tag{1.6}$$
$$\tilde{M} := D - |B| \tag{1.7}$$
$$\tilde{q} := \tilde{M}\tilde{x} + \tilde{y}. \tag{1.8}$$

The linear complementarity problem $LCP(\tilde{M}, \tilde{q})$ is said to be *feasible* (see [4]) if

$$FEA(\tilde{M}, \tilde{q}) := \{x \in R^n |\ x \geq 0,\ \tilde{M}x + \tilde{q} \geq 0\} \neq \emptyset.$$

An element $u \in FEA(\tilde{M}, \tilde{q})$ is called a *feasible vector* of the problem $LCP(\tilde{M}, \tilde{q})$. Let a feasible vector $u \in FEA(\tilde{M}, \tilde{q})$ be known, let

$$r = \tilde{x} + u \tag{1.9}$$

and define

$$\mathcal{D} := \{x \in R^n |\ |x - \hat{x}| \leq r\}.$$

Suppose that there exist constants γ_i such that for any $x = (x_i) \in \mathcal{D}$

$$\frac{\varphi_i(x_i) - \varphi_i(\hat{x}_i)}{x_i - \hat{x}_i} \leq \gamma_i \qquad \text{for} \quad x_i \neq \hat{x}_i, \tag{1.10}$$

$i = 1, \ldots, n$. We will show in Theorem 2.1 that $NCP(F)$ has a solution $x^* = (x_i^*) \in \mathcal{D}$. The following componentwise error bound is then straightforward:

$$|\hat{x}_i - x_i^*| \leq r_i. \tag{1.11}$$

Moreover, let M be an H-matrix (see below) whose diagonal elements are all positive. Then the linear complementarity problem $LCP(\tilde{M}, \tilde{q})$ is feasible, since $u = \tilde{M}^{-1}\max\{0, -\tilde{q}\}$ is a feasible vector of $LCP(\tilde{M}, \tilde{q})$. Therefore, by Theorem 2.1, the complementarity problem $NCP(F)$ has a solution $x^* \in \mathcal{D}$ (which is known to be unique). For this case we have the following error bound which follows immediately from (1.9) and (1.11):

$$|\hat{x} - x^*| \leq r = \tilde{x} + \tilde{M}^{-1}\max\{0, -\tilde{q}\}. \tag{1.12}$$

This error bound can be computed by solving a system of linear equations.

Very recently the error estimation

$$|\hat{x} - x^*| \;\; \leq \;\; \tilde{M}^{-1}\max\{D, I\}|\min\{\hat{x}, M\hat{x} + \varphi(\hat{x})\}| \tag{1.13}$$

was given in [1], when M is an H-matrix whose diagonal elements are all positive, D denotes the diagonal part of M, I denotes the identity matrix, and "max" has to be understood componentwise. From (1.13), the error estimation

$$\|\hat{x} - x^*\|_p \;\; \leq \;\; \|\tilde{M}^{-1}\max\{D, I\}\|_p\| \min\{\hat{x}, M\hat{x} + q\}\|_p$$

follows for $LCP(M, q)$. This has already been obtained in [3] for $1 \leq p \leq \infty$. It was proven to be more accurate than that given by Mathias and Pang [8]. We will show that our error bound (1.12) is more accurate than (1.13). Precisely speaking, we will prove that

$$\tilde{x} + \tilde{M}^{-1}\max\{0, -\tilde{q}\} \;\; \leq \;\; \tilde{M}^{-1}\max\{D, I\}|\min\{\hat{x}, M\hat{x} + \varphi(\hat{x})\}|.$$

It will be demonstrated that this inequality could hold strictly by several orders of magnitude (See Table 1 in Section 5).

We also give error bounds for $NCP(F)$ with an F, for which (1.10) does not hold, if we have

$$\hat{x}_i \;\; \neq \;\; (M\hat{x})_i + \varphi_i(\hat{x}_i)$$

for each $i = 1, \ldots, n$.

A word on the notations used in this paper. Let $x = (x_i), y = (y_i) \in R^n$. Let $x \leq y$ stand for $x_i \leq y_i$, $i = 1, \ldots, n$. We denote by $\max\{x, y\}$ and $\min\{x, y\}$ the componentwise maximum and minimum of x and y, respectively. Let $M = (m_{ij}) \in R^{n \times n}$. M is called a Z-matrix if each of its off-diagonal elements is non-positive. M is called a P-matrix if each of its principal minors is positive. M is called an H-matrix if its comparison matrix $\langle M \rangle = (\langle m_{ij} \rangle)$ has a nonnegative inverse, that is, each element of the inverse of $\langle M \rangle$ is nonnegative, where

$$\langle m_{ij} \rangle = \begin{cases} |m_{ii}| & \text{if} \quad i = j, \\ -|m_{ij}| & \text{if} \quad i \neq j. \end{cases}$$

Note that $\tilde{M} = \langle M \rangle$ in (1.7) if M is a matrix whose diagonal elements are all positive. It is well known that an H-matrix whose diagonal elements are all positive is a P-matrix. See [4, 10], for example. We define $|M| = (|m_{ij}|)$.

2 Error Estimation

At first we give an existence theorem for the complementarity problem $NCP(F)$, where F is defined by (1.2). This existence theorem simultaneously delivers a componentwise error estimation.

Theorem 2.1. *Let $M \in R^{n \times n}$ be a given matrix, and let $\varphi(x) = (\varphi_i(x_i))$ be a given diagonal mapping with each $\varphi_i(\cdot)$ being continuous and monotonically increasing. Let $\hat{x} = (\hat{x}_i) \in R^n_+$ be given, where R^n_+ denotes the set of vectors with nonnegative components. Let α, β, $\tilde{x}$ and $\tilde{y}$, $\tilde{M}$ and $\tilde{q}$ be defined by (1.3)-(1.8), respectively. Let an element $u \in \mathrm{FEA}(\tilde{q}, \tilde{M})$ be known, and let $r = \tilde{x} + u$. Denote*

$$\mathcal{D} := \{x \in R^n | \; |x - \hat{x}| \leq r\}. \tag{2.14}$$

Suppose that there are constants $\gamma_i \geq 0$, $i = 1, \ldots, n$, such that for any $x \in \mathcal{D}$

$$\frac{\varphi_i(x_i) - \varphi_i(\hat{x}_i)}{x_i - \hat{x}_i} \;\; \leq \;\; \gamma_i \qquad \textit{for} \quad x_i \neq \hat{x}_i. \tag{2.15}$$

Then the problem $NCP(F)$ has a solution $x^* \in \mathcal{D}$, where $F(x) = Mx + \varphi(x)$.

Proof. If (2.15) holds, then it holds for arbitrarily large constants γ_i and therefore we can assume without the loss of generality that the γ_i are chosen in such a manner that $m_{ii}+\gamma_i>0$, $i=1,\ldots,n$. Then we choose the elements of the diagonal matrix $\Delta = diag(\delta_i)$ such that

$$0 \;<\; \delta_i \;\leq\; \frac{1}{m_{ii}+\gamma_i}, \qquad i=1,\ldots,n. \tag{2.16}$$

We note that x^* is a solution of $NCP(F)$ if and only if x^* is a fixed point of the mapping

$$\Gamma(x) := \max\{0, x-\Delta(Mx+\varphi(x))\}.$$

We show that under our assumptions, Γ has a fixed point x^* in $\mathcal{D}$. Let $x\in\mathcal{D}$ be fixed. Assume that $x_i \neq \hat{x}_i$. From the fact that φ is monotonically increasing, it follows that

$$\frac{\varphi_i(x_i)-\varphi_i(\hat{x}_i)}{x_i-\hat{x}_i} \geq 0.$$

This, together with (2.16), yields

$$1-\delta_i m_{ii} \;\geq\; 1-\delta_i(m_{ii}+\frac{\varphi_i(x_i)-\varphi_i(\hat{x}_i)}{x_i-\hat{x}_i}) \;\geq\; 1-\delta_i(m_{ii}+\gamma_i) \;\geq\; 0,$$

and therefore we have for $x_i \neq \hat{x}_i$

$$\begin{aligned} & [(I-\Delta M)(x-\hat{x})-\Delta(\varphi(x)-\varphi(\hat{x}))]_i \\ = \;& \left(1-\delta_i(m_{ii}+\frac{\varphi_i(x_i)-\varphi_i(\hat{x}_i)}{x_i-\hat{x}_i})\right)(x_i-\hat{x}_i)-\delta_i\sum_{j\neq i} m_{ij}(x_j-\hat{x}_j) \\ \leq \;& (1-\delta_i m_{ii})r_i+\delta_i\sum_{j\neq i}|m_{ij}|r_j. \end{aligned}$$

If $x_i=\hat{x}_i$, then, using again (2.16)

$$\begin{aligned} & [(I-\Delta M)(x-\hat{x})-\Delta(\varphi(x)-\varphi(\hat{x}))]_i \\ = \;& x_i-\hat{x}_i-\delta_i\sum_{j=1}^{n} m_{ij}(x_j-\hat{x}_j) \;=\; -\delta_i\sum_{j=1}^{n} m_{ij}(x_j-\hat{x}_j) \\ \leq \;& \delta_i\sum_{j\neq i}^{n}|m_{ij}|r_j \;\leq\; (1-\delta_i m_{ii})r_i+\delta_i\sum_{j\neq i}|m_{ij}|r_j. \end{aligned}$$

Summarizing, we have for $x\in\mathcal{D}$

$$(I-\Delta M)(x-\hat{x})-\Delta(\varphi(x)-\varphi(\hat{x})) \;\leq\; (I-\Delta D+\Delta|B|)r.$$

From this we achieve

$$\begin{aligned} \Gamma(x) &= \max\{0, x-\Delta(Mx+\varphi(x))\} \\ &= \max\{0, \hat{x}-\Delta(M\hat{x}+\varphi(\hat{x}))+(I-\Delta M)(x-\hat{x})-\Delta(\varphi(x)-\varphi(\hat{x}))\} \\ &\leq \max\{0, \hat{x}-\Delta(M\hat{x}+\varphi(\hat{x}))+(I-\Delta D+\Delta|B|)r\}. \end{aligned}$$

Noting that $M\hat{x}+\varphi(\hat{x})\geq\tilde{y}$ and $u\in FEA(\tilde{M},\tilde{q})$, using (1.7), (1.8) and (1.9) we have

$$\begin{aligned} (D-|B|)r+(M\hat{x}+\varphi(\hat{x})) &= \tilde{M}(\tilde{x}+u)+(M\hat{x}+\varphi(\hat{x})) \\ &\geq \tilde{M}(\tilde{x}+u)+\tilde{y} \\ &= \tilde{M}u+(\tilde{M}\tilde{x}+\tilde{y}) = \tilde{M}u+\tilde{q}\geq 0. \end{aligned}$$

From this we obtain

$$-\Delta(M\hat{x}+\varphi(\hat{x}))+\Delta(-D+|B|)r \leq 0,$$

therefore

$$\hat{x}-\Delta(M\hat{x}+\varphi(\hat{x}))+(I-\Delta D+\Delta|B|)r \;\leq\; \hat{x}+r,$$

which, together with $\hat{x}+r \geq 0$, yields $\Gamma(x) \leq \hat{x}+r$.
We move on to prove $\Gamma(x) \geq \hat{x}-r$. Similarly as above, we can show

$$(I-\Delta M)(x-\hat{x})-\Delta(\varphi(x)-\varphi(\hat{x})) \ \geq\ -(I-\Delta D+\Delta|B|)r.$$

Consequently

$$\begin{aligned}\Gamma(x) &= \max\{0, x-\Delta(Mx+\varphi(x))\}\\ &= \max\{0, \hat{x}-\Delta(M\hat{x}+\varphi(\hat{x}))+(I-\Delta M)(x-\hat{x})-(\varphi(x)-\varphi(\hat{x}))\}\\ &\geq \max\{0, \hat{x}-\Delta(M\hat{x}+\varphi(\hat{x}))-(I-\Delta D+\Delta|B|)r\}.\end{aligned}$$

For an index $i \in \alpha$, we have $(\hat{x}-r)_i = -u_i \leq 0 \leq [\Gamma(x)]_i$. For an index $i \in \beta$, noting that $-(M\hat{x}+\varphi(\hat{x})) \geq \tilde{y}$ and $u \in FEA(\tilde{M},\tilde{q})$, we have

$$\begin{aligned}[(D-|B|)r-(M\hat{x}+\varphi(\hat{x}))]_i &= [\tilde{M}(\tilde{x}+u)-(M\hat{x}+\varphi(\hat{x}))]_i\\ &\geq [\tilde{M}(\tilde{x}+u)+\tilde{y}]_i\\ &= [\tilde{M}u+(\tilde{M}\tilde{x}+\tilde{y})]_i = [\tilde{M}u+\tilde{q}]_i \geq 0.\end{aligned}$$

Therefore

$$[\hat{x}-\Delta(M\hat{x}+\varphi(\hat{x}))+(I-\Delta D+\Delta|B|)r]_i \geq (\hat{x}-r)_i.$$

This yields

$$[\Gamma(x)]_i \ \geq\ \max\{0, [\hat{x}-\Delta(M\hat{x}+\varphi(\hat{x}))-(I-\Delta D+\Delta|B|)r]_i\} \ \geq\ (\hat{x}-r)_i.$$

Summerarizing, we have $\Gamma(x) \geq \hat{x}-r$. Hence we have shown that Γ maps $\mathcal{D}$ into itself. Since Γ is continuous and $\mathcal{D}$ is compact, convex and not empty, it follows from the Brouwer fixed-point theorem [9] that Γ has a fixed point $x^* \in \mathcal{D}$, which is a solution of $NCP(F)$. □

The existence domain $\mathcal{D}$, defined by (2.14), delivers a componentwise error estimation

$$|x^*-\hat{x}| \ \leq\ r. \tag{2.17}$$

The error bound $r=\tilde{x}+u$ can be obtained once a feasible vector $u \in FEA(\tilde{M},\tilde{q})$ is known. However, there might be more than one feasible vector $u \in FEA(\tilde{M},\tilde{q})$. The following result on the "sharpest" error bound given in Theorem 2.1 is well known.

Theorem 2.2. *Let $\tilde{M}$ and $\tilde{q}$ be defined by (1.7) and (1.8), respectively. If $FEA(\tilde{M},\tilde{q}) \neq \emptyset$, then there is a unique vector $u^* \in FEA(\tilde{M},\tilde{q})$ such that*

$$u^* \ \leq\ u, \qquad \forall u \in FEA(\tilde{M},\tilde{q}).$$

Moreover u^* is a solution of $LCP(\tilde{M},\tilde{q})$.

Proof. For the proof we refer to Theorem 3.11.6, pp.201, [4]. □

The vector u^* from the preceding theorem is called *the least element* of the set $FEA(\tilde{M},\tilde{q})$.

Theorem 2.2 indicates that we can obtain a "sharpest" error bound via the least element of $FEA(\tilde{M},\tilde{q})$, since for a given vector $\hat{x} \in R^n$, we have

$$|\hat{x}-x^*| \ \leq\ \tilde{x}+u^* \ \leq\ \tilde{x}+u, \qquad \forall u \in FEA(\tilde{M},\tilde{q}).$$

In Theorem 2.1 we impose no requirement on the matrix M. The problem $NCP(F)$ may have no solution (in this case the problem $LCP(\tilde{M},\tilde{q})$ is not feasible for any $\hat{x}$), or may have a unique solution, or may have more than one solution. The following well known result is on the unique solvability of the problem $NCP(F)$.

Theorem 2.3 ([1]). *Let $\varphi(x) = (\varphi_i(x_i))$ be a given diagonal mapping with each $\varphi_i(\cdot)$ being continuous and monotonically increasing, let $M \in R^{n\times n}$ be a P-matrix. Then $NCP(F)$ has a unique solution x^*.*

We know that if M is an H-matrix with positive diagonal elements, then M is a P-matrix, and the corresponding problem $NCP(F)$ has a unique solution. We show that $LCP(\tilde{M}, \tilde{q})$ is always feasible for this case.

Theorem 2.4. *Let the conditions of Theorem 2.1 hold. Moreover suppose that M is an H-matrix with positive diagonal elements. Let $\tilde{x}$, $\tilde{M}$ and $\tilde{q}$ are defined by (1.5), (1.7) and (1.8), respectively. Then*

$$u \;=\; \langle M\rangle^{-1}\max\{0, -\tilde{q}\} \in FEA(\tilde{M}, \tilde{q}),$$

and the problem $NCP(F)$ has a unique solution x^* with the estimation

$$|\hat{x} - x^*| \;\le\; r \;=\; \tilde{x} + \langle M\rangle^{-1}\max\{0, -\tilde{q}\}. \tag{2.18}$$

(Note that (2.18) is identical to (1.12) because $\tilde{M} = \langle M\rangle$.)

Proof. Since M is assumed to be an H-matrix with positive diagonal elements, we have $\langle M\rangle^{-1} \ge 0$, that is, each element of $\langle M\rangle^{-1}$ is nonnegative. So

$$u \;=\; \langle M\rangle^{-1}\max\{0, -\tilde{q}\} \ge 0.$$

Moreover, noting $\tilde{M} = \langle M\rangle$, we have

$$\tilde{M}u + \tilde{q} \;=\; \tilde{M}\langle M\rangle^{-1}\max\{0, -\tilde{q}\} + \tilde{q} \;=\; \max\{0, -\tilde{q}\} + \tilde{q} \;\ge\; 0.$$

Hence $u = \langle M\rangle^{-1}\max\{0, -\tilde{q}\} \in FEA(\tilde{M}, \tilde{q})$, and from Theorem 2.1 it follows that there is a solution x^* of the problem $NCP(F)$ with $|x^* - \hat{x}| \le r$. The uniqueness of the solution can be guaranteed by Theorem 2.3. □

From the following example, we can see that for the case that M is a P-matrix without being an H-matrix, the problem $LCP(\tilde{M}, \tilde{q})$ can be feasible, and Theorem 2.1 can be applied.

Example 2.5. *Let*

$$M \;=\; \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \qquad q \;=\; \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

M is a P-matrix but not an H-matrix. $LCP(M, q)$ has the unique solution $x^* = [0, 1]^T$. We compute for $\hat{x} = [1, 7]^T$ that $\tilde{x} = [1, 0]^T$ and

$$\tilde{M} \;=\; \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}, \qquad \tilde{q} \;=\; \begin{bmatrix} 15 \\ -6 \end{bmatrix}.$$

It is easy to verify that $u = [1, 8]^T \in FEA(\tilde{M}, \tilde{q})$, and from (2.17) we have the error bound

$$|\hat{x} - x^*| \;\le\; r \;=\; \tilde{x} + u \;=\; \begin{bmatrix} 2 \\ 8 \end{bmatrix}.$$

Unfortunately, we are not in the position to prove that for the whole set of P-matrices, we can find a feasible vector of $LCP(\tilde{M}, \tilde{q})$, which would allow the application of Theorem 2.1.
On the other hand we now present an example to demonstrate that Theorem 2.1 can also be applied even if M is not a P-matrix.

Example 2.6. *Let*

$$M = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \qquad q = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Note that M is not a P-matrix since $m_{11} = 0$. $LCP(M,q)$ has the unique solution $x^* = [0,1]^T$. We compute for $\hat{x} = [1,6]^T$ that $\tilde{x} = [1,6]^T$ and

$$\tilde{M} = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix}, \qquad \tilde{q} = \begin{bmatrix} 0 \\ 11 \end{bmatrix}.$$

It is easy to verify that $u = [0,0]^T \in FEA(\tilde{M}, \tilde{q})$, and from (2.17) we have the error bound

$$|\hat{x} - x^*| \leq r = \tilde{x} + u = \begin{bmatrix} 1 \\ 6 \end{bmatrix}.$$

3 Discussion for the Non-Lipschitz Case

Let $\hat{x} = (\hat{x}_i) \in R^n$ be given, let $\mathcal{D}$ be defined by (2.14). In Theorem 2.1 the condition (2.15) is required, that is, it is supposed that there are constants $\gamma_i \geq 0$, $i = 1, \ldots, n$, such that for any $x = (x_i) \in \mathcal{D}$

$$\frac{\varphi_i(x_i) - \varphi_i(\hat{x}_i)}{x_i - \hat{x}_i} \leq \gamma_i \qquad \text{for} \quad x_i \neq \hat{x}_i.$$

This condition is fulfilled if each φ_i is locally Lipschitz at $\hat{x}_i$. A function $f : R^n \to R^n$ is said to be locally Lipschitz at a vector $\hat{x} \in R^n$ if there are constants $\epsilon > 0$ and $\gamma \geq 0$ such that for any $x \in \mathcal{N}(\hat{x}) = \{x \in R^n | \ \|x - \hat{x}\| \leq \epsilon\}$

$$\|f(x) - f(\hat{x})\| \leq \gamma \|x - \hat{x}\|.$$

For the case that φ fails to be Lipschitz continuous at $\hat{x}$, we have the following result on the error estimation for the problem $NCP(F)$.

Theorem 3.1. *Let $M \in R^{n \times n}$ be an H-matrix with positive diagonal elements. Let $\varphi(x) = (\varphi_i(x_i))$ be a given diagonal mapping with each $\varphi_i(\cdot)$ continuous and monotonically increasing. Denote by x^* the unique solution of $NCP(F)$ (see Theorem 2.3), where $F(x) = Mx + \varphi(x)$. Let a vector $\hat{x} \in R^n$ be given such that*

$$\hat{x}_i \neq (M\hat{x})_i + \varphi_i(\hat{x}_i), \tag{3.19}$$

$i = 1, \ldots, n$. Let $\tilde{x}$, $\tilde{M}$ and $\tilde{q}$ be defined by (1.5), (1.7) and (1.8), respectively. If there is a sequence $\{\hat{x}^m\}_{m=1}^{\infty}$ with $\lim_{m \to \infty} \hat{x}^m = \hat{x}$ such that $\varphi(\cdot)$ is Lipschitz continuous at each $\hat{x}^m$, then we have

$$|\hat{x} - x^*| \leq \tilde{x} + \langle M \rangle^{-1} \max\{0, -\tilde{q}\}.$$

Remark 3.2. A solution x^* of $NCP(F)$ is called non-degenerate (see [4]) if

$$x_i^* \neq F_i(x^*), \qquad i = 1, \ldots, n.$$

Condition (3.19) is fulfilled when $\hat{x}$ is sufficiently close to a non-degenerate solution x^*.

Proof. Let the index subsets α and β be defined by (1.3) and (1.4), respectively. Under the condition (3.19) they can be written as

$$\alpha := \{i : \ \hat{x}_i < (M\hat{x})_i + \varphi_i(\hat{x}_i)\}, \qquad \beta := \{i : \ \hat{x}_i > (M\hat{x})_i + \varphi_i(\hat{x}_i)\}.$$

Since $\lim_{m\to\infty} \hat{x}^m = \hat{x}$ and $Mx + \varphi(x)$ is continuous, we have for a sufficiently large m

$$\begin{array}{ll} \hat{x}_i^m < (M\hat{x}^m)_i + \varphi_i(\hat{x}_i^m) & \text{if } i \in \alpha, \\ \hat{x}_i^m > (M\hat{x}^m)_i + \varphi_i(\hat{x}_i^m) & \text{if } i \in \beta. \end{array}$$

Let $\tilde{x}^m$ and $\tilde{y}^m$ be defined as in (1.5) and (1.6), respectively. Under the condition (3.19) they can be written as

$$\begin{aligned} \tilde{x}^m &= (\tilde{x}_i^m) \text{ with } \tilde{x}_i^m = \begin{cases} |\hat{x}_i^m| & \text{if } i \in \alpha, \\ 0 & \text{if } i \in \beta, \end{cases} \\ \tilde{y}^m &= (\tilde{y}_i^m) \text{ with } \tilde{y}_i^m = \begin{cases} (M\hat{x}^m)_i + \varphi_i(\hat{x}_i^m) & \text{if } i \in \alpha, \\ -|(M\hat{x}^m)_i + \varphi_i(\hat{x}_i^m)| & \text{if } i \in \beta. \end{cases} \end{aligned}$$

We have $\lim_{m\to\infty} \tilde{x}^m = \tilde{x}$, $\lim_{m\to\infty} \tilde{y}^m = \tilde{y}$, where $\tilde{x}$ and $\tilde{y}$ are defined by (1.5) and (1.6), respectively. Let $\tilde{q}^m := \tilde{M}\tilde{x}^m + \tilde{y}^m$. Therefore

$$\lim_{m\to\infty} \tilde{q}^m = \tilde{M}\tilde{x} + \tilde{y} = \tilde{q}.$$

Since $\varphi(\cdot)$ is Lipschitz continuous at each $\hat{x}^m$, from Theorem 2.1 we have

$$|\hat{x}^m - x^*| \le \tilde{x}^m + \langle M\rangle^{-1} \max\{0, -\tilde{q}^m\}$$

if m is sufficiently large. Taking $m \to \infty$, this yields the conclusion. □

4 Comparison of Error Bounds

In this section we show for the case that M is an H-matrix with positive diagonal elements that the error bound (2.18) is more accurate than (1.13), which was given in [1].

Theorem 4.1. *Let $M \in R^{n\times n}$ be an H-matrix whose diagonal elements are all positive, and let $\varphi(x) = (\varphi_i(x_i))$ be a given diagonal mapping. Let $\hat{x} \in R_+^n$ be given, where R_+^n denotes the set of vectors with nonnegative components. Let $\tilde{x}$ and $\tilde{q}$ be defined by (1.5) and (1.8), respectively. Let D denote the diagonal part of M. Then we have*

$$\tilde{x} + \langle M\rangle^{-1} \max\{0, -\tilde{q}\} \le \langle M\rangle^{-1} \max\{D, I\} |\min\{\hat{x}, M\hat{x} + \varphi(\hat{x})\}|. \tag{4.20}$$

(The right hand side of (4.20) is identical to that of (1.13) because $\tilde{M} = \langle M\rangle$.)

Proof. Let $\tilde{y}$ be defined by (1.6). At first we show

$$\max\{\tilde{M}\tilde{x}, -\tilde{y}\} \le \max\{D, I\} |\min\{\hat{x}, M\hat{x} + \varphi(x)\}|.$$

From (1.5) we know that $\tilde{x} \ge 0$. We have

$$(\langle M\rangle\tilde{x})_i = m_{ii}\tilde{x}_i - \sum_{j\ne i} |m_{ij}|\tilde{x}_j \le m_{ii}|\hat{x}_i| \le \max\{1, m_{ii}\}|\hat{x}_i|.$$

From the definition of $\tilde{y}$ in (1.6) we know that for $i \in \alpha$

$$-\tilde{y}_i = -(M\hat{x})_i - \varphi_i(\hat{x}_i) \le -\hat{x}_i \le \max\{1, m_{ii}\}|\hat{x}_i|.$$

Noting for $i \in \alpha$ that $\min\{\hat{x}_i, (M\hat{x})_i + \varphi_i(\hat{x}_i)\} = \hat{x}_i$, we have for $i \in \alpha$

$$\max\{(\langle M\rangle\tilde{x})_i, -\tilde{y}_i\} \le \max\{1, m_{ii}\} |\min\{\hat{x}_i, (M\hat{x})_i + \varphi_i(\hat{x}_i)\}|. \tag{4.21}$$

For $i \in \beta$, from the definition of $\tilde{x}$ in (1.5) we know that $\tilde{x}_i = 0$. Therefore we have

$$(\langle M\rangle \tilde{x})_i = m_{ii}\tilde{x}_i - \sum_{j\neq i}|m_{ij}|\tilde{x}_j \leq 0 \leq \max\{1, m_{ii}\}|(M\tilde{x})_i + \varphi_i(\hat{x}_i)|$$

for $i \in \beta$. From the definition of $\tilde{y}$ in (1.6) we know that for $i \in \beta$

$$\tilde{y}_i = -|(M\hat{x})_i + \varphi_i(\hat{x}_i)|,$$

therefore

$$-\tilde{y}_i = |(M\hat{x})_i + \varphi_i(\hat{x}_i)| \leq \max\{1, m_{ii}\}|(M\hat{x})_i + \varphi_i(\hat{x}_i)|.$$

Noting for $i \in \beta$ that $\min\{\hat{x}_i, (M\hat{x})_i + \varphi_i(\hat{x}_i)\} = (M\hat{x})_i + \varphi_i(\hat{x}_i)$. We have for $i \in \beta$

$$\max\{(\langle M\rangle \tilde{x})_i, -\tilde{y}_i\} \leq \max\{1, m_{ii}\}|\min\{\hat{x}_i, (M\hat{x})_i + \varphi_i(\hat{x}_i)\}|. \tag{4.22}$$

Summarizing (4.21) and (4.22) we have

$$\max\{\langle M\rangle \tilde{x}, -\tilde{y}\} \leq \max\{I, D\}|\min\{\hat{x}, M\hat{x} + \varphi(\hat{x})\}|.$$

Since M is an H-matrix with positive diagonal elements, each element of $\langle M\rangle^{-1}$ is nonnegative. So

$$\langle M\rangle^{-1}\max\{\langle M\rangle \tilde{x}, -\tilde{y}\} \leq \langle M\rangle^{-1}\max\{D, I\}|\min\{\hat{x}, M\hat{x} + \varphi(x)\}|.$$

From the definition of $\tilde{q} = \tilde{M}\tilde{x} + \tilde{y}$ and from the fact that $\tilde{M} = \langle M\rangle$ it follows that

$$\begin{aligned}\langle M\rangle^{-1}\max\{\langle M\rangle \tilde{x}, -\tilde{y}\} &= \langle M\rangle^{-1}(\tilde{M}\tilde{x} + \max\{0, -\tilde{M}\tilde{x} - \tilde{y}\}) \\ &= \tilde{x} + \langle M\rangle^{-1}\max\{0, -\tilde{q}\},\end{aligned}$$

which completes the proof. □

5 Numerical Experiment and Remarks

In this section we perform numerical experiments for Example 5.1 to demonstrate that the bound (2.18) is more accurate than (1.13), that is, to demonstrate (4.20):

$$\tilde{x} + \langle M\rangle^{-1}\max\{0, -\tilde{q}\} \leq \langle M\rangle^{-1}\max\{D, I\}|\min\{\hat{x}, M\hat{x} + \varphi(\hat{x})\}|.$$

Example 5.1. *Let $\Omega = (0,1)^2 \subseteq R^2$. Let $\phi(s,t,\nu) : R^3 \to R_+$ and $\psi(s,t) : R^2 \to R$ be given functions. We consider finding a function $u(s,t) : R^2 \to R$ such that*

$$\begin{cases} \triangle u = \phi(s,t,u) & \text{in } \Omega_+ \\ u = \psi(s,t) & \text{in } \partial\Omega \\ u \geq 0 & \text{in } \Omega, \end{cases}$$

where the domain

$$\Omega_+ := \{(s,t) \in \Omega |\ u(s,t) > 0\}$$

is unknown. This free boundary problem is formulated from a Dirichlet problem [7], and it models some reaction-diffusion procedures. We impose a uniform square mesh of $n = k^2$ grid points (s_l, t_m) with the coordinates (lh, mh), $h = 1/(k+1)$, $l, m = 1, \ldots, k$. The solution u can be approximated by a vector $x^* = (x_i^*) \in R^n$, which solves NCP(F), where $F(x) = Mx + \varphi(x)$, $M \in R^{n\times n}$, $\varphi(x) = (\varphi_i(x_i))$. The matrix M has the form

$$M = \frac{1}{h^2}\begin{bmatrix} H & -I & & \\ -I & H & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & H \end{bmatrix} \quad \text{with} \quad H = \begin{bmatrix} 4 & -1 & & \\ -1 & 4 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{bmatrix}.$$

	$\mu = 0.1$	$\mu = 0.3$	$\mu = 0.5$	$\mu = 0.7$	$\mu = 0.9$
$n = 100$	1.0687e-02	1.2832e-02	6.6929e-03	1.0399e-02	8.2744e-03
$n = 225$	2.8664e-03	2.8058e-03	3.9246e-03	2.5748e-03	2.5866e-03
$n = 400$	1.9845e-03	1.6332e-03	1.4918e-03	1.6383e-03	1.7056e-03
$n = 900$	8.4605e-04	8.4262e-04	9.0013e-04	9.0624e-04	7.5197e-04
$n = 1600$	3.9945e-04	3.8633e-04	4.1893e-04	4.3989e-04	4.0299e-04

Table 1: Values of κ from (5.23)

The diagonal function has the form

$$\varphi_i(x_i) = \phi(s_l, t_m, x_i) + c_i,$$

where c_i is constant and is dependent on the values of ψ on the boundary of Ω, $i = (l-1)k + m$, $l, m = 1, \ldots, k$. It can be verified that M is an H-matrix whose diagonal elements are all positive. A similar discrete analogue can also be obtained by the finite element method for a more complex domain Ω. See [2].

Let $\mu \in (0,1)$ be given, set $\lambda = \frac{9}{(1-\mu)^2}$. In our numerical experiments we choose ϕ and φ in Example 5.1 as follows

$$\phi(s,t,u) = \frac{9}{(1-\mu)^2\sqrt{s^2+t^2}} \left(\frac{3\sqrt{s^2+t^2}-1}{2}\right)^{\frac{2\mu}{1-\mu}} \max\{0, \sqrt{s^2+t^2} - \frac{1}{3}\} + \lambda \max\{0,u\}^{\mu},$$

$$\psi(s,t) = \left(\frac{3\sqrt{s^2+t^2}-1}{2}\right)^{\frac{2}{1-\mu}} \max\{0, \sqrt{s^2+t^2} - \frac{1}{3}\}.$$

Let $\hat{x} = (\hat{x}_i) \in R^n_+$ be generated in the following way

$$\hat{x}_i = \max\{0, v_i - 0.5\} \times 10^{w_i - 0.5},$$

where v_i and w_i are random numbers in $[0,1]$, $i = 1, \ldots, n$. The assumptions in Theorem 3.1 concerning the sequence $\{\hat{x}^m\}_{m=1}^{\infty}$ are fulfilled for this choice of $\hat{x}$. In Table 1 we report for the vectors $\hat{x}$ which fulfill the condition (3.19) the following values

$$\kappa := \max_{i=1,\ldots,n} \frac{(\tilde{x} + \langle M\rangle^{-1}\max\{0, -\tilde{q}\})_i}{(\langle M\rangle^{-1}\max\{D, I\}|\min\{\hat{x}, M\hat{x} + \varphi(\hat{x})\}|)_i}. \tag{5.23}$$

The numerical results indicate that our error bound (2.18) is more accurate than (1.13), which was given by Alefeld and Chen in [1], by several orders of magnitude.

Remark 5.2. In the course of numerical computation it might be difficult because of rounding errors to determine the sets α and β, defined by (1.3) and (1.4). Without going into details we mention that by using Intlab [11] rounding errors can be taken into account by modifying (1.3) and (1.4), and correspondingly also Theorem 2.1.

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