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NEW CRITERIA FOR THE FEASIBILITY OF THE CHOLESKY METHOD WITH INTERVAL DATA*

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Dedicated to Prof. Dr. Ulrich Kulisch of the University of Karlsruhe on the occasion of his 75th birthday

Abstract. We present some new criteria for the feasibility of the interval Cholesky method. In particular, we relate this feasibility to that of the interval Gaussian algorithm.

Key words. linear interval equations, Cholesky method, interval Cholesky algorithm, Gaussian algorithm, interval Gaussian algorithm, linear system of equations, criteria of feasibility

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1. Introduction. In [2] we introduced the interval Cholesky method in order to enclose the symmetric solution set

$$S_{\mathrm{sym}} = \{x \in \mathbb{R}^n \mid Ax = b,\ A = A^T \in [A] = [A]^T,\ b \in [b]\},$$

where $[A] = [\underline{A}, \overline{A}]$ is a given $n \times n$ interval matrix and $[b]$ is a corresponding interval vector. The algorithm uses the formulae of the classical Cholesky method, replacing the real entries and arithmetic by interval ones. It terminates with an interval vector $[x]^C = \mathrm{ICh}([A], [b])$ which encloses S_{sym} but not necessarily the general solution set

$$S = \{x \in \mathbb{R}^n \mid Ax = b,\ A \in [A],\ b \in [b]\},$$

which also contains the solutions of linear systems with unsymmetric matrices from $[A]$. A criterion necessary for $[x]^C$ to exist is the positive definiteness of all symmetric matrices in $[A]$—independently of any right-hand side $[b]$. Unfortunately, this property is not sufficient as Reichmann's example in [13] shows which originally was constructed for a different situation. This example caused the necessity of criteria which guarantee the existence of $[x]^C$ or, equivalently, the feasibility of the interval Cholesky method for arbitrary right-hand interval sides. In [2] we proved that $[x]^C$ exists for a variety of structured matrices, among them H-matrices, M-matrices, diagonal dominant matrices, and tridiagonal ones, all with appropriate additional properties. In [3] we extended these criteria of feasibility by perturbation results analogously to those in [11]. In [15] further results of feasibility were presented for block variants of the algorithm which were introduced there. It is the purpose of the present paper to add others. In particular, we will show that the feasibility of the interval Gaussian algorithm [1] implies the existence of $[x]^C$ provided that $[A]$ contains at least one positive definite element matrix. Based on this crucial result a

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great deal of criteria for the interval Gaussian algorithm carries over to the interval Cholesky method. Unfortunately, the feasibility of the interval Cholesky method does not necessarily imply that of the interval Gaussian algorithm. We will illustrate this phenomenon by an example. It was unexpected, since we can show that the existence of $[x]^C$ for each *symmetric* matrix $\tilde{A} \in [A]$ implies the feasibility of the Gaussian algorithm for *any* matrix $A \in [A]$ and not only for the symmetric ones.

We have organized this paper as follows: In section 2 we recall the formulae for the algorithm and a recursive representation. In addition we introduce our notation and some basic facts that are used later on. In section 3 we state and prove our new results illustrating them by examples.

2. Preliminaries. By $\mathbb{R}^n, \mathbb{R}^{n\times n}, \mathbb{IR}, \mathbb{IR}^n, \mathbb{IR}^{n\times n}$ we denote the set of real vectors with n components, the set of real $n\times n$ matrices, the set of intervals, the set of interval vectors with n components, and the set of $n \times n$ interval matrices, respectively. By "interval" we always mean a real compact interval. We write interval quantities in brackets with the exception of point quantities (i.e., degenerate interval quantities) which we identify with the element they contain. Examples are the zero matrix O, the identity matrix I, and the vector $e = (1, 1, \ldots, 1)^T$. We use the notation $[A] = [\underline{A}, \overline{A}] = ([a]_{ij}) = ([\underline{a}_{ij}, \overline{a}_{ij}]) \in \mathbb{IR}^{n\times n}$ simultaneously without further reference, and we proceed similarly for the elements of $\mathbb{R}^n, \mathbb{R}^{n\times n}, \mathbb{IR}$, and $\mathbb{IR}^n$. We also mention the standard notation from interval analysis ([1], [11]),

$$\check{a} = \mathrm{mid}([a]) = (\underline{a} + \overline{a})/2 \qquad \text{(midpoint)},$$

$$|[a]| = \max\{|\tilde{a}| \mid \tilde{a} \in [a]\} = \max\{|\underline{a}|, |\overline{a}|\} \qquad \text{(absolute value)},$$

$$\langle [a] \rangle = \min\{|\tilde{a}| \mid \tilde{a} \in [a]\} = \begin{cases} \min\{|\underline{a}|, |\overline{a}|\} & \text{if } 0 \notin [a], \\ 0 & \text{otherwise} \end{cases} \qquad \text{(minimal absolute value)}$$

for intervals $[a]$. For $[A] \in \mathbb{IR}^{n\times n}$ we obtain $|[A]| \in \mathbb{R}^{n\times n}$ by applying the operator $|\cdot|$ entrywise, and we define the comparison matrix $\langle [A] \rangle = (c_{ij}) \in \mathbb{R}^{n\times n}$ by setting

$$c_{ij} = \begin{cases} -|[a]_{ij}| & \text{if } i \neq j, \\ \langle [a]_{ii} \rangle & \text{if } i = j. \end{cases}$$

Since real numbers can be viewed as degenerate intervals, $|\cdot|$ and $\langle \cdot \rangle$ can also be used for them. In this case they coincide with their well known real counterpart.

By $A \geq O$ we denote a nonnegative $n \times n$ matrix, i.e., $a_{ij} \geq 0$ for $i, j = 1, \ldots, n$. Analogously, we define $x \geq 0$ for $x \in \mathbb{R}^n$. We call $x \in \mathbb{R}^n$ positive writing $x > 0$ if $x_i > 0$, $i = 1, \ldots, n$. We use $Z^{n\times n}$ for the set of real $n \times n$ matrices with nonpositive off-diagonal entries. Trivially, $Z^{n\times n}$ contains the $n \times n$ matrix $\langle A \rangle$. As usual we call $A \in \mathbb{R}^{n\times n}$ an M-matrix if A is nonsingular with $A^{-1} \geq O$ and $A \in Z^{n\times n}$. It is an H-matrix if $\langle A \rangle$ is an M-matrix.

An interval matrix $[A] \in \mathbb{IR}^{n\times n}$ is defined to be an M-matrix if each element $\tilde{A} \in [A]$ is an M-matrix. In the same way the term "H-matrix" can be extended to $\mathbb{IR}^{n\times n}$. It is easy to verify that $[A] \in \mathbb{IR}^{n\times n}$ is an M-matrix if and only if $\underline{A}$ is an M-matrix and $\overline{a}_{ij} \leq 0$ for $i \neq j$, and that $[A] \in \mathbb{IR}^{n\times n}$ is an H-matrix if and only if $\langle [A] \rangle$ is an M-matrix.

We call $[A] \in \mathbb{IR}^{n\times n}$ irreducible if $\langle [A] \rangle$ is irreducible. In the same way we define $[A]$ to be diagonally dominant, strictly diagonally dominant, and irreducibly diagonally dominant, respectively. If there is a positive vector x such that

$$\langle [A] \rangle x \geq 0 \tag{2.1}$$

holds, then we call $[A]$ generalized diagonally dominant. Moreover, we define $[A]$ to be generalized strictly diagonally dominant if strict inequality holds in (2.1). Analogously, a generalized irreducibly diagonally dominant matrix $[A]$ is irreducible and generalized diagonally dominant with $(\langle [A] \rangle x)_i > 0$ in (2.1) for at least one component i. It is well known that generalized strictly diagonally dominant matrices are H-matrices and vice versa.

We equip $\mathbb{IR}, \mathbb{IR}^n, \mathbb{IR}^{n\times n}$ with the usual real interval arithmetic as described in [1], [11]. We assume that the reader is familiar with the basic properties of this arithmetic. For $[a] \in \mathbb{IR}$ we define

$$\sqrt{[a]} = \{\sqrt{a} \mid a \in [a]\} \text{ for } 0 \leq \underline{a}$$

and

$$[a]^2 = \{a^2 \mid a \in [a]\}. \tag{2.2}$$

Instead of $\sqrt{[a]}$ we also write $[a]^{1/2}$.

Then the interval Cholesky method reads as follows.

Given $[A] = [A]^T \in \mathbb{IR}^{n\times n}$ and $[b] \in \mathbb{IR}^n$, define the lower triangular matrix $[L]$ and the vectors $[y], [x]^C = ([x]_i^C) = \mathrm{ICh}([A],[b]) \in \mathbb{IR}^n$ by

$$\left.\begin{aligned} [l]_{jj} &= \left([a]_{jj} - \sum_{k=1}^{j-1} [l]_{jk}^2 \right)^{1/2}, \\ [l]_{ij} &= \left([a]_{ij} - \sum_{k=1}^{j-1} [l]_{ik}[l]_{jk} \right) / [l]_{jj}, \quad i = j+1, \ldots, n, \end{aligned}\right\} \quad j = 1, \ldots, n,$$

$$[y]_i = \left([b]_i - \sum_{j=1}^{i-1} [l]_{ij}[y]_j \right) / [l]_{ii}, \qquad i = 1, \ldots, n,$$

$$[x]_i^C = \left([y]_i - \sum_{j=i+1}^{n} [l]_{ji}[x]_j^C \right) / [l]_{ii}, \quad i = n, n-1, \ldots, 1.$$

Sums with an upper bound smaller than the lower one are defined to be zero; the squares in the first formula are evaluated by applying the interval square function (2.2).

Apparently $[x]^C$ exists if and only if $0 < \underline{l}_{ii}$, $i = 1, \ldots, n$. In this case we call the algorithm feasible. Note that this feasibility does not depend on the choice of $[b]$. For the interval Cholesky method we assume, without loss of generality, $[A]$ to be symmetric, i.e., $[A] = [A]^T$. (In the case $[A] \neq [A]^T$ we replace $[A]$ by the largest interval matrix $[B] \subseteq [A]$ which satisfies $[B] = [B]^T$ and rename $[B]$ to $[A]$.) By the overestimation of the interval arithmetic only

$$[A] \subseteq [L][L]^T$$

can be guaranteed; cf. [2] for details. Nevertheless the pair $([L], [L]^T)$ is called the Cholesky decomposition of $[A]$. This decomposition can also be defined in a recursive way. To this end write $[A] \in \mathbb{IR}^{n\times n}$ as

$$[A] = \begin{pmatrix} [a]_{11} & [c]^T \\ [c] & [A]' \end{pmatrix}$$

and use its Schur complement $\Sigma^C_{[A]} = [A]' - [c][c]^T/[a]_{11}$ if $n > 1$, $0 \notin [a]_{11}$, where $[c]_i[c]_i$ is evaluated as $[c]_i^2$.

DEFINITION 2.1 (equivalent definition of $([L], [L]^T)$). *The pair* $([L], [L]^T)$ *is called the Cholesky decomposition of* $[A] = [A]^T \in \mathbb{IR}^{n\times n}$ *if* $0 < \underline{a}_{11}$ *and if either* $n = 1$, $[L] = (\sqrt{[a]_{11}})$, *or if* $n > 1$ *and*

$$[L] = \begin{pmatrix} \sqrt{[a]_{11}} & 0 \\ [c]/\sqrt{[a]_{11}} & [L]' \end{pmatrix}, \tag{2.3}$$

where $([L]', ([L]')^T)$ *is the Cholesky decomposition of* $\Sigma^C_{[A]}$. *If* $0 \in [a]_{11}$, *then the Cholesky decomposition does not exist.*

In [2] we showed that the matrix $[L]$ in Definition 2.1 is the same as that defined by the interval Cholesky method. In particular, the existence of the Cholesky decomposition is equivalent to the feasibility of the interval Cholesky method. We will exploit this fact later. It is a basic fact of matrix analysis that the existence of the Cholesky decomposition of a symmetric point matrix $A \in \mathbb{R}^{n\times n}$ is equivalent to A being positive definite, to A having only positive eigenvalues, and to A having only positive leading principal minors; cf., for instance, [7].

Directly from the formulae of the interval Cholesky method we obtain the following result which corresponds to Lemma 3.1 (b) in [8].

LEMMA 2.1. *Let* $[A] = [A]^T \in \mathbb{IR}^{n\times n}$, $[b] \in \mathbb{IR}^n$, *and let* $[x]^C = \mathrm{ICh}([A], [b])$ *exist. If* $D = \mathrm{diag}(d_1, \ldots, d_n) \in \mathbb{R}^{n\times n}$ *has positive entries* d_i, $i = 1, \ldots, n$, *in the diagonal, then* $[\tilde{x}]^C = \mathrm{ICh}([D[A]D, D[b])$ *exists and satisfies* $[\tilde{x}]^C = D^{-1}[x]^C$.

Proof. Denote by a tilde all items which belong to $[\tilde{x}]^C$. Then, by induction, the formulae of the interval Cholesky method yield $[\tilde{L}] = D[L]$, hence $[\tilde{y}] = [y]$ and $[\tilde{x}]^C = D^{-1}[x]^C$. □

We continue by recalling some results from [2].

THEOREM 2.1. *Let* $[A] = [A]^T \in \mathbb{IR}^{n\times n}$ *be an H-matrix with* $0 < \underline{a}_{ii}$, $i = 1, \ldots, n$. *Then the following statements hold.*

(a) *The vector* $[x]^C$ *exists, and* $[L]$ *is again an H-matrix.*

(b) *Each symmetric matrix* $\tilde{A} \in [A]$ *is positive definite.*

From Theorem 2.1 we easily get the following corollary.

COROLLARY 2.1. *Let* $[A] = [A]^T \in \mathbb{IR}^{n\times n}$ *be an H-matrix. Then the following statements are equivalent.*

(i) *The vector* $[x]^C$ *exists.*

(ii) *The sign condition* $\underline{a}_{ii} > 0$, $i = 1, \ldots, n$, *holds.*

(iii) *The matrix* $[A]$ *contains at least one symmetric and positive definite element* $\tilde{A} \in [A]$.

Proof. (i) $\Rightarrow$ (ii). Since $\langle [A] \rangle$ is an M-matrix we have $\langle [a]_{ii} \rangle > 0$, $i = 1, \ldots, n$, whence $0 \notin [a]_{ii}$. The existence of $[x]^C$ then implies $\underline{a}_{ii} > 0$.

The implications (ii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii) follow directly from Theorem 2.1.

(iii) $\Rightarrow$ (ii). As in the first implication above, one gets $0 \notin [a]_{ii}$, and the sign condition for $\underline{a}_{ii}$ follows from the positive definiteness of $\tilde{A}$. □

THEOREM 2.2. *Let $[A] = [A]^T \in \mathbb{IR}^{n\times n}$ be a tridiagonal matrix, and let $\tilde{A} \in [A]$ be any symmetric matrix which satisfies $\langle \tilde{A} \rangle = \langle [A] \rangle$ and is positive definite. Then $[A]$ is an H-matrix; in particular, all symmetric matrices $A \in [A]$ are positive definite, and $[x]^C$ exists.*

THEOREM 2.3. *Let*

$$A = \begin{pmatrix} a_{11} & c^T \\ c & A' \end{pmatrix} \in \mathbb{R}^{n\times n}$$

be symmetric and positive definite. Then the Schur complement $\Sigma_A^C = A' - cc^T/a_{11}$ of A is symmetric and positive definite.

Proof. Use $0 < x^T Ax = (x')^T \Sigma_A^C x'$ for $x = \left(-c^T x'/a_{11}, (x')^T\right)^T$ and any nonzero vector $x' \in \mathbb{R}^{n-1}$. $\square$

Since we will also use results of the interval Gaussian algorithm we will recall its formulae, too.

Given $[A] \in \mathbb{IR}^{n\times n}$ and $[b] \in \mathbb{IR}^n$, define $[A]^{(k)} = ([a]_{ij}^{(k)}) \in \mathbb{IR}^{n\times n}$, $[b]^{(k)} = ([b]_i^{(k)}) \in \mathbb{IR}^n$, $k = 1, \ldots, n$, and $[x]^G = ([x]_i^G) = \mathrm{IGA}([A], [b]) \in \mathbb{IR}^n$ by

$$[A]^{(1)} = [A], \quad [b]^{(1)} = [b],$$

$$[a]_{ij}^{(k+1)} = \begin{cases} [a]_{ij}^{(k)}, & i = 1, \ldots, k,\ j = 1, \ldots, n, \\ [a]_{ij}^{(k)} - \dfrac{[a]_{ik}^{(k)} \cdot [a]_{kj}^{(k)}}{[a]_{kk}^{(k)}}, & i = k+1, \ldots, n,\ j = k+1, \ldots, n, \\ 0 & \text{otherwise}, \end{cases}$$

$$[b]_i^{(k+1)} = \begin{cases} [b]_i^{(k)}, & i = 1, \ldots, k, \\ [b]_i^{(k)} - \dfrac{[a]_{ik}^{(k)}}{[a]_{kk}^{(k)}} \cdot [b]_k^{(k)}, & i = k+1, \ldots, n, \end{cases}$$

$$k = 1, \ldots, n-1,$$

$$[x]_i^G = \left([b]_i^{(n)} - \sum_{j=i+1}^{n} [a]_{ij}^{(n)} [x]_j^G \right) \Big/ [a]_{ii}^{(n)}, \quad i = n, n-1, \ldots, 1.$$

For $i = n$ the sum is set equal to zero.

Note that $[x]^G$ is defined without permuting rows or columns. The algorithm is feasible if and only if $0 \notin [a]_{kk}^{(k)}$, $k = 1, \ldots, n$, where again the feasibility does not depend on the choice of $[b]$. Define the lower triangular matrix $[\hat{L}]$ by $[\hat{l}]_{ii} = 1$, $[\hat{l}]_{ij} = [a]_{ij}^{(j)}/[a]_{jj}^{(j)}$ for $i > j$, and the upper triangular matrix $[\hat{U}]$ by $[\hat{u}]_{ij} = [a]_{ij}^{(n)}$ for $i \le j$. According to [11] the pair $([\hat{L}], [\hat{U}])$ is called the triangular decomposition of $[A]$.

Similar to Definition 2.1 there is an equivalent recursive definition of that decomposition. It uses the partition

$$[A] = \begin{pmatrix} [a]_{11} & [c]^T \\ [d] & [A]' \end{pmatrix}$$

and its Schur complement $\Sigma^G_{[A]} = [A]' - [d][c]^T/[a]_{11}$ if $n > 1$, $0 \notin [a]_{11}$. Note that for $[A] = [A]^T$ we have $[c] = [d]$. In this case we assume that $[c]_i[c]_i$ in the product $[d][c]^T = [c][c]^T$ is evaluated as a product of intervals and not as in (2.2). This implies

$$\Sigma^C_{[A]} \subseteq \Sigma^G_{[A]}, \tag{2.4}$$

where both matrices may differ from each other. For symmetric *point matrices* $A \equiv [A]$, however, equality always holds in (2.4), provided that $a_{11} > 0$.

DEFINITION 2.2 (equivalent definition of $([\hat{L}], [\hat{U}])$). *The pair $([\hat{L}], [\hat{U}])$ is called triangular decomposition of $[A] \in \mathbb{IR}^{n\times n}$ if $0 \notin [a]_{11}$ and if either $n = 1$, $[\hat{L}] = 1$, $[\hat{U}] = ([a]_{11})$, or if $n > 1$ and*

$$[\hat{L}] = \begin{pmatrix} 1 & 0 \\ [d]/[a]_{11} & [\hat{L}]' \end{pmatrix}, \qquad [\hat{U}] = \begin{pmatrix} [a]_{11} & [c]^T \\ 0 & [\hat{U}]' \end{pmatrix},$$

where $([\hat{L}]', [\hat{U}]')$ is the triangular decomposition of $\Sigma^G_{[A]}$. If $0 \in [a]_{11}$, then the triangular decomposition does not exist.

In what follows we will use the notation of section 2 without further reference.

3. New results. In this section we will present some new criteria for the feasibility of the interval Cholesky method. Since neither the existence of $[x]^C$ nor that of $[x]^G$ depends on the right-hand side $[b]$, we do not refer to $[b]$ in our results.

Assume now that $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite. Then from the Cholesky decomposition (L, L^T) of A we define the diagonal matrix $D = \mathrm{diag}(l_{11}, \ldots \ldots, l_{nn})$. It is well known that D has positive diagonal entries. Hence $A = LL^T = (LD^{-1})(DL^T)$ yields the unique $(\hat{L}, \hat{U})$-decomposition of A with $\hat{L} = LD^{-1}$ and $\hat{U} = DL^T$. Conversely, from the $(\hat{L}, \hat{U})$-decomposition of a symmetric matrix $A \in \mathbb{R}^{n\times n}$ with positive diagonal entries $\hat{u}_{ii}$, $i = 1, \ldots, n$, one easily verifies positive definiteness of A and hence the existence of the Cholesky decomposition. Therefore, the question arises at once whether a similar result also holds in the interval case. In one direction the answer is positive.

THEOREM 3.1. *Let $[A] = [A]^T \in \mathbb{IR}^{n\times n}$ contain a symmetric and positive definite matrix $\tilde{A}$. If $[x]^G$ exists, then $[x]^C$ exists, too.*

Proof. Since $\tilde{A} \in [A]$ is symmetric and positive definite we have $\tilde{a}_{11} > 0$. Moreover, since by assumption $[x]^G$ exists we obtain $\underline{a}_{11} > 0$. We now proceed by induction on the dimension of $[A]$.

If $n = 1$, then the assertion is obvious. If $n > 1$, then let it hold for dimensions less than n, and let

$$[A] = \begin{pmatrix} [a]_{11} & [c]^T \\ [c] & [A'] \end{pmatrix}.$$

From $\underline{a}_{11} > 0$, the Schur complements

$$\Sigma^C_{[A]} = [A]' - [c][c]^T/[a]_{11} \quad (\text{with } [c]_i[c]_i \text{ being evaluated as } [c]_i^2)$$

for the Cholesky method and

$$\Sigma^G_{[A]} = [A]' - [c][c]^T/[a]_{11} \supseteq \Sigma^C_{[A]}$$

(with $[c]_i[c]_i$ being evaluated as a product of two intervals) for the Gaussian algorithm exist. Since, by assumption, the interval Gaussian algorithm is feasible for $\Sigma^G_{[A]}$

and since this interval matrix contains the symmetric and positive definite matrix $\Sigma_A^G = \Sigma_A^C$ (cf. Theorem 2.3), the induction hypothesis applies for $\Sigma_{[A]}^G$. Therefore, the Cholesky decomposition exists for this interval matrix and thus exists for the (possibly proper) subset $\Sigma_{[A]}^C$, too. $\square$

We will prove now a result on point matrices which originally increased our hope for a converse of Theorem 3.1.

THEOREM 3.2. *Let all symmetric matrices $\tilde{A} \in [A] = [A]^T \in \mathbb{IR}^{n\times n}$ be positive definite. Then the Gaussian algorithm is feasible without pivoting for all matrices $A \in [A]$ (and not only for the symmetric ones).*

Proof. Let $A \in [A]$. Then the symmetric part[1] $A_{\text{sym}} = (A + A^T)/2$ of A is contained in $[A]$, and hence it is positive definite by assumption. For $x \neq 0$ we have

$$0 < x^T A_{\text{sym}} x = (x^T A x + x^T A^T x)/2 = x^T A x, \tag{3.1}$$

where we used $x^T A^T x = (x^T A^T x)^T = x^T A x$. From (3.1) we immediately get $\det A \neq 0$. Since this implication applies also to all leading submatrices of A the assertion follows from Theorem 9.1.2 in [12]. $\square$

Despite this positive result the converse of Theorem 3.1 does not hold. This is illustrated by the following example.

Example 3.1. Consider the matrix

$$[A] = \begin{pmatrix} 1 & [-1,1] & 0 & 0 \\ [-1,1] & 2 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 2 & 5+\varepsilon \end{pmatrix}$$

with a positive parameter ε which will be chosen below. Then for the interval Cholesky method we get

$$[L] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ [-1,1] & [1,\sqrt{2}] & 0 & 0 \\ 0 & [1/\sqrt{2},1] & [1,\sqrt{3/2}] & 0 \\ 0 & [2/\sqrt{2},2] & [0,1] & [\sqrt{\varepsilon},\sqrt{3+\varepsilon}] \end{pmatrix},$$

i.e., $[x]^C$ exists for any positive value of ε. On the other hand we obtain

$$[\hat{U}] = [A]^{(4)} = \begin{pmatrix} 1 & [-1,1] & 0 & 0 \\ 0 & [1,3] & 1 & 2 \\ 0 & 0 & [1,5/3] & [0,4/3] \\ 0 & 0 & 0 & [\varepsilon-7/9,\varepsilon+11/3] \end{pmatrix}$$

for the upper triangular matrix of the interval Gaussian algorithm. Choosing $\varepsilon = 1/3$ results in the interval $[a]_{44}^{(4)} = [-4/9, 4]$ which contains zero. Hence $[x]^G$ does not exist although $[x]^C$ does. In particular, the assumptions of Theorem 3.2 are fulfilled. Therefore, the Gaussian algorithm is feasible for any matrix $\tilde{A} \in [A]$, and our example is also a counterexample for the interval Gaussian algorithm.

[1] We thank Prof. M. Plum of the University of Karlsruhe for his suggestion to apply the symmetric part of A, which made our original proof more elementary.

The dimension $n = 4$ in Example 3.1 is minimal for a counterexample. This can be seen from our next result.

THEOREM 3.3. *Let $[A] = [A]^T \in \mathbb{IR}^{n\times n}$ contain a symmetric and positive definite matrix $\tilde{A}$, and let $n \le 3$. Then $[x]^C$ exists if and only if $[x]^G$ exists.*

Proof. By virtue of Theorem 3.1 we must only show that the existence of $[x]^C$ implies that of $[x]^G$. Therefore, from now on we assume that $[x]^C$ exists. In particular, $0 < \underline{a}_{11}$ holds, which immediately guarantees the existence of $[x]^G$ in the case $n = 1$.

$n = 2$: By Theorem 3.2 no matrix $\tilde{A} \in [A] \in \mathbb{IR}^{2\times 2}$ is singular, hence $[x]^G$ exists by Proposition 4.5.4 in [11].

$n = 3$: From $\underline{a}_{11} > 0$ we know that $\Sigma^G_{[A]}$ exists. Since any interval $[c]$ satisfies

$$[c]^2 \subseteq [c] \cdot [c] = [c]^2 + [-d, 0]$$

with an appropriate nonnegative number d, we obtain

$$\Sigma^G_{[A]} = \Sigma^C_{[A]} + [D] \quad \text{with } [D] = \mathrm{diag}([0, d_1], [0, d_2], [0, d_3]),$$

where d_1, d_2, d_3 are appropriate nonnegative real numbers. Note that $\min(\Sigma^G_{[A]})_{11} = \min(\Sigma^C_{[A]})_{11} > 0$. Choose $x \in \mathbb{R}^2\backslash\{0\}$ and $\tilde{\Sigma}^G = (\tilde{\Sigma}^G)^T \in \Sigma^G_{[A]}$. Then $\tilde{\Sigma}^G$ can be written as $\tilde{\Sigma}^G = \tilde{\Sigma}^C + \tilde{D}$ with $\tilde{\Sigma}^C = (\tilde{\Sigma}^C)^T \in \Sigma^C_{[A]}$ and $O \le \tilde{D} \in [D]$, whence

$$x^T\tilde{\Sigma}^G x = x^T\tilde{\Sigma}^C x + x^T\tilde{D}x \ge x^T\tilde{\Sigma}^C x > 0. \tag{3.2}$$

Thus any symmetric matrix $\tilde{\Sigma}^G \in \Sigma^G_{[A]}$ is positive definite, and Theorem 3.2 applies to $\Sigma^G_{[A]}$. Therefore, no matrix $\Sigma \in \Sigma^G_{[A]}$ is singular, and $[x]^G \in \mathbb{IR}^3$ exists again by virtue of Proposition 4.5.4 in [11] applied to $\Sigma^G_{[A]} \in \mathbb{IR}^{2\times 2}$. $\square$

Another interesting negative result can be seen from Example 3.1: For symmetric and positive definite matrices $\tilde{A} \in \mathbb{R}^{n\times n}$, one proves similarly as for $\tilde{\Sigma}^G$ in (3.2) that $\tilde{A} + \tilde{D}$ with $\tilde{D} \ge O$ is positive definite, hence the Cholesky method is feasible for $\tilde{A} + \tilde{D}$, too. For interval matrices $[A] + [O, D]$, $D \ge O$, an analogous result does not hold if one merely knows that $[x]^C$ exists for $[A]$. Otherwise apply this result to $\Sigma^G_{[A]} = \Sigma^C_{[A]} + [O, D]$; it would guarantee that $\Sigma^G_{[A]}$ has a Cholesky decomposition if $[A]$ has one, and an inductive argument would show that Theorem 3.1 has a converse. This contradicts Example 3.1.

There are more classes of matrices for which one can prove the converse of Theorem 3.1. In order to characterize some of them we use the concept of an undirected graph of a real matrix $A \in \mathbb{R}^{n\times n}$ with the nodes $1, \dots, n$ and the edges $\{i, j\}$ whenever $|a_{ij}| + |a_{ji}| \ne 0$; cf. for instance [6]. We call j a neighbor of the node i ($\ne j$) if i and j are connected by an edge. The number of neighbors of i are the degree of i in the underlying graph. Let G_k denote the kth elimination graph of $[A]$, i.e., the undirected graph of $|[A]^{(k)}|$ in which the nodes $1, \dots, k-1$ and the corresponding edges have been removed and for which we assume that $[a]^{(k-1)}_{ij} \ne 0$ implies $[a]^{(k)}_{ij} \ne 0$, $i, j \ge k$ (no accidental zeros!); cf. [6]. If in G_k the node k has the smallest degree and if this holds for all $k = 1, \dots, n$, then we say that $[A]$ is ordered by minimum degree. If the graph of such a matrix has tree structure (i.e., it is a connected graph with no cycles of length ≥ 3; cf. [4]), then the following result holds.

THEOREM 3.4. *Let $[A] = [A]^T \in \mathbb{IR}^{n\times n}$ contain a symmetric and positive definite matrix $\tilde{A}$. If the undirected graph of $\langle [A] \rangle$ is a tree and if it is ordered by minimum degree, then the following statements are equivalent.*

(i) *The vector* $[x]^G$ *exists.*
(ii) *The vector* $[x]^C$ *exists.*
(iii) *Each symmetric matrix in* $[A]$ *is positive definite.*

Proof. (i) $\Rightarrow$ (ii) follows from Theorem 3.1.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) follows from Theorem 3.2 and Theorem 4 in [4]. $\square$

For a variant of the interval Cholesky method, Theorem 3.4 was proved in [4]. Note that symmetric tridiagonal interval matrices and symmetric arrowhead interval matrices [14] belong to the class of matrices characterized in Theorem 3.4 (provided that they contain a symmetric and positive definite matrix $\tilde{A}$).

Example 3.2. Consider the arrowhead matrix

$$[A] = \begin{pmatrix} 2 & 0 & [-1,1] \\ 0 & 2 & [-1,1] \\ [-1,1] & [-1,1] & 2 \end{pmatrix}.$$

Then Gerschgorin's theorem shows that the eigenvalues of each symmetric matrix $\tilde{A} \in [A]$ are nonnegative. They are even positive as can be seen in most cases by the same theorem. For the remaining cases $\tilde{A}$ is irreducibly diagonally dominant and thus an H-matrix. Since such a matrix is regular it cannot have zero as an eigenvalue. Therefore, each symmetric matrix $\tilde{A} \in [A]$ is positive definite, and $[x]^C$ exists for $[A]$ by Theorem 3.4.

In order to formulate our next result we need the extended sign matrix S' which we define recursively as in [8].

DEFINITION 3.1 (sign matrix S and extended sign matrix S' for $[A]$). *Let* $[A] \in \mathbb{IR}^{n\times n}$. *Then we have the following.*

(a) *The matrix* $S \in \mathbb{R}^{n\times n}$ *with* $s_{ij} = \operatorname{sign}\check{a}_{ij}$ *is called the sign matrix of* $[A]$.
(b) *With* S *from (a) the extended sign matrix* S' *is defined as follows:*

$S' = S$
for $k = 1 : (n-1)$
  for $i = (k+1) : n$
    for $j = (k+1) : n$
      if $s'_{ij} == 0$ *then* $s'_{ij} = -s'_{ik}s'_{kk}s'_{kj}$.

Note that the values of s'_{ij} depend only on S. Any other matrix $[\hat{A}]$ with the same sign matrix S as $[A]$ yields the same extended sign matrix S'.

THEOREM 3.5. *Let* $[A] = [A]^T \in \mathbb{IR}^{n\times n}$ *be irreducible and generalized diagonally dominant with* $0 < \underline{a}_{ii}$, $i = 1,\ldots,n$. *Moreover, let* S' *be the extended sign matrix of* $[A]$ *defined in Definition* 3.1. *Then the following statements are equivalent.*

(i) *The vector* $[x]^C$ *exists.*
(ii) *The vector* $[x]^G$ *exists.*
(iii) *The matrix* $[A]$ *is generalized irreducibly diagonally dominant or the sign condition*

$$s'_{ij}\, s'_{ik}\, s'_{kj}\, s'_{kk} = \begin{cases} 1 & \text{if } i \neq j, \\ -1 & \text{if } i = j \end{cases} \tag{3.3}$$

holds for some triple (i,j,k) *with* $k < i, j$.

(iv) *The matrix* $[A]$ *is generalized irreducibly diagonally dominant or the sign condition*

$$s'_{ij}\, s'_{ik}\, s'_{kj} = 1 \tag{3.4}$$

holds for some triple (i, j, k) *with* $k < j < i$.

Proof. The case $n = 1$ is trivial since $\underline{a}_{11} > 0$. Therefore, from now on we assume $n > 1$.

(ii) $\Leftrightarrow$ (iii) holds by virtue of Theorem 4.7 in [8].

(iii) $\Leftrightarrow$ (iv). From the general assumptions of the theorem we get $s_{ii} = 1 = s'_{ii}$, $i = 1, \ldots, n$, and $S = S^T$, whence $S' = (S')^T$. Therefore,

$$s'_{ii}\, s'_{ik}\, s'_{ki}\, s'_{kk} = (s'_{ik})^2 \neq -1$$

holds, i.e., the second sign condition in (3.3) can never be fulfilled. Moreover, a factor $s'_{kk} = 1$ can always be added in (3.4) which results in the first sign condition in (3.3). Hence the existence of some triple (i, j, k) as required in (iii) is equivalent to the existence of some triple as required in (iv).

(ii) $\Rightarrow$ (i). Since $[x]^G$ exists by assumption, each matrix $\tilde{A} \in [A]$ is regular. Consider the matrix $[A] + \varepsilon I$, $\varepsilon > 0$. Since $\underline{a}_{ii} > 0$, $i = 1, \ldots, n$, we get $\langle [A] + \varepsilon I \rangle = \langle [A] \rangle + \varepsilon I$ which shows that $[A] + \varepsilon I$ is generalized strictly diagonally dominant. Therefore, it is an H-matrix by Theorem 4.4 (a) in [8] and Theorem 2.1 guarantees that $\tilde{A} + \varepsilon I$ is positive definite for each symmetric matrix $\tilde{A} \in [A]$. Hence $\tilde{A} + \varepsilon I$ has only positive real eigenvalues which remain positive in the limit $\varepsilon \to 0$ since $\tilde{A}$ is regular and since the eigenvalues behave continuously when changing the entries of a matrix continuously. Therefore, $\tilde{A}$ is positive definite for each symmetric matrix $\tilde{A} \in [A]$. In particular, $[A]$ contains at least one such matrix, and Theorem 3.1 finishes the proof.

(i) $\Rightarrow$ (ii). Let $[x]^C$ exist and assume that $[x]^G$ does not exist. Then $[A]$ cannot be an H-matrix; in particular, by Theorem 4.4 (b) in [8] it cannot be generalized irreducibly diagonally dominant. However, since it is generalized diagonally dominant by assumption, there must exist a positive vector x such that $\langle [A] \rangle x = 0$. Without loss of generality, we can assume $x = e$, i.e.,

$$\langle [A] \rangle\, e = 0. \tag{3.5}$$

Otherwise consider the matrix $D[A]D$ with $D = \operatorname{diag}(x_1, \ldots, x_n) \in \mathbb{R}^{n \times n}$. This matrix has the same extended sign matrix S' as $[A]$, is irreducible and diagonally dominant, but not irreducibly diagonally dominant. Moreover, it fulfills (3.5), and by Lemma 2.1 the interval Cholesky method is feasible for it since it is for $[A]$ by assumption.

Since we assumed that $[x]^G$ does not exist, the equivalence of (ii) and (iii) shows that the sign condition (3.3) does not hold. Choose $k = 1$ for the moment and let S be the sign matrix of $[A]$. If $s_{ij} \neq s'_{ij}$, then s_{ij} must be zero by the construction of S' in Definition 3.1. (Note that at the beginning of this definition we have $S' = S$. Later s'_{ij} is changed only if it was equal to zero.) Therefore, $s_{ij} = s'_{ij}$ or $s_{ij} = 0$. Hence (3.3) does not hold if s'_{ij} is replaced there by s_{ij}. By Lemma 2.1 in [9] this implies

$$\left| [a]_{ij} - \frac{[a]_{i1} \cdot [a]_{1j}}{[a]_{11}} \right| = |[a]_{ij}| + \frac{|[a]_{i1}| \cdot |[a]_{1j}|}{\langle [a]_{11} \rangle} \quad \text{if } i \neq j \text{ and } i, j > 1. \tag{3.6}$$

Next we remark that the equality $|[a]^2| = |[a]|^2 = |[a]\cdot[a]|$ holds for any interval $[a]$. Since $\underline{a}_{ii}$ is positive and since $[x]^C$ exists we have $0 < \underline{l}_{ii}$ and

$$0 < (\underline{l}_{ii})^2 = \underline{a}_{ii} - \sum_{k=1}^{i-1} |[l]_{ik}^2| \le \underline{a}_{ii} - |[l]_{i1}^2| = \underline{a}_{ii} - \left|\left(\frac{[a]_{i1}}{[l]_{11}}\right)^2\right| = \langle [a]_{ii}\rangle - \left|\frac{[a]_{i1}^2}{[a]_{11}}\right|$$

$$= \langle [a]_{ii}\rangle - \frac{|[a]_{i1}^2|}{\langle [a]_{11}\rangle} = \langle [a]_{ii}\rangle - \frac{|[a]_{i1}|^2}{\langle [a]_{11}\rangle}, \qquad i > 1.$$

In particular, $\langle [a]_{ii}\rangle > \dfrac{|[a]_{i1}|^2}{\langle [a]_{11}\rangle}$ holds, and Lemma 2.1(b) in [9] implies

$$\left\langle [a]_{ii} - \frac{[a]_{i1}^2}{[a]_{11}}\right\rangle = \left\langle [a]_{ii} - \frac{[a]_{i1}\cdot[a]_{i1}}{[a]_{11}}\right\rangle = \langle [a]_{ii}\rangle - \frac{|[a]_{i1}|^2}{\langle [a]_{11}\rangle}, \qquad i > 1. \tag{3.7}$$

From (3.6) and (3.7) we directly get

$$\langle \Sigma_{[A]}^C\rangle = \Sigma_{\langle [A]\rangle}^C = \Sigma_{\langle [A]\rangle}^G = \langle \Sigma_{[A]}^G\rangle \tag{3.8}$$

although $\Sigma_{[A]}^C \subsetneqq \Sigma_{[A]}^G$ may hold. In fact, by construction both matrices can differ at most in the diagonal because $[c]_i^2 \subsetneqq [c]_i \cdot [c]_i$ can occur. Since $[x]^C$ exists the diagonal entries of $\Sigma_{[A]}^C$ are positive; hence the sign matrices of $\Sigma_{[A]}^C$ and $\Sigma_{[A]}^G$ coincide and the same holds for the extended sign matrices.

With $e = \binom{1}{e'}$ and (3.5) we obtain

$$\left(\langle \Sigma_{[A]}^C\rangle e'\right)_i = \left(\Sigma_{\langle [A]\rangle}^C e'\right)_i = \left(\langle [a]_{ii}\rangle - \frac{|[a]_{i1}|^2}{\langle [a]_{11}\rangle}\right) - \sum_{\substack{j=2\\ j\ne i}}^{n} \left(|[a]_{ij}| + \frac{|[a]_{i1}|\cdot|[a]_{j1}|}{\langle [a]_{11}\rangle}\right)$$

$$= \left\{\langle [a]_{ii}\rangle - \sum_{\substack{j=1\\ j\ne i}}^{n} |[a]_{ij}|\right\} + \frac{|[a]_{i1}|}{\langle [a]_{11}\rangle}\left\{\langle [a]_{11}\rangle - \sum_{j=2}^{n} |[a]_{j1}|\right\}$$

$$= (\langle [A]\rangle e)_i + \frac{|[a]_{i1}|}{\langle [a]_{11}\rangle}(\langle [A]\rangle e)_1 = 0, \qquad i = 2,\dots,n.$$

Hence

$$\langle \Sigma_{[A]}^C\rangle\, e' = \Sigma_{\langle [A]\rangle}^C\, e' = \Sigma_{\langle [A]\rangle}^G e' = \langle \Sigma_{[A]}^G\rangle\, e' = 0.$$

Moreover, from (3.8) together with Lemma 3.3 in [5] we know that $\Sigma_{[A]}^C$ is irreducible provided that $n \ge 3$.

Since we assumed that $[x]^G$ does not exist, the interval Gaussian algorithm cannot be feasible for $\Sigma_{[A]}^G$. Therefore, (3.3) cannot hold when formulated for the extended sign matrix of $\Sigma_{[A]}^G$. (In fact, deleting the first row and column of S' for $[A]$ results in the corresponding extended sign matrix for the Schur complement.) Since we already showed that $\Sigma_{[A]}^C$ and $\Sigma_{[A]}^G$ have the same extended sign matrices the equivalence of (ii) and (iii) implies that the interval Gaussian algorithm is not feasible for $\Sigma_{[A]}^C$. Thus the assumptions of Theorem 3.5 for $[A]$ are also fulfilled for $\Sigma_{[A]}^C = (\Sigma_{[A]}^C)^T$. Therefore, the previous conclusions can be repeated up to the dimension $n = 2$ for

$\Sigma^C_{[A]}$. (Note that the restriction of the dimension n concerns only the irreducibility.) For ease of notation assume that $[A]$ plays the role of $\Sigma^C_{[A]}$ if $n = 2$, i.e., it is an irreducible symmetric 2×2 interval matrix satisfying $\langle [A] \rangle c = 0$. As before we obtain $\langle \Sigma^C_{[A]} \rangle c' = 0$, i.e., $0 \in \Sigma^C_{[A]} \in \mathbb{IR}^{1\times 1}$, which contradicts the feasibility of the interval Cholesky method and which finally shows that (3.3) must hold for some triple (i, j, k) unless $[A] \in \mathbb{IR}^{n\times n}$ is generalized irreducibly diagonally dominant. (In this case the sign condition (3.3) may be hurt as the example $[A] = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ shows.) $\square$

Example 3.3.

(a) Let

$$[A]_\alpha = \begin{pmatrix} 4 & [\alpha, 2] & [\alpha, 2] \\ [\alpha, 2] & 4 & 2 \\ [\alpha, 2] & 2 & 4 \end{pmatrix}, \qquad -2 \le \alpha \le 2.$$

Then $\langle [A]_\alpha \rangle e = 0$. For $-2 < \alpha \le 2$ we obtain $S = ee^T = S'$. Thus (3.4) is fulfilled with $(i, j, k) = (3, 2, 1)$, and $[x]^C$ exists.

If $\alpha = -2$, then things change. Here

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = S'$$

and (3.4) does not hold as one can easily check. Thus $[x]^C$ does not exist. In fact, $[A]_{-2}$ contains the singular matrix

$$\tilde{A} = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 4 & 2 \\ -2 & 2 & 4 \end{pmatrix}.$$

(b) Let

$$[A] = \begin{pmatrix} 4 & 0 & [0, 2] & [-2, 0] \\ 0 & 4 & [0, 2] & [0, 2] \\ [0, 2] & [0, 2] & [6, 9] & [-2, 2] \\ [-2, 0] & [0, 2] & [-2, 2] & [6, 9] \end{pmatrix}.$$

Then $[A]$ is irreducible and diagonally dominant. In particular, it satisfies the assumptions of Theorem 3.5. Moreover, we have

$$S = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix} \ne S' = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

with (3.4) for $(i, j, k) = (4, 3, 2)$. Hence $[x]^C$ exists.

It is easy to see by Example 3.3 (b) that (3.4) does not hold if the entries of S' are replaced there by the corresponding entries of S. Doing so, nevertheless, yields a sufficient criterion analogously to Theorem 5.3 in [5]. We state this result as a corollary which follows directly from Theorem 3.5(iv) since (3.9) below can be written as (3.4).

COROLLARY 3.1. *Let $[A] = [A]^T \in \mathbb{IR}^{n\times n}, n \geq 3$, be irreducible and generalized diagonally dominant with $0 < \underline{a}_{ii}$, $i = 1, \ldots, n$. Moreover, let S be the sign matrix of $[A]$ defined in Definition 3.1. If*

$$s_{ij}\, s_{ik}\, s_{kj} = 1 \tag{3.9}$$

for some triple (i, j, k) with $k < j < i$, then $[x]^C$ exists.

Now we consider tridiagonal matrices.

THEOREM 3.6. *Let $[A] = [A]^T \in \mathbb{IR}^{n\times n}$ be tridiagonal. Then the following statements are equivalent.*

(i) *The vector $[x]^G$ exists and $[A]$ contains at least one symmetric and positive definite matrix.*

(ii) *The vector $[x]^C$ exists.*

(iii) *Each symmetric matrix $\tilde{A} \in [A]$ is positive definite.*

Proof. (i) $\Rightarrow$ (ii) follows from Theorem 3.1.

(ii) $\Rightarrow$ (iii) follows from the feasibility of the Cholesky method for each symmetric matrix $\tilde{A} \in [A]$.

(iii) $\Rightarrow$ (i) follows from Theorem 2.2 and the feasibility of the interval Gaussian algorithm for H-matrices; cf. [1] or [11]. $\square$

Example 3.4. Let $[A] = \text{tridiag}([-1, 1], 2, [-1, 1]) \in \mathbb{IR}^{n\times n}$. Then Gershgorin's theorem shows that the eigenvalues of each symmetric matrix $\tilde{A} \in [A]$ are nonnegative. Since $\tilde{A}$ is either irreducibly diagonally dominant or consists of blocks of such matrices, it is an H-matrix. Therefore, no eigenvalue can be zero, each symmetric matrix $\tilde{A} \in [A]$ is positive definite, and $[x]^C$ exists for $[A]$ by Theorem 3.6.

Our final result deals with matrices of the form $[A] = I + [-R, R]$, which at first glance look very specific. However, preconditioning any regular interval matrix by its midpoint inverse $\check{A}^{-1}$ finally results in such a matrix.

THEOREM 3.7. *Let $[A] = I + [-R, R]$ with $O \leq R = R^T \in \mathbb{R}^{n\times n}$ and $0 < \underline{a}_{ii}$, $i = 1, \ldots, n$. Then the following statements are equivalent.*

(i) *The vector $[x]^G$ exists.*

(ii) *The vector $[x]^C$ exists.*

(iii) *The spectral radius of R is less than one.*

(iv) *The matrix $[A]$ is an H-matrix.*

Proof. The equivalence of (i), (iii), and (iv) is contained in Theorem 3.1 of [10]; cf. also Theorem 4.2 in [8]. The implication (iv) $\Rightarrow$ (ii) follows from Theorem 2.1. For the implication (ii) $\Rightarrow$ (iv), let $[x]^C$ exist. Then the Cholesky method is feasible for $\tilde{A} = I - R = \langle [A] \rangle \in [A]$, hence $\tilde{A}$ is symmetric and positive definite. Moreover, it is an M-matrix whence $[A]$ is an H-matrix. $\square$

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