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ERROR ESTIMATION FOR NONLINEAR COMPLEMENTARITY PROBLEMS VIA LINEAR SYSTEMS WITH INTERVAL DATA

Götz Alefeld[1] and Zhengyu Wang[2]

[1]*Institute of Applied and Numerical Mathematics, University of Karlsruhe (Karlsruhe Institute of Technology), Karlsruhe, Germany*
[2]*Institute of Computational Mathematics and Scientific/Engineering Computing, Chinese Academy of Sciences, Beijing, China; Department of Mathematics, Nanjing University, Nanjing, China*

□ *For the nonlinear complementarity problem, we derive norm bounds for the error of an approximate solution, generalizing the known results for the linear case. Furthermore, we present a linear system with interval data, whose solution set contains the error of an approximate solution. We perform extensive numerical tests and compare the different approaches.*

Keywords Error bound; Linear interval system; Nonlinear complementarity problem; P-matrix.

AMS Subject Classification 90C33; 65G30; 65K10.

1. INTRODUCTION

Let the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ be given. The nonlinear complementarity problem, denoted by *NCP*(f), is to find a vector x^* such that

$$x^* \geq 0, \quad f(x^*) \geq 0, \quad (x^*)^T f(x^*) = 0, \tag{1.1}$$

where the inequalities are defined componentwise. *NCP*(f) models many real problems in economics, engineering, and so forth. For its source problems, see [13, 15], for example.

Address correspondence to Götz Alefeld, Institute of Applied and Numerical Mathematics, University of Karlsruhe (Karlsruhe Institute of Technology), Karlsruhe 76128, Germany; E-mail: goetz.alefeld@math.uni-karlsruhe.de

Let $NCP(f)$ be given by the mapping

$$f(x) = Mx + \Phi(x),$$

where $M \in \mathbb{R}^{n\times n}$ is a given matrix, and

$$\Phi(x) = (\Phi_i(x_i)).$$

We call such an $NCP(f)$ an almost linear complementarity problem and denote it by $ALCP(\Phi, M)$. When $\Phi(x) = q \in \mathbb{R}^n$ is constant, $ALCP(\Phi, M)$ reduces to a so-called linear complementarity problem, which we denote by $LCP(q, M)$. $ALCP(\Phi, M)$ has wide applications, especially in engineering, for example in the obstacle Bratu problem [24], which models the nonlinear diffusion phenomena taking place in combustion and in semiconductors.

Error estimation plays an important role both in numerical solution and in theoretical analysis for $NCP(f)$. Error estimation has been extensively studied up to now, see [7, 9, 14, 16, 17, 22] and the monograph [13]. In the papers [1, 3], a verification test for the existence of a solution of $LCP(q, M)$ and $NCP(f)$, respectively, was given. If the test is successful, error bounds are delivered automatically. The idea is as follows. Given some interval vector $[x]$ and an $\hat{x} \in [x]$, an enclosure of all slopes formed with $\hat{x}$ and all $x \in [x]$ is computed. Using this slope enclosure, it is checked (computationally) whether the so-called Krawczyk-operator maps the interval vector into itself. If this is the case, then by the Brouwer fixed point theorem, the existence of a solution of the complementarity problem is guaranteed, and we have a componentwise error bound. It turns out that this procedure is surprisingly successful if $\hat{x}$ is a good approximation. For $LCP(q, M)$ and $ALCP(\Phi, M)$, a verification procedure was given in [4] and [5], using the special structure of these problems.

In the current article, we propose two different approaches for getting error bounds. In the first case, we can deliver norm bounds for the error by using properties of the generalized Jacobian in the sense of Clarke. A modified approach leads to a linear system with an interval matrix, whose solution set contains the error vector.

The paper is organized as follows: we include some frequently used notations and results in Section 2. In Section 3, two different approaches of error estimation are proposed for $NCP(f)$. Special cases of $ALCP(\Phi, M)$ and $LCP(q, M)$ are studied in Section 4. Extensive numerical experiments are performed in Section 5 to support the theoretical analysis. We complete the paper with some concluding remarks in Section 6.

2. PRELIMINARIES AND NOTATIONS

Denote by $\mathbb{R}^n_+$ the nonnegative orthant of $\mathbb{R}^n$, and denote by $\mathbb{R}^n_{++}$ the interior of $\mathbb{R}^n_+$. Denote by "$\leq$" the natural (or componentwise) partial ordering in $\mathbb{R}^n$, and let $(x_i) = x < y = (y_i)$ stand for $x_i < y_i$, $i = 1, \dots, n$. For any $x, y \in \mathbb{R}^n$, we denote by $\max\{x, y\}$ and $\min\{x, y\}$ the componentwise maximum and minimum of the two vectors, respectively.

We denote by I_n the $n \times n$ identity matrix, denote the ith row vector of I_n by e_i^T, and denote $e = (1, \dots, 1)^T$. We define $\mathcal{I} := \{1, \dots, n\}$. For any $\tau \subseteq \mathcal{I}$, we denote by $\bar{\tau}$ the complement of τ, and $|\tau|$ denotes the cardinality of τ. For any $A \in \mathbb{R}^{n\times n}$ and for any $\tau, \kappa \subseteq \mathcal{I}$ with $\tau, \kappa \neq \emptyset$, we denote by $A_{\tau\kappa}$ the submatrix of A with its rows and columns indexed by the elements of τ and κ, respectively. For the diagonal matrix D, we also write $D_{\tau\tau}$ as D_τ for convenience. For $x \in \mathbb{R}^n$ and $\tau \in \mathcal{I}$, we denote by x_τ the subvector of x with its components indexed by the elements of τ.

Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$. A is called a P-matrix if for any nonzero $x \in \mathbb{R}^n$

$$\max_i x_i(Ax)_i > 0.$$

We denote the set of all $n \times n$ P-matrices by $\mathcal{P}_n$. A is called an H-matrix if the so-called comparison matrix $\langle A \rangle = (\langle a_{ij} \rangle)_{n\times n}$ has a nonnegative inverse, where

$$\langle a_{ij} \rangle := \begin{cases} |a_{ii}| & i = j, \\ -|a_{ij}| & i \neq j. \end{cases}$$

We denote the set of all $n \times n$ H-matrices by $\mathcal{H}_n$ and denote the set of all $n \times n$ H-matrices with positive diagonal elements by $\mathcal{H}_n^+$. A is called a Z-matrix if each off-diagonal element of A is nonpositive. We denote the set of all $n \times n$ Z-matrices by $\mathcal{Z}_n$. A is called an M-matrix if A is a Z-matrix and has a nonnegative inverse. The set of all $n \times n$ M-matrices is denoted by $\mathcal{M}_n$.

The following theorem holds.

Theorem 2.1. *For P-matrices, H-matrices, and M-matrices, we have the following properties:*

1. *A is nonsingular if $A \in \mathcal{P}_n$;*
2. *$\mathcal{P}_n \supset \mathcal{H}_n^+ \supset \mathcal{M}_n$;*
3. *$A \in \mathcal{P}_n$ if and only if each of its principal minors is positive;*
4. *$A \in \mathcal{H}_n$ if and only if there is a vector $x > 0$ such that for the comparison matrix $\langle A \rangle$, we have $\langle A \rangle x > 0$;*
5. *$A \in \mathcal{Z}_n$ is an M-matrix if there is a $B \in \mathcal{M}_n$ such that $B \leq A$;*
6. *$I_n - D + DA \in \mathcal{P}_n$ if $A \in \mathcal{P}_n$ and $D = diag(d_i)$ with $d_i \in [0, 1]$.*

The proof of statements 1–5 can be found in [23], for example. Statement 6 can be proved by using 1–5.

We recall some notations from interval analysis, see also [2] or [19], for example. Let $[a] = [\underline{a}, \overline{a}]$ with $-\infty < \underline{a} \leq \overline{a} < \infty$ be a compact interval in $\mathbb{R}$. Then we denote by $\mathbb{IR}$ the set of all real compact intervals in $\mathbb{R}$. Let $\underline{A} = (\underline{a}_{ij})$, $\overline{A} = (\overline{a}_{ij}) \in \mathbb{R}^{m\times n}$ with $\underline{a}_{ij} \leq \overline{a}_{ij}$ for any indices i and j. An interval matrix, denoted by $[A] = [\underline{A}, \overline{A}]$, is defined as a matrix with each element $[A_{ij}] = [\underline{a}_{ij}, \overline{a}_{ij}] \in \mathbb{IR}$. An interval matrix $[A]$ is the set of the matrices that are element-wise bounded by $\underline{A}$ from below and bounded by $\overline{A}$ from above. Denote by $\mathbb{IR}^{m\times n}$ the set of all $m \times n$ real interval matrices. For the case of $n = 1$, the interval matrix is also called an m-dimensional interval vector; we denote by $\mathbb{IR}^m$ the set of all m-dimensional interval vectors.

Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitzian, i.e., for any $x \in \mathbb{R}^n$ there is a neighborhood $\mathcal{N}(x)$ and a constant L such that:

$$\|h(u) - h(v)\| \leq L\|u - v\| \quad \forall u, v \in \mathcal{N}(x).$$

From Rademacher's theorem [12], it follows that if h is locally Lipschitzian, then h is differentiable almost everywhere. The generalized Jacobian of h in the sense of Clarke, denoted by $\partial h(x)$, is defined as the set of matrices

$$\partial h(x) := \overline{co}\left\{H = \lim_{k\to\infty} h'(x^k) : x^k \to x \text{ with } h \text{ differentiable at each } x^k\right\},$$

where $\overline{co}$ denotes the convex hull.

Theorem 2.2. *For the generalized Jacobian $\partial h(\cdot)$, we have:*

1. $\partial h(x)$ *is nonempty, convex and compact;*
2. *(mean-value theorem)*

$$h(x) - h(y) \in \partial h(\overline{co}\{x, y\})(x - y),$$

where

$$\partial h(\overline{co}\{x, y\}) := \bigcup_{z\in\overline{co}\{x,y\}} \partial h(z).$$

For the proof, see [10].

For completeness, we recall that a mapping $h : \mathbb{R}^n \to \mathbb{R}^n$ is called isotone if from $x \leq y$ it follows that $h(x) \leq h(y)$. The matrix norm used in the paper is always assumed to be subordinate to given vector norm.

3. ERROR ESTIMATION FOR *NCP(f)*

Let x^* be a solution of $NCP(f)$, let $\hat{x} \in \mathbb{R}^n$ be a given fixed vector, which could be the result of an iterative method for approximating x^*, for example. We are interested in the problem of estimating the error

$$e(\hat{x}) = \hat{x} - x^*. \tag{3.1}$$

Subsequently, we always assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable at any point of interest.

Let $\Delta = diag(\delta_i)$ be an arbitrary but fixed diagonal matrix with $\delta_i > 0$, $i = 1, \ldots, n$. Clearly, x^* solves $NCP(f)$ if and only if x^* is a solution of the equation

$$h_\Delta(x) := \min\{x, \Delta f(x)\} = 0. \tag{3.2}$$

Remark 3.1. Usually, the case $\Delta = I_n$ (identity matrix) is only considered in the literature (see [21]). The choice of a Δ different from I_n can have a striking effect concerning the quality of computed error bounds. See Example 5.1, e.g., and the second to the last column in Tables 1 and 2. A theoretical discussion of the dependency of the error bounds on the choice of Δ is nontrivial problem, which must be left for future research.

It is noted that $h_\Delta(\cdot)$ is locally Lipschitzian, so from Rademacher's theorem it follows that $h_\Delta(\cdot)$ is differentiable almost everywhere. We study the generalized Jacobian $\partial h_\Delta(x)$ of $h_\Delta(\cdot)$ in the sense of Clarke [10].

Definition 3.2. Let $\hat{x} \in \mathbb{R}^n$ be fixed. We define three index subsets of $\mathcal{J}$:

$$\alpha = \alpha_\Delta(\hat{x}) := \{i : \hat{x}_i < \delta_i f_i(\hat{x})\},$$
$$\beta = \beta_\Delta(\hat{x}) := \{i : \hat{x}_i = \delta_i f_i(\hat{x})\},$$
$$\gamma = \gamma_\Delta(\hat{x}) := \{i : \hat{x}_i > \delta_i f_i(\hat{x})\}.$$

It is clear that $\alpha_\Delta(\hat{x}) \cup \beta_\Delta(\hat{x}) \cup \gamma_\Delta(\hat{x}) = \mathcal{J}$.

Proposition 3.3. *Define the set of matrices*

$$\Pi_\Delta(\hat{x}) := \left\{ I_n - D + D\Delta f'(\hat{x}) : D = diag(d_i),\ d_i \begin{cases} = 0 & i \in \alpha_\Delta(\hat{x}) \\ \in [0,1] & i \in \beta_\Delta(\hat{x}) \\ = 1 & i \in \gamma_\Delta(\hat{x}) \end{cases} \right\}.$$

Then we have:

1. $\Pi_\Delta(\hat{x})$ *is compact and convex;*
2. $\Pi_\Delta(\hat{x}) \subset \mathcal{P}_n$ *if* $(f'(\hat{x}))_{\bar{\alpha}\bar{\alpha}} \in \mathcal{P}_{|\bar{\alpha}|}$;
3. $\partial h_\Delta(\hat{x}) \subseteq \Pi_\Delta(\hat{x})$.

Remark 3.4. Let $W \in \mathbb{R}^{n\times n}$ be a matrix contained in $\Pi_\Delta(\hat{x})$. Denote by w_i^T and $(f'(\hat{x}))_i^T$ the ith row vector of W and $f'(\hat{x})$, respectively. Then it holds

$$w_i = \begin{cases} e_i & i \in \alpha_\Delta(\hat{x}), \\ (1-d_i)e_i + d_i\delta_i(f'(\hat{x}))_i & i \in \beta_\Delta(\hat{x}), \\ \delta_i(f'(\hat{x}))_i & i \in \gamma_\Delta(\hat{x}). \end{cases}$$

Remark 3.5. The proof of Proposition 3.3 is a special case of (2.5) in [8], as was pointed out by Chen (personal communication) and by an anonymous referee.

The next proposition shows that the error $e(\hat{x}) = \hat{x} - x^*$ can be represented as the solution of a linear system of equations.

Proposition 3.6. *Suppose that for a solution x^* of NCP(f) formulated by* (3.2) *and a fixed $\hat{x} \in \mathbb{R}^n$, a set $\mathcal{D}$ is known with $\overline{co}\{x^*, \hat{x}\} \subseteq \mathcal{D} \subseteq \mathbb{R}^n$. Then there are $\xi \in \mathcal{D}$ and $D = diag(d_i)$ with $d_i \in [0,1]$ such that*

$$(I_n - D + D\Delta f'(\xi))(\hat{x} - x^*) = h_\Delta(\hat{x}). \tag{3.3}$$

Proof. From Theorem 2.2 and Proposition 3.3 and from $h_\Delta(x^*) = 0$, it follows

$$\begin{aligned} h_\Delta(\hat{x}) = h_\Delta(\hat{x}) - h_\Delta(x^*) &\in \partial h_\Delta(\overline{co}\{\hat{x}, x^*\})(\hat{x} - x^*) \\ &\in \bigcup_{y\in\mathcal{D}} \Pi_\Delta(y)(\hat{x} - x^*), \end{aligned}$$

which, together with the expression for $\Pi_\Delta(\cdot)$ yields (3.3). □

Remark 3.7. If $f'(x) \in \mathcal{P}_n$ holds for any $x \in \mathcal{D}$, then from Proposition 3.3 we know that $(I_n - D + D\Delta f'(\xi)) \in \mathcal{P}_n$, and so it is nonsingular by Theorem 2.1. This guarantees the unique solvability of the system (3.3).

System (3.3) has the unknown data $\xi \in \mathcal{D}$ and $D = diag(d_i)$ in its coefficient matrix. We establish an interval matrix $[J]_{\mathcal{D},\Delta}$ such that

$$e(\hat{x}) = \hat{x} - x^* \in \{x \in \mathbb{R}^n : Jx = h_\Delta(\hat{x}), J \in [J]_{\mathcal{D},\Delta}\}$$

for a fixed $\hat{x} \in \mathcal{D}$ and a solution x^* of $NCP(f)$ contained also in $\mathcal{D}$.

Theorem 3.8. *Suppose that for a solution x^* of NCP(f) and a fixed $\hat{x} \in \mathbb{R}^n$, $\overline{co}\{x^*, \hat{x}\} \subseteq \mathcal{D} \subseteq \mathbb{R}^n$ for some given set $\mathcal{D}$. Suppose*

$$\left.\begin{aligned} -\infty < \underline{f'}^{\mathcal{D}}_{ij} \leq \inf\{(f'(u))_{ij} : u \in \mathcal{D}\} \\ +\infty > \overline{f'}^{\mathcal{D}}_{ij} \geq \sup\{(f'(u))_{ij} : u \in \mathcal{D}\} \end{aligned}\right\} \quad i, j = 1, 2, \ldots, n. \tag{3.4}$$

Define the matrices

$$\underline{f'}^{\mathcal{D}} := \left(\underline{f'}^{\mathcal{D}}_{ij}\right) \quad \text{and} \quad \overline{f'}^{\mathcal{D}} := \left(\overline{f'}^{\mathcal{D}}_{ij}\right).$$

Then the error $e(\hat{x}) = \hat{x} - x^*$ is included in the solution set

$$\sum([J]_{\mathcal{D},\Delta}, h_\Delta(\hat{x})) := \{x \in \mathbb{R}^n : Jx = h_\Delta(\hat{x}), J \in [J]_{\mathcal{D},\Delta}\},$$

where the interval matrix $[J]_{\mathcal{D},\Delta}$ is defined by

$$([J]_{\mathcal{D},\Delta})_{ij} = \begin{cases} \left[\delta_i \min\left\{0, \underline{f'}^{\mathcal{D}}_{ij}\right\}, \delta_i \max\left\{0, \overline{f'}^{\mathcal{D}}_{ij}\right\}\right] & j \neq i, \\ \left[\min\left\{1, \delta_i \underline{f'}^{\mathcal{D}}_{ii}\right\}, \max\left\{1, \delta_i \overline{f'}^{\mathcal{D}}_{ii}\right\}\right] & j = i. \end{cases} \tag{3.5}$$

Proof. Observe that the elements of the matrix $I_n - D + D\Delta f'(\xi)$ from (3.3) are

$$(I_n - D + D\Delta f'(\xi))_{ij} = \begin{cases} d_i \delta_i (f'(\xi))_{ij} & j \neq i, \\ 1 - d_i + d_i \delta_i (f'(\xi))_{ii} & j = i. \end{cases}$$

Because $d_i \in [0, 1]$ and $(f'(\xi))_{ij} \in [\underline{f'}^{\mathcal{D}}_{ij}, \overline{f'}^{\mathcal{D}}_{ij}]$, we have

$$\begin{aligned} d_i \delta_i (f'(\xi))_{ij} &\in [0, 1]\left[\delta_i \underline{f'}^{\mathcal{D}}_{ij}, \delta_i \overline{f'}^{\mathcal{D}}_{ij}\right] & j \neq i, \\ 1 - d_i + d_i \delta_i (f'(\xi))_{ii} &\in \left[\min\left\{1, \delta_i \underline{f'}^{\mathcal{D}}_{ii}\right\}, \max\left\{1, \delta_i \overline{f'}^{\mathcal{D}}_{ii}\right\}\right] & j = i, \end{aligned}$$

and noting that

$$[0, 1]\left[\delta_i \underline{f'}^{\mathcal{D}}_{ij}, \delta_i \overline{f'}^{\mathcal{D}}_{ij}\right] = \left[\delta_i \min\left\{0, \underline{f'}^{\mathcal{D}}_{ij}\right\}, \delta_i \max\left\{0, \overline{f'}^{\mathcal{D}}_{ij}\right\}\right], \tag{3.6}$$

we conclude that for any $\xi \in \mathcal{D}$

$$I_n - D + D\Delta f'(\xi) \in [J]_{\mathcal{D},\Delta},$$

from which, together with Proposition 3.6, the assertion follows. □

Remark 3.9. The assumption (3.4) will be replaced by a different one at the end of the section. For $LCP(q, M)$, we have for any $\mathcal{D} \subseteq \mathbb{R}^n$

$$\underline{f'}^{\mathcal{D}} = \overline{f'}^{\mathcal{D}} = M.$$

Remark 3.10. In general, we cannot guarantee that $[J]_{\mathcal{D},\Delta}$ contains no singular matrices, even if $[\underline{f'}^{\mathcal{D}}, \overline{f'}^{\mathcal{D}}] \subset \mathcal{P}_n$. Consider, for example, the matrix

$$M = \begin{pmatrix} 5 & 3 & 2 \\ 5 & 5 & 3 \\ 2 & 1 & 1 \end{pmatrix} \in \mathcal{P}_n.$$

For any $\mathcal{D} \subseteq \mathbb{R}^3$, we have $\underline{f'}^{\mathcal{D}} = \overline{f'}^{\mathcal{D}} = M$. With $\Delta = I_3$, we find for (3.5)

$$[J]_{\mathcal{D},\Delta} = \begin{pmatrix} [1,5] & [0,3] & [0,2] \\ [0,5] & [1,5] & [0,3] \\ [0,2] & [0,1] & [1,1] \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

is contained in $[J]_{\mathcal{D},\Delta}$ and is singular.

The diagonal matrix $\Delta = diag(\delta_i)$, $\delta_i > 0$, $i = 1, \ldots, n$, was chosen arbitrarily but fixed in (3.2), and therefore also in Theorem 3.8. We now discuss how to choose Δ to ensure that $[J]_{\mathcal{D},\Delta}$ contains no singular matrices.

Theorem 3.11. *Let $\underline{f'}^{\mathcal{D}} = (\underline{f'}^{\mathcal{D}}_{ij})$ and $\overline{f'}^{\mathcal{D}} = (\overline{f'}^{\mathcal{D}}_{ij})$ be defined by (3.4), and suppose $[\underline{f'}^{\mathcal{D}}, \overline{f'}^{\mathcal{D}}] \subset \mathcal{H}_n^+$. Then $[J]_{\mathcal{D},\Delta} \subset \mathcal{H}_n^+$ if $\Delta = diag(\delta_i)$ with*

$$0 < \delta_i \leq \delta_i^* := \frac{1}{\overline{f'}^{\mathcal{D}}_{ii}}, \quad i = 1, \ldots, n. \tag{3.7}$$

Proof. Let $J \in [J]_{\mathcal{D},\Delta}$. From (3.5), (3.6), and (3.7), it follows

$$(J)_{ij} \in \begin{cases} ([J]_{\mathcal{D},\Delta})_{ij} = [0,1]\left[\delta_i \underline{f'}^{\mathcal{D}}_{ij}, \delta_i \overline{f'}^{\mathcal{D}}_{ij}\right] & \text{if } i \neq j, \\ ([J]_{\mathcal{D},\Delta})_{ii} = \left[\delta_i \underline{f'}^{\mathcal{D}}_{ii}, 1\right] & \text{if } i = j. \end{cases}$$

Therefore we have

$$|(J)_{ij}| \le \delta_i \max\left\{\left|\underline{f'}^{\mathcal{D}}_{ij}\right|, \left|\overline{f'}^{\mathcal{D}}_{ij}\right|\right\},$$
$$|(J)_{ii}| \ge \delta_i \underline{f'}^{\mathcal{D}}_{ii},$$

and so $\langle J\rangle \ge \Delta\langle R\rangle$, where $\Delta = diag(\delta_i)$, $R = (r_{ij})$ with $r_{ii} = \underline{f'}^{\mathcal{D}}_{ii}$ and for $j \ne i$

$$r_{ij} = \begin{cases} \overline{f'}^{\mathcal{D}}_{ij} & \text{if } \left|\overline{f'}^{\mathcal{D}}_{ij}\right| > \left|\underline{f'}^{\mathcal{D}}_{ij}\right|, \\ \underline{f'}^{\mathcal{D}}_{ij} & \text{if } \left|\overline{f'}^{\mathcal{D}}_{ij}\right| \le \left|\underline{f'}^{\mathcal{D}}_{ij}\right|. \end{cases}$$

It is clear that $R \in [\underline{f'}^{\mathcal{D}}, \overline{f'}^{\mathcal{D}}]$, and from the assumption $[\underline{f'}^{\mathcal{D}}, \overline{f'}^{\mathcal{D}}] \subset \mathcal{H}_n^+$, we know that $R \in \mathcal{H}_n^+$, and so $\Delta\langle R\rangle \in \mathcal{M}_n$. Together with the fact that $\langle J\rangle \ge \Delta\langle R\rangle$, we obtain from statement 5 of Theorem 2.1 that $\langle J\rangle \in \mathcal{M}_n$, hence $J \in \mathcal{H}_n^+$. □

Remark 3.12. $[J]_{\mathcal{D},\Delta}$ may contain singular matrices if the condition (3.7) is not fulfilled. Consider, for example, $LCP(q, M)$ with the matrix (see [9])

$$M = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \in \mathcal{H}_n^+.$$

For any $\mathcal{D} \subseteq \mathbb{R}^2$, we obtain $\underline{f'}^{\mathcal{D}} = \overline{f'}^{\mathcal{D}} = M$. With the choice $\delta_i = 1 > \delta_i^* = \frac{1}{2}$ for $i = 1, 2$, we find

$$[J]_{\mathcal{D},\Delta} = \begin{pmatrix} [1,2] & [-1,0] \\ [-1,0] & [1,2] \end{pmatrix}$$

for the matrix $[J]_{\mathcal{D},\Delta}$ defined by (3.5). The singular matrix

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

is contained in $[J]_{\mathcal{D},\Delta}$.

Remark 3.13. If $[\underline{f'}^{\mathcal{D}}, \overline{f'}^{\mathcal{D}}]$ is contained in $\mathcal{P}_n$ but not contained in $\mathcal{H}_n^+$, $[J]_{\mathcal{D},\Delta}$ might contain singular matrices even if the condition (3.7) is fulfilled. To demonstrate this, we consider the matrix from Remark 3.10 with the choice $\Delta = diag(\frac{1}{5}, \frac{1}{5}, 1)$, for which (3.7) is fulfilled with $\delta_i = \delta_i^*$.

We find for (3.5)

$$[J]_{\mathcal{D},\Delta} = \begin{pmatrix} 1 & [0,0.6] & [0,0.4] \\ [0,1] & 1 & [0,0.6] \\ [0,2] & [0,1] & 1 \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} 1 & 0 & 0.4 \\ 1 & 1 & 0.6 \\ 2 & 1 & 1 \end{pmatrix}$$

is contained in $[J]_{\mathcal{D},\Delta}$ and is singular.

Let us go back to the system (3.3):

$$(I_n - D + D\Delta f'(\xi))(\hat{x} - x^*) = h_\Delta(\hat{x}).$$

Because $I_n - D + D\Delta f'(\xi)$ is dependent on the unknown point $\xi \in \mathcal{D}$ and the unknown diagonal matrix D, we consider the mapping $J_{\mathcal{D},\Delta} : [0,1]^n \times \mathcal{D} \to \mathbb{R}^{n\times n}$ with

$$J_{\mathcal{D},\Delta}(d,u) = I_n - D + D\Delta f'(u) \tag{3.8}$$

where $D = diag(d_i)$ and $d = (d_i) \in [0,1]^n$, $u \in \mathcal{D}$.

Lemma 3.14. *Let $f'(u) \in \mathcal{P}_n$ for any $u \in \mathcal{D}$. Then $J_{\mathcal{D},\Delta}(d,u) \in \mathcal{P}_n$, and $J_{\mathcal{D},\Delta}(d,u)^{-1}$ is continuous w.r.t. (d,u).*

Proof. Because $f'(u) \in \mathcal{P}_n$, it follows from Theorem 2.1 that $J_{\mathcal{D},\Delta}(d,u) \in \mathcal{P}_n$. Because f is continuously differentiable, $J_{\mathcal{D},\Delta}(d,u)$ is continuous w.r.t. (d,u), from which together with 2.3.3 of [20] the conclusion follows. □

Theorem 3.15. *Let $\overline{co}\{x^*, \hat{x}\} \subseteq \mathcal{D} \subseteq \mathbb{R}^n$ for some fixed $\hat{x} \in \mathcal{D}$ and let $x^* \in \mathcal{D}$ be a solution of NCP(f). Let $f'(u) \in \mathcal{P}_n$ for any $u \in \mathcal{D}$ and assume that (3.4) holds. Let $\Omega = [0,1]^n \times \mathcal{D}$. Then we have for any $\Delta = diag(\delta_i)$ with $\delta_i > 0$, $i = 1, \ldots, n$, the error bounds*

$$\frac{\|h_\Delta(\hat{x})\|}{\max_{(d,u)\in\Omega}\|J_{\mathcal{D},\Delta}(d,u)\|} \le \|\hat{x} - x^*\| \le \max_{(d,u)\in\Omega}\|(J_{\mathcal{D},\Delta}(d,u))^{-1}\| \cdot \|h_\Delta(\hat{x})\|. \tag{3.9}$$

Proof. The error bound is the direct result of (3.3) and Lemma 3.14. □

Remark 3.16. The error bound (2.3) in [9] for the linear complementarity problem is the special case of (3.9) with the choice $\Delta = I_n$.

Remark 3.17. In general, the error bounds (3.9) are not easy to compute. However, as we will see in the next section, the difficulty of the computation is greatly reduced for $ALCP(\Phi, M)$ and $LCP(q, M)$.

Thus far, we have established two approaches of error estimation for $NCP(f)$:

1. Componentwise error estimation via the solution set $\Sigma([J]_{\mathcal{D},\Delta}, h_\Delta(\hat{x}))$ for the interval matrix $[J]_{\mathcal{D},\Delta}$ and the vector $h_\Delta(\hat{x})$ (see Theorem 3.8), and
2. Computing bounds of $\|\hat{x} - x^*\|$ (see Theorem 3.15).

For both of them, we have to suppose (3.4). In the special case that $\mathcal{D} = \mathbb{R}^n$, Theorem 3.8 and Theorem 3.11 simplify to the following two theorems, respectively.

Theorem 3.18. *Let x^* be a solution of $NCP(f)$, and $\hat{x} \in \mathbb{R}^n$ be fixed. Suppose that f' is bounded in $\mathbb{R}^n$, that is*

$$\left.\begin{aligned} -\infty < \underline{f'}^{\mathbb{R}^n}_{ij} &\le \inf\{(f'(u))_{ij} : u \in \mathbb{R}^n\} \\ +\infty > \overline{f'}^{\mathbb{R}^n}_{ij} &\ge \sup\{(f'(u))_{ij} : u \in \mathbb{R}^n\} \end{aligned}\right\} \quad i, j = 1, \ldots, n. \tag{3.10}$$

Define the matrices

$$\underline{f'}^{\mathbb{R}^n} := \left(\underline{f'}^{\mathbb{R}^n}_{ij}\right) \quad \text{and} \quad \overline{f'}^{\mathbb{R}^n} := \left(\overline{f'}^{\mathbb{R}^n}_{ij}\right).$$

Then the error $e(\hat{x}) = \hat{x} - x^*$ is included in the solution set

$$\Sigma([J]_{\mathbb{R}^n,\Delta}, h_\Delta(\hat{x})) := \{x \in \mathbb{R}^n : Jx = h_\Delta(\hat{x}), J \in [J]_{\mathbb{R}^n,\Delta}\}$$

where the interval matrix $[J]_{\mathbb{R}^n,\Delta}$ is defined by

$$([J]_{\mathbb{R}^n,\Delta})_{ij} = \begin{cases} \left[\delta_i \min\left\{0, \underline{f'}^{\mathbb{R}^n}_{ij}\right\}, \delta_i \max\left\{0, \overline{f'}^{\mathbb{R}^n}_{ij}\right\}\right] & j \ne i, \\ \left[\min\left\{1, \delta_i \underline{f'}^{\mathbb{R}^n}_{ij}\right\}, \max\left\{1, \delta_i \overline{f'}^{\mathbb{R}^n}_{ij}\right\}\right] & j = i. \end{cases}$$

Theorem 3.19. *Let* $\underline{f'}^{\mathbb{R}^n} := \left(\underline{f'}^{\mathbb{R}^n}_{ij}\right)$ *and* $\overline{f'}^{\mathbb{R}^n} := \left(\overline{f'}^{\mathbb{R}^n}_{ij}\right)$ *be given by* (3.10), *and suppose* $[\underline{f'}^{\mathbb{R}^n}, \overline{f'}^{\mathbb{R}^n}] \subset \mathcal{K}_n^+$. *Then* $[J]_{\mathbb{R}^n,\Delta} \subset \mathcal{K}_n^+$ *if* $\Delta = diag(\delta_i)$ *with*

$$0 < \delta_i \leq \delta_i^* := \frac{1}{\overline{f'}^{\mathbb{R}^n}_{ii}}, \quad i = 1, \ldots, n.$$

In the case that $\mathcal{D} = \mathbb{R}^n$, Theorem 3.15 simplifies to the following result.

Theorem 3.20. *Let* $\underline{f'}^{\mathbb{R}^n} := \left(\underline{f'}^{\mathbb{R}^n}_{ij}\right)$ *and* $\overline{f'}^{\mathbb{R}^n} := \left(\overline{f'}^{\mathbb{R}^n}_{ij}\right)$ *be given by* (3.10), *and let* x^* *be a solution of* $NCP(f)$ *and let* $\hat{x} \in \mathbb{R}^n$ *be given. Define the mapping*

$$J_\Delta(d, U) := I_n - D + D\Delta U$$

where $D = diag(d_i)$ *with* $d = (d_i) \in [0, 1]^n$, *and* $U \in [\underline{f'}^{\mathbb{R}^n}, \overline{f'}^{\mathbb{R}^n}]$. *Suppose* $[\underline{f'}^{\mathbb{R}^n}, \overline{f'}^{\mathbb{R}^n}] \subset \mathcal{P}_n$. *Denote* $\Omega := [0, 1]^n \times [\underline{f'}^{\mathbb{R}^n}, \overline{f'}^{\mathbb{R}^n}]$. *Then we have for any* $\Delta = diag(\delta_i)$, $\delta_i > 0$, $i = 1, \ldots, n$,

$$\frac{\|h_\Delta(\hat{x})\|}{\max_{(d,U)\in\Omega}\|J_\Delta(d, U)\|} \leq \|\hat{x} - x^*\| \leq \max_{(d,U)\in\Omega}\|(J_\Delta(d, U))^{-1}\| \cdot \|h_\Delta(\hat{x})\|.$$

4. THE SPECIAL CASE *ALCP*(Φ, *M*)

In the preceding section, we gave a pointwise inclusion of the error (3.1) by Theorems 3.8 and 3.18, respectively. After that, we gave lower and upper norm bounds of the error by Theorems 3.15 and 3.20, respectively. In this section, we specialize the results to $ALCP(\Phi, M)$ and $LCP(q, M)$, respectively. We first construct a convex set $\mathcal{D} \subseteq \mathbb{R}^n$ containing a solution x^*. From $\mathcal{D}$, the approximation $\hat{x}$ is also chosen. The condition $\overline{co}\{x^*, \hat{x}\} \subseteq \mathcal{D}$ was required in Theorem 3.8.

Theorem 4.1. *Let* $\Phi = (\Phi_i(x_i))$ *be isotone and continuously differentiable. Let* $M \in \mathcal{K}_n^+$, *and denote by* Λ *and* $-B$ *the diagonal and off-diagonal parts of* M, *respectively. Then* $ALCP(\Phi, M)$ *has a unique solution* x^*, *which is included in the interval vector* $\mathcal{D} := [\hat{x} - r, \hat{x} + r]$, *where* $\hat{x}$ *is a certain fixed vector and*

$$r := |\hat{x}| + \langle M\rangle^{-1}\max\{0, -M\hat{x} - \Phi(\hat{x}) - \langle M\rangle|\hat{x}|\}. \tag{4.1}$$

Proof. Let $[\hat{x}] = [\hat{x} - r, \hat{x} + r]$ and assume that for any $x \in [\hat{x}]$

$$0 \leq \phi_{1i} \leq \frac{d\Phi_i(x_i)}{dx_i} \leq \phi_{2i}, \quad i = 1, \ldots, n,$$

and let $\Phi_1' = diag(\phi_{1i})$, $\Phi_2' = diag(\phi_{2i})$, and set $\Delta = (\Lambda + \Phi_2')^{-1}$. Let

$$\Gamma(\hat{x}, [\hat{x}], \Delta) := \max\{0, \hat{x} - \Delta(M\hat{x} + \Phi(\hat{x})) + (I_n - \Delta(M + [\Phi_1', \Phi_2']))([\hat{x}] - \hat{x})\}.$$

In Theorem 2.1 of [5], it was proved for the interval vector $\Gamma(\hat{x}, [\hat{x}], \Delta)$ that if the interval inclusion

$$\Gamma(\hat{x}, [\hat{x}], \Delta) \subseteq [\hat{x} - r, \hat{x} + r]$$

holds, then $x^* \in [\hat{x} - r, \hat{x} + r]$, where x^* is a solution of $ALCP(\Phi, M)$. Note that

$$\begin{aligned} I_n - \Delta(M + [\Phi_1', \Phi_2']) &= \Delta(\Delta^{-1} - (\Lambda - B + [\Phi_1', \Phi_2'])) \\ &= \Delta(\Lambda + \Phi_2' - [\Lambda - B + \Phi_1', \Lambda - B + \Phi_2']) \\ &= \Delta[B, B + \Phi_2' - \Phi_1'] \end{aligned}$$

and

$$\begin{aligned} (I_n - \Delta(M + [\Phi_1', \Phi_2']))([\hat{x}] - \hat{x}) &= \Delta[B, B + \Phi_2' - \Phi_1'][-r, r] \\ &= \Delta(|B| + \Phi_2' - \Phi_1')[-r, r]. \end{aligned}$$

Let $\Gamma(\hat{x}, [\hat{x}], \Delta) = [\underline{\Gamma(\hat{x}, [\hat{x}], \Delta)}, \overline{\Gamma(\hat{x}, [\hat{x}], \Delta)}]$. Then we have

$$\begin{aligned} \overline{\Gamma(\hat{x}, [\hat{x}], \Delta)} &= \max\{0, \hat{x} - \Delta(M\hat{x} + \Phi(\hat{x})) + \Delta(|B| + \Phi_2' - \Phi_1')r\} \\ \underline{\Gamma(\hat{x}, [\hat{x}], \Delta)} &= \max\{0, \hat{x} - \Delta(M\hat{x} + \Phi(\hat{x})) - \Delta(|B| + \Phi_2' - \Phi_1')r\}. \end{aligned}$$

We verify at first $\overline{\Gamma(\hat{x}, [\hat{x}], \Delta)} \leq \hat{x} + r$. Considering

$$\begin{aligned} (I_n - \Delta(|B| + \Phi_2' - \Phi_1'))r &= \Delta(\Delta^{-1} - |B| - \Phi_2' + \Phi_1')r \\ &= \Delta(\Lambda - \Phi_2' - |B| - \Phi_2' + \Phi_1')r \\ &= \Delta(\Lambda + \Phi_1' - |B|)r \\ &\geq \Delta(\Lambda - |B|)r = \Delta\langle M\rangle r \end{aligned}$$

and

$$\begin{aligned} \langle M\rangle r &= \langle M\rangle|\hat{x}| + \max\{0, -M\hat{x} - \Phi(\hat{x}) - \langle M\rangle|\hat{x}|\} \\ &\geq \langle M\rangle|\hat{x}| - M\hat{x} - \Phi(\hat{x}) - \langle M\rangle|\hat{x}| \\ &\geq -M\hat{x} - \Phi(\hat{x}) \end{aligned}$$

we have

$$(I_n - \Delta(|B| + \Phi_2' - \Phi_1'))r \geq -\Delta(M\hat{x} + \Phi(\hat{x}))$$

and so

$$\hat{x} - \Delta(M\hat{x} + \Phi(\hat{x})) + \Delta(|B| + \Phi_2' - \Phi_1')r \leq \hat{x} + r,$$

from which, together with the fact that $\hat{x} + r \geq \hat{x} + |\hat{x}| \geq 0$ by (4.1), it follows that

$$\max\{0, \hat{x} - \Delta(M\hat{x} + \Phi(\hat{x})) + \Delta(|B| + \Phi_2' - \Phi_1')r\} \leq \hat{x} + r.$$

Using again $r \geq |\hat{x}|$, and so $\hat{x} - r \leq \hat{x} - |\hat{x}| \leq 0$, we have

$$\max\{0, \hat{x} - \Delta(M\hat{x} + \Phi(\hat{x})) - \Delta(|B| + \Phi_2' - \Phi_1')r\} \geq \hat{x} - r.$$

The proof is complete. □

Remark 4.2. If $\Phi(x) = (\Phi_i(x_i))$ is isotone, $M \in \mathcal{H}_n^+$, then $ALCP(\Phi, M)$ has a unique solution. The proof can be found in [5].

Theorem 4.3. *Let $\Phi(x) = (\Phi_i(x_i))$ be isotone and continuously differentiable, $M \in \mathcal{H}_n^+$, and denote by x^* the unique solution of $ALCP(\Phi, M)$. Let $\hat{x} \in \mathbb{R}^n$ be fixed, let r be defined by (4.1). Suppose that for any $x \in \mathcal{D} := [\hat{x} - r, \hat{x} + r]$*

$$0 \leq \phi_{1i} \leq \frac{d\Phi_i(x_i)}{dx_i} \leq \phi_{2i}, \quad i = 1, \ldots, n, \tag{4.2}$$

and let $\Phi_1' := diag(\phi_{1i})$, $\Phi_2' := diag(\phi_{2i})$. Let $\Delta = diag(\delta_i)$ with

$$0 < \delta_i \leq \delta_i^* := \frac{1}{m_{ij} + \phi_{2i}}, \quad i = 1, \ldots, n. \tag{4.3}$$

Then

$$e(\hat{x}) = \hat{x} - x^* \in \Sigma([J]_{\mathcal{D},\Delta}, h_\Delta(\hat{x})) := \{x \in \mathbb{R}^n : Jx = h_\Delta(\hat{x}), J \in [J]_{\mathcal{D},\Delta}\}, \tag{4.4}$$

where the interval matrix $[J]_{\mathcal{D},\Delta}$ is defined by

$$([J]_{\mathcal{D},\Delta})_{ij} = \begin{cases} [\delta_i \min\{0, m_{ij}\}, \delta_i \max\{0, m_{ij}\}] & j \neq i, \\ [\delta_i(m_{ii} + \phi_{i1}), 1] & j = i. \end{cases}$$

Moreover, $[J]_{\mathcal{D},\Delta} \subset \mathcal{H}_n^+$, and so contains no singular matrices.

Proof. The proof is an immediate consequence of Theorems 3.8 and 3.11. □

The next discussion is concerned with *LCP*(q, M). In this case, $\Phi'(x) \equiv 0$, and we obtain from Theorem 4.3 the following result.

Corollary 4.4. *Let $M \in \mathcal{H}_n^+$, $q \in \mathbb{R}^n$. Let x^* denote the unique solution of LCP(q, M), and let $\hat{x} \in \mathbb{R}^n$ be given. Let $\Delta = diag(\delta_i)$ with*

$$0 < \delta_i \le \delta_i^* := \frac{1}{m_{ii}}, \quad i = 1, \ldots, n. \tag{4.5}$$

Then

$$e(x) = \hat{x} - x^* \in \Sigma([J]_{\mathbb{R}^n,\Delta}, h_\Delta(\hat{x})) := \{x \in \mathbb{R}^n : Jx = h_\Delta(\hat{x}), J \in [J]_{\mathbb{R}^n,\Delta}\}, \tag{4.6}$$

where the interval matrix $[J]_{\mathbb{R}^n,\Delta}$ is defined by

$$([J]_{\mathbb{R}^n,\Delta})_{ij} = \begin{cases} [\delta_i \min\{0, m_{ij}\}, \delta_i \max\{0, m_{ij}\}] & j \ne i, \\ [\delta_i m_{ii}, 1] & j = i. \end{cases}$$

Moreover, $[J]_{\mathbb{R}^n,\Delta} \in \mathcal{H}_n^+$, and so contains no singular matrices.

Now we reconsider the norm estimation for $e(\hat{x}) = \hat{x} - x^*$ given in Theorem 3.15. This result is in general not easy to apply. For the case of *ALCP*(Φ, M) with $M \in \mathcal{H}_n^+$ and Φ isotone, an efficient and computational bound can be given, however. We need the following theorem. See [9], Theorem 2.1.

Theorem 4.5. *Let $M \in \mathcal{H}_n^+$ with diagonal part Λ. Then for $1 \le p \le +\infty$ and $D = diag(d_i)$, $d_i \in [0, 1]$, we have*

$$\max_{d \in [0,1]^n} \|(I_n - D + DM)^{-1}\|_p \le \|\langle M\rangle^{-1} \max\{\Lambda, I_n\}\|_p.$$

Theorem 4.6. *Let $M \in \mathcal{H}_n^+$ with diagonal part Λ, and let $\Phi(x) = (\Phi_i(x_i))$ be isotone and continuously differentiable. Let $\hat{x} \in \mathbb{R}^n$ be fixed. Let r be defined by (4.1), $\mathcal{D} := [\hat{x} - r, \hat{x} + r]$, and let $\Phi_1' = diag(\phi_{1i})$ and $\Phi_2' = diag(\phi_{2i})$ be defined by (4.2). Then for any $\Delta = diag(\delta_i)$ with $\delta_i > 0$, $i = 1, \ldots, n$, we have for the solution x^* of ALCP(Φ, M)*

$$\|\hat{x} - x^*\|_p \le \|(\langle M + \Phi_1'\rangle)^{-1} \max\{\Lambda + \Phi_2', \Delta^{-1}\}\|_p \cdot \|h_\Delta(\hat{x})\|_p =: E_{\Delta,p}^{bnd}(\hat{x}). \tag{4.7}$$

Proof. Because $M \in \mathcal{H}_n^+$ and Φ is isotone, it is clear that $\Delta(M + \Phi'(u)) \in \mathcal{H}_n^+$. For the matrix (3.8) we obtain

$$J_{\mathcal{D},\Delta}(d, u) = I_n - D + D\Delta(M + \Phi'(u)),$$

which together with Theorem 4.5 and (4.2) yields

$$\|(J_{\mathcal{D},\Delta}(d, u))^{-1}\|_p \le \|\langle\Delta(M + \Phi'(u))\rangle^{-1}\max\{\Delta(\Lambda + \Phi'(u)), I_n\}\|_p.$$

Because

$$\langle\Delta(M + \Phi'(u))\rangle^{-1} \le (\Delta(\langle M\rangle + \Phi_1'))^{-1}$$

and

$$\max\{\Delta(\Lambda + \Phi'(u)), I_n\} \le \max\{\Delta(\Lambda + \Phi_2'), I_n\},$$

we obtain, using the monotonicity of $\|\cdot\|_p$,

$$\begin{aligned} \|(J_{\mathcal{D},\Delta}(d, u))^{-1}\|_p &\le \|(\Delta(\langle M\rangle + \Phi_1'))^{-1}\max\{\Delta(\Lambda + \Phi_2'), I_n\}\|_p \\ &= \|(\langle M\rangle + \Phi_1')^{-1}\Delta^{-1}\max\{\Delta(\Lambda + \Phi_2'), I_n\}\|_p \\ &\le \|(\langle M\rangle + \Phi_1')^{-1}\max\{\Lambda + \Phi_2', \Delta^{-1}\}\|_p. \end{aligned}$$

Therefore we obtain (4.7) from (3.9). □

As a special case of Theorem 4.6, we obtain the following result for *LCP*(q, M).

Corollary 4.7. *Let $M \in \mathcal{H}_n^+$ with the diagonal part Λ. Let $\Delta = diag(\delta_i)$ with $\delta_i > 0$, $i = 1, 2, \ldots, n$. For any $\hat{x} \in \mathbb{R}^n$, we have the following error bound for the solution of LCP(q, M):*

$$\|\hat{x} - x^*\|_p \le \|\langle M\rangle^{-1}\max\{\Lambda, \Delta^{-1}\}\|_p \cdot \|h_\Delta(\hat{x})\|_p =: E_{\Delta,p}^{bnd}(\hat{x}). \tag{4.8}$$

For *ALCP*(Φ, M) with Φ having a bounded derivative for all $x \in \mathbb{R}^n$, we have the following results.

Corollary 4.8. *Let $\Phi(x) = (\Phi_i(x_i))$ be isotone, $M \in \mathcal{H}_n^+$, and denote by x^* the unique solution of ALCP(Φ, M). Let $\hat{x} \in \mathbb{R}^n$ be given. Suppose that for all $x \in \mathbb{R}^n$*

$$0 \le \phi_{1i} \le \frac{d\Phi_i(x_i)}{dx_i} \le \phi_{2i}, \quad i = 1, \ldots, n, \tag{4.9}$$

and let

$$\Phi_1' := diag(\phi_{1i}), \quad \Phi_2' := diag(\phi_{2i}).$$

Let $\Delta = diag(\delta_i)$ *with*

$$0 < \delta_i \le \delta_i^* := \frac{1}{m_{ii} + \phi_{2i}}, \quad i = 1, \dots, n. \tag{4.10}$$

Then

$$e(\hat{x}) = \hat{x} - x^* \in \Sigma([J]_{\mathbb{R}^n,\Delta}, h_\Delta(\hat{x})) := \{x \in \mathbb{R}^n : Jx = h_\Delta(\hat{x}), J \in [J]_{\mathbb{R}^n,\Delta}\}, \tag{4.11}$$

where the interval matrix $[J]_{\mathbb{R}^n,\Delta}$ *is defined by:*

$$([J]_{\mathbb{R}^n,\Delta})_{ij} = \begin{cases} [\delta_i \min\{0, m_{ij}\}), \delta_i \max\{0, m_{ij}\}] & j \ne i, \\ [\delta_i(m_{ii} + \phi_{i1}), 1] & j = i. \end{cases}$$

Moreover, $[J]_{\mathbb{R}^n,\Delta} \in \mathscr{H}_n^+$, *and so contains no singular matrices.*

Corollary 4.8 is a special case of Theorems 3.18 and 3.19.

Theorem 4.9. *Let* $M \in \mathscr{H}_n^+$ *with diagonal part* Λ, *and let* $\Phi(x) = (\Phi_i(x_i))$ *be isotone and continuously differentiable. Let* $\hat{x} \in \mathbb{R}^n$ *be given, and assume that* (4.9) *holds for all* $x \in \mathbb{R}^n$. *Then for any* $\Delta = diag(\delta_i)$ *with* $\delta_i > 0$, $i = 1, \dots, n$, *we have*

$$\|\hat{x} - x^*\|_p \le \|(\langle M + \Phi_1' \rangle^{-1} \max\{\Lambda + \Phi_2', \Delta^{-1}\}\|_p \cdot \|h_\Delta(\hat{x})\|_p =: E_{\Delta,p}^{bnd}(\hat{x}). \tag{4.12}$$

Proof. It can be proved in a similar manner as in Theorem 4.6. □

5. NUMERICAL EXPERIMENTS

In this section, by using MATLAB with the support of INTLAB [25], we perform the numerical experiments for five test problems:

- one $LCP(q, M)$ with $M \in \mathscr{P}_n$ but $M \notin \mathscr{H}_n^+$ (Example 5.1);
- two $LCP(q, M)$ with $M \in \mathscr{H}_n^+$ (Examples 5.2 and 5.3);
- one $ALCP(\Phi, M)$ with $M \in \mathscr{H}_n^+$ and Φ diagonal isotone (Example 5.4);
- one $ALCP(\Phi, M)$ with $M \in \mathscr{H}_n^+$ and Φ diagonal isotone and having bounded derivative (Example 5.5).

TABLE 1 ϱ_ϵ, κ_ϵ, and $\tilde{\kappa}$ for Example 5.2

	$\varrho_\epsilon, \epsilon = 0.001$	$\varrho_\epsilon, \epsilon = 0.01$	$\varrho_\epsilon, \epsilon = 0.1$	$\varrho_\epsilon, \epsilon = 1$	$\kappa_\epsilon, \epsilon = 1$	$\tilde{\kappa}$
π_1	1.2376e+03	1.2867e+02	1.7016e+01	4.7560e+00	4.0126e+02	8.0251e+04
π_2	3.0961e+03	3.1067e+02	3.4189e+01	5.8566e+00	7.8831e+02	1.5766e+03
π_3	6.4719e+02	6.6442e+01	8.1716e+00	1.5259e+00	1.1000e+01	2.4200e+01
π_4	4.3448e+02	4.4590e+01	6.4868e+00	3.1684e+00	7.3882e+01	2.5787e+02

The exact solution x^* of each test problem is known beforehand for the numerical experiment. For Examples 5.2 to 5.5 the test point $\hat{x} = (\hat{x}_i)$ is generated in the following way:

$$\hat{x}_i := \max\{0, v_i - 0.5\} \times 10^{10(w_i - 0.5)},$$

where w_i and v_i are random numbers in [0,1]. The function "verifylss.m" of INTLAB is used to include the solution sets (4.4) and (4.11) for $ALCP(\Phi, M)$, and (4.6) for $LCP(q, M)$, respectively. Denote by $[\hat{x} - x^*]_\Delta$ the enclosure returned by "verifylss.m," and define

$$E_\Delta^{lis}(\hat{x}) := \max\{\|y\|_\infty : y \in [\hat{x} - x^*]_\Delta\}.$$

Here the acronym '*lis*' means 'linear interval system' which is a linear system with interval data. Subsequently, we denote $E_{\Delta,\infty}^{bnd}(\hat{x})$ (see Eq. (4.12)) by $E_\Delta^{bnd}(\hat{x})$ for convenience. We choose $\Delta = \epsilon\Delta^*$ with $0 < \epsilon \leq 1$, where $\Delta^* = diag(\delta_i^*)$ with δ_i^*, $i = 1, \ldots, n$, is defined by (4.3) or (4.10) for $ALCP(\Phi, M)$ and by (4.5) for $LCP(q, M)$, respectively. The goal of the numerical experiments is to investigate

- the impact of Δ on the enclosure of $\Sigma([J]_{\mathcal{D},\Delta}, h_\Delta(\hat{x}))$;
- the impact of Δ on $E_\Delta^{bnd}(\hat{x})$.

In order to demonstrate the impact of Δ, we report the ratio

$$\varrho_\epsilon := \frac{E_{\epsilon\Delta^*}^{lis}(\hat{x})}{\|\hat{x} - x^*\|_\infty}$$

for $\epsilon \in (0, 1]$.

TABLE 2 ϱ_ϵ, κ_ϵ, and $\tilde{\kappa}$ for Example 5.3

	$\varrho_\epsilon, \epsilon = 0.001$	$\varrho_\epsilon, \epsilon = 0.01$	$\varrho_\epsilon, \epsilon = 0.1$	$\varrho_\epsilon, \epsilon = 1$	$\kappa_\epsilon, \epsilon = 1$	$\tilde{\kappa}$
$n = 20$	2.1954e+01	3.6319e+00	2.9310e+00	1.5218e+00	3.1440e+01	1.2619e+03
$n = 50$	5.3897e+03	5.4831e+02	6.2486e+01	1.9011e+00	1.2389e+02	6.3177e+03
$n = 100$	1.1618e+03	1.1801e+02	1.4986e+01	3.1231e+00	7.3378e+02	5.1208e+04
$n = 200$	8.8857e+03	5.9513e+03	6.5251e+02	6.9192e+00	2.8929e+03	1.7705e+05
$n = 500$	6.5660e+03	2.6104e+03	3.0057e+02	8.6819e+00	1.5437e+04	1.0642e+06

In order to demonstrate the impact of Δ on $E^{bnd}_{\Delta,\infty}(\hat{x})$, we plot the logarithm of the ratio

$$\kappa_\epsilon := \frac{E^{bnd}_{\epsilon\Delta^*}(\hat{x})}{\|\hat{x} - x^*\|_\infty}$$

for $\epsilon \in (0, a]$ with $a \gg 1$. The data are plotted for Example 5.2 in Figure 1 and for Example 5.4 in Figure 2. Very similar numerical results are obtained for Examples 5.2 to 5.5.

The values of κ_ϵ for $\epsilon = 1$ are listed in the tables to compare the preciseness of $E^{lis}_{\epsilon\Delta^*}(\hat{x})$ and $E^{bnd}_{\epsilon\Delta^*}(\hat{x})$. We also list the value

$$\tilde{\kappa} := \frac{E^{bnd}_{I_n}(\hat{x})}{\|\hat{x} - x^*\|_\infty},$$

where $E^{bnd}_{I_n}(\hat{x})$ is the error bound (2.4) given by Chen et al. in [9], which is obtained from (4.7) by choosing $\Delta = I_n$ and $p = \infty$.

In reporting the numerical results, the notation "NaN" indicates that no meaningful result is returned by "verifylss.m."

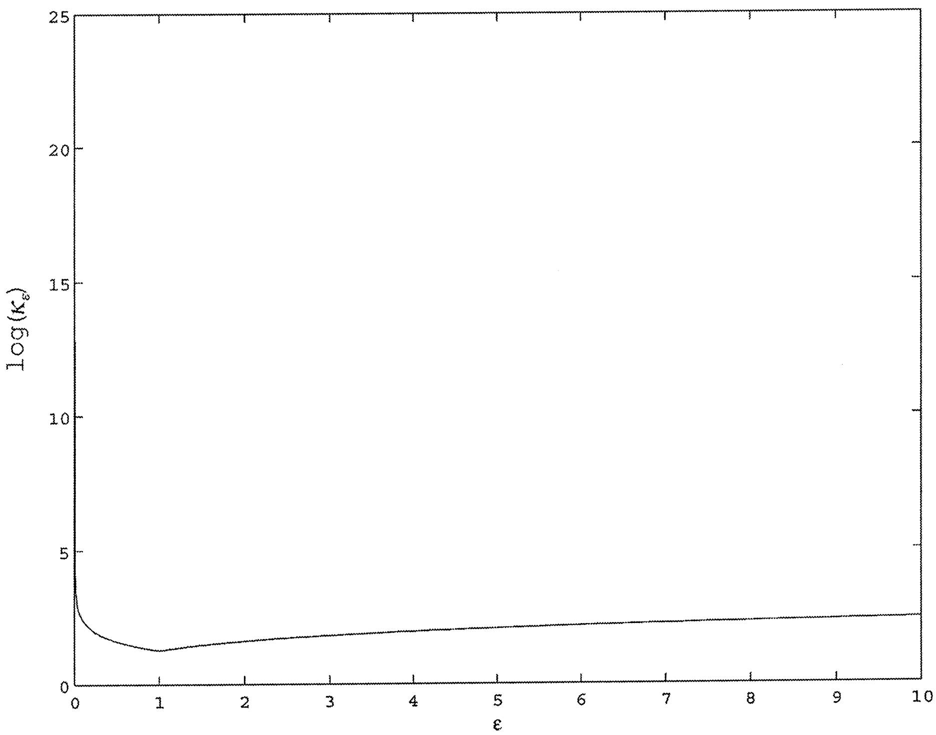

FIGURE 1 log(κ_ϵ) for Example 5.2.

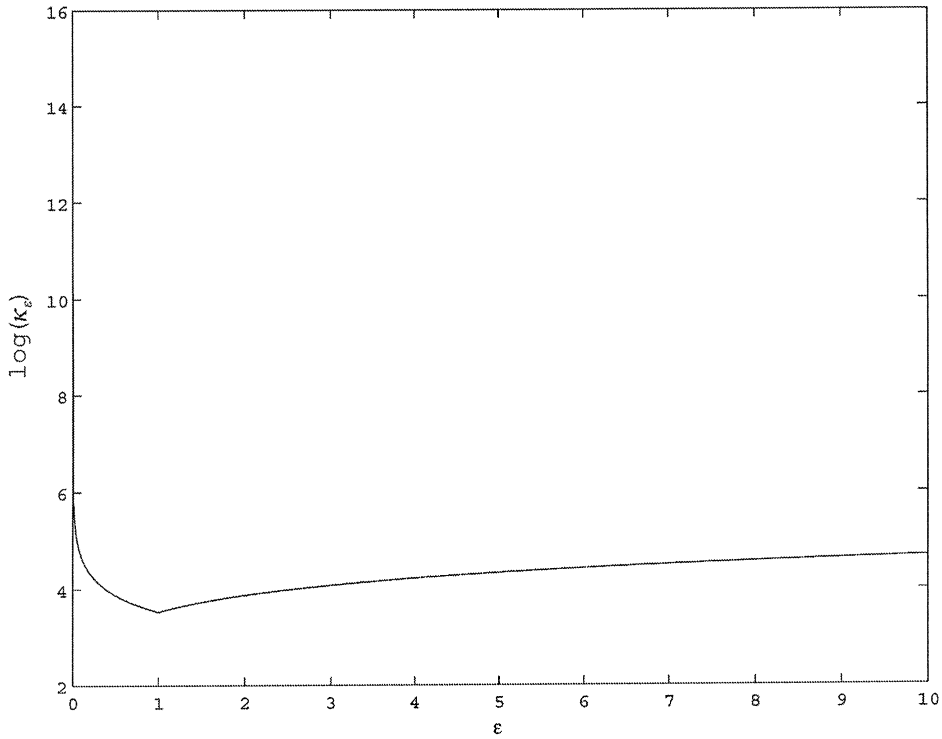

FIGURE 2 $\log(\kappa_\epsilon)$ for Example 5.4.

Example 5.1. An LCP with a P-Matrix. We consider an $LCP(q, M)$ with the data

$$M = \begin{pmatrix} 1 & -4 \\ 5 & 7 \end{pmatrix}, \quad q = \begin{pmatrix} -1 \\ 3 \end{pmatrix},$$

which has the unique solution $x^* = (1, 0)^T$. One can verify that $M \in \mathcal{P}_n$ but $M \notin \mathcal{H}_n^+$. This example was studied in [9]. We obtain for $\hat{x} = (1, 1)^T$

$$E_{I_2}^{bnd}(\hat{x}) = \max_{d \in [0,1]^2} \|(I - D + DM)^{-1}\|_\infty \|\min\{x, Mx + q\}\|_\infty = 20.$$

Considering now the solution $\Sigma([J]_{\mathbb{R}^n, \epsilon\Delta^*}, h_{\epsilon\Delta^*}(\hat{x}))$, where $0 < \epsilon \leq 1$,

$$[J]_{\mathbb{R}^n, \epsilon\Delta^*} = \begin{pmatrix} [\epsilon, 1] & [-4\epsilon, 0] \\ [0, \frac{5}{7}\epsilon] & [\epsilon, 1] \end{pmatrix}, \quad h_{\epsilon\Delta^*}(\hat{x}) = \begin{pmatrix} -4\epsilon \\ \min\{1, \frac{15}{7}\epsilon\} \end{pmatrix}.$$

It can be verified that $[J]_{\mathbb{R}^n, \epsilon\Delta^*}$ contains no singular matrices. For any $0 < \epsilon \leq 1$, we compute the enclosure of its solution set by

Cramer's rule:

$$[\hat{x} - x^*]_{\epsilon\Delta^*} = \begin{pmatrix} [-\frac{4}{\epsilon}, -4 + \frac{4}{\epsilon}\min\{1, \frac{15}{7}\epsilon\}] \\ [\frac{\epsilon\min\{7,15\epsilon\}}{7+20\epsilon^2}, \frac{20}{7} + \frac{1}{\epsilon^2}\min\{1, \frac{15}{7}\epsilon\}] \end{pmatrix}.$$

For $\epsilon \in (0, 1]$, it holds

$$[\hat{x} - x^*]_{\epsilon\Delta^*} \supseteq [\hat{x} - x^*]_{\Delta^*} = \begin{pmatrix} [-4, 0] \\ [\frac{7}{27}, \frac{27}{7}] \end{pmatrix}.$$

From this we get

$$E^{lis}_{\Delta^*}(\hat{x}) = 4 < 20 = E^{bnd}_{I_2}(\hat{x}).$$

By choosing Δ equal to the inverse of the diagonal of M, we get from (10) $\|\hat{x} - x^*\|_\infty = 4$. This is also obtained from (2.2) in [9] if applied to $r(x) = \min\{x, \Delta(Mx + q)\}$, as was pointed out by the referee.

Example 5.2. An LCP with an H-Matrix. Let $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ with

$$m_{ij} = \begin{cases} c, & j = i + 1, \\ b + \mu \sin\left(\dfrac{i}{n}\right), & j = i, \\ a, & j = i - 1, \\ 0, & \text{otherwise.} \end{cases}$$

The example was also studied in [9]. We generate the exact solution $x^* = (x_i^*)$ by setting

$$x_i^* := \max\{0, v_i - 0.5\} \times 10^{10(w_i - 0.5)}.$$

Then the column vector $q = (q_i)$ is generated in the following way:

$$q_i := \begin{cases} -(Mx^*)_i, & x_i^* > 0, \\ -(Mx^*)_i + \max\{0, \tilde{v}_i - 0.5\} \times 10^{10(\tilde{w}_i - 0.5)}, & x_i^* = 0. \end{cases}$$

Here w_i, v_i, $\tilde{w}_i$, and $\tilde{v}_i$ are random numbers in $[0, 1]$. We report the numerical results in Table 1 for the following choices of the parameters $\pi = (\mu, a, b, c)$:

$$\pi_1 = (0, -1, 2, -1), \quad \pi_2 = (n^{-2}, -1.5, 2, -0.5),$$
$$\pi_3 = (1, -1.5, 3, -1.5), \quad \pi_4 = (n^{-2}, -1.5, 2.2, -0.5).$$

For these choices, we even obtain M-matrices.

Example 5.3. An LCP with H-Matrix Arising from Journal Bearing Problem. The following problem arises in discretizing the free boundary problem for a journal bearing by a finite difference method [6]. Let $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ with

$$m_{ij} = \begin{cases} -h^3_{i+\frac{1}{2}}, & j = i+1, \\ h^3_{i-\frac{1}{2}} + h^3_{i+\frac{1}{2}}, & j = i, \\ -h^3_{i-\frac{1}{2}}, & j = i-1, \\ 0, & \text{otherwise}, \end{cases}$$

and let $q = (q_i)$ with

$$q_i = \mu\left(h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}}\right), \quad i = 1, 2, \ldots, n.$$

The details of computing μ and $h_{i-\frac{1}{2}}$ can be found in [11]. The numerical results for $\mu = 0.8$ are reported in Table 2 for the choice of $n = 20, 50, 100, 200, 500$.

Example 5.4. An ALCP Arising from Obstacle Bratu Problem. Let n be the square of a positive integer k, $c \in \mathbb{R}^n$ be constant, and let $\Phi(x) = (e^{x_1}, e^{x_2}, \ldots, e^{x_n})^T + c$,

$$M = \frac{1}{h^2}\begin{pmatrix} H & -I & & \\ -I & H & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & H \end{pmatrix}$$

where $h = \frac{1}{n+1}$,

$$H = \begin{pmatrix} 4 & -1 & & \\ -1 & 4 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{pmatrix} \in \mathcal{M}_{\sqrt{n}}.$$

Set $x^* = (0, 1, 0, 1, \ldots, 1)^T \in \mathbb{R}^n$ and choose $c = (c_i)^T \in \mathbb{R}^n$ as in [3]:

$$c_i = -\begin{cases} (Mx^*)_i + e^{x_i^*} & x_i^* > 0, \\ (Mx^*)_i + e^{x_i^*} - \xi_i & \text{otherwise}, \end{cases}$$

TABLE 3 ϱ_ϵ, κ_ϵ, and $\|r\|_\infty$ (See Ref. 4.1) for Example 5.4

	$\varrho_\epsilon, \epsilon = 0.001$	$\varrho_\epsilon, \epsilon = 0.01$	$\varrho_\epsilon, \epsilon = 0.1$	$\varrho_\epsilon, \epsilon = 1$	$\kappa_\epsilon, \epsilon = 1$	$\|r\|_\infty$
$n = 25$	1.2693e+03	1.2704e+02	1.2831e+01	1.4950e+00	4.2502e+00	8.9571e+00
$n = 49$	1.2310e+03	1.2497e+02	1.4253e+01	2.6267e+00	1.0962e+01	2.7047e+00
$n = 81$	2.8601e+03	2.8863e+02	3.1093e+01	4.7700e+00	2.0011e+01	3.1352e+00
$n = 225$	4.5441e+03	4.5613e+02	4.7048e+01	5.3182e+00	3.1561e+01	1.2153e+01
$n = 400$	NaN	NaN	NaN	NaN	3.8036e+01	1.1382e+01

where ξ_i is a random nonnegative number. The $ALCP(\Phi, M)$ models the obstacle Bratu problem [24] and was studied in [3]. The matrix M is an H-matrix with positive diagonal elements, and Φ is an isotone diagonal mapping. We treat the problem as an $ALCP(\Phi, M)$ with the enclosure computed by using Theorem 4.1 and report the numerical results in Table 3 for the different choices of the dimension $n = 5^2, 7^2, 9^2, 15^2, 20^2$.

Example 5.5. An $ALCP(\Phi, M)$ with Bounded Derivative. We study an NCP with all the data being randomly generated. Take $f(x) = D(x) + Mx + p$ with $M = A^T A + B$, where the elements of $A \in \mathbb{R}^{n\times n}$ are randomly generated in the interval $[-5, 5]$, and B is a skew symmetric matrix generated in a similar way. The vector $p \in \mathbb{R}^n$ is generated from a uniform distribution in the interval $[-500, 500]$. We take $D(x) = diag(a_0 + a_j \arctan(x_j))$ with a_j generated randomly in $[0, 1]$ and $a_0 > 0$ large enough such that $f'(x)$ is an H-matrix for all $x \in \mathbb{R}^n$. Then we have $\Phi(x) = D(x) + p$. Similar problems were studied in [18, 26]. Obviously, f has a bounded derivative. The numerical results are reported in Table 4 for the choices of the dimension $n = 10, 20, 50, 100, 200$.

6. CONCLUDING REMARKS

In the paper, we formulate the error estimation for $NCP(f)$ as enclosing the solution of a linear system of equations with its coefficient matrix contained in a known interval matrix. Based on this formulation, upper bounds of the error of an approximate solution $\hat{x}$ for $ALCP(\Phi, M)$

TABLE 4 ϱ_ϵ, κ_ϵ, and $\|r\|_\infty$ (See Ref. 4.1) for Example 5.5

	$\varrho_\epsilon, \epsilon = 0.001$	$\varrho_\epsilon, \epsilon = 0.01$	$\varrho_\epsilon, \epsilon = 0.1$	$\varrho_\epsilon, \epsilon = 1$	$\kappa_\epsilon, \epsilon = 1$	$\|r\|_\infty$
$n = 10$	3.8574e+02	6.6436e+01	9.6348e+00	1.2362e+00	4.6768e+00	9.5349e+00
$n = 20$	1.2310e+03	1.2497e+02	1.4253e+01	2.6267e+00	1.0513e+01	2.4407e+00
$n = 50$	NaN	NaN	NaN	1.6596e+00	1.5936e+01	1.9735e+01
$n = 100$	NaN	NaN	NaN	NaN	4.4943e+01	1.1382e+01
$n = 200$	NaN	NaN	NaN	NaN	7.8151e+01	4.8426e+01

and $LCP(q, M)$ are given. The following phenomena can be observed in the numerical experiments without exception.

- The error estimation obtained from the formulation of LIS is quite precise, in fact it is mostly of the same order of magnitude as that of the exact error when choosing $\Delta = \epsilon\Delta^*$ with $\epsilon \to 1^-$.
- When $\epsilon \ll 1$, then the interval matrix contains a matrix that is approximately singular. The estimation becomes bad. Numerical results show that the estimation delivered by "verifylss.m" becomes worse and worse as $\epsilon \to 0$ and cannot return meaningful results completely when ϵ is relatively close to 0.
- For both $ALCP(\Phi, M)$ and $LCP(q, M)$, with the choice of $\Delta = \epsilon\Delta^*$, the upper bounds (4.7) and (4.8) of the error obtain a minimum at $\epsilon = 1$ (i.e., with the choice $\Delta = \Delta^*$). They are always sharper than the bound with the choice $\Delta = I_n$. For Example 5.3, the bounds are sharper by two orders of magnitude. This phenomena is observed for all the cases in the numerical experiments, although the data is plotted just for Example 5.2 and for Example 5.4 (Figs. 1 and 2).
- Recently, Rohn developed a software (a MATLAB function "intervalhull.m" based on INTLAB) for computing the smallest interval vector containing the solution set of a linear system with interval data. This vector is usually called the interval hull of the solution set. This software can be downloaded at http://www.cs.cas.cz/rohn/matlab/index.html. The interval

$$[\hat{x} - x^*]_{\Delta^*} = \begin{pmatrix} [-4, 0] \\ [\frac{7}{27}, \frac{27}{7}] \end{pmatrix}$$

for Example 5.1 can be computed with this software. For the other examples, the preciseness of "intervalhull.m" is better compared with that of "verifylss.m." The difference is not obvious, especially for problems with large dimension.

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