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Enclosing Solutions of Singular Interval Systems Iteratively

Dedicated to Professor G. Maeß on the occasion of his 65th birthday.

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Abstract. Richardson splitting applied to a consistent system of linear equations $Cx = b$ with a singular matrix C yields to an iterative method $x^{k+1} = Ax^k + b$ where A has the eigenvalue one. It is known that each sequence of iterates is convergent to a vector $x^* = x^*(x^0)$ if and only if A is semi-convergent. In order to enclose such vectors we consider the corresponding interval iteration $[x]^{k+1} = [A][x]^k + [b]$ with $\rho(|[A]|) = 1$ where $|[A]|$ denotes the absolute value of the interval matrix $[A]$. If $|[A]|$ is irreducible we derive a necessary and sufficient criterion for the existence of a limit $[x]^* = [x]^*([x]^0)$ of each sequence of interval iterates. We describe the shape of $[x]^*$ and give a connection between the convergence of $([x]^k)$ and the convergence of the powers $[A]^k$ of $[A]$.

1. Introduction

Many practical problems finally lead to systems of linear equations

$$Cx = b, \qquad C \in \mathbb{R}^{n\times n}, \;\; b \in \mathbb{R}^n. \tag{1.1}$$

Mostly C is regular and therefore (1.1) is uniquely solvable. Sometimes, however, C is singular but the system is consistent, i.e., it is solvable. This situation occurs, e.g., when discretizing a Neumann problem, problems for elastic bodies with free surfaces or Poisson's equation with periodic boundary conditions. The stationary distribution vector of a finite homogeneous Markov chain grows out from a singular system as well as sometimes the production vector of a Leontief input-output economic model. Details can be found in [2], Chapters 7.6, 8.4, 9.4.

When solving linear systems of equations (1.1) the Richardson splitting $C = I - A$ (see the discussion in Sections 3.3 and 3.4 of [15]) leads to the equivalent fixed point form

$$x = Ax + b$$

which is the starting point for the iterative method

$$x^{k+1} = Ax^k + b, \qquad k = 0, 1, \ldots \tag{1.2}$$

For consistent systems (1.1) it is well-known (cf. [2, Lemma 6.13, p. 198], e.g.) that each sequence of such iterates converges to some solution $x^*(x^0)$ of (1.1) if and only if A is semi-convergent, i.e., if $A^\infty := \lim_{k\to\infty} A^k$ exists. (By the notation $x^*(x^0)$ we express the fact that the limit x^* may depend on the starting vector x^0.) This criterion is certainly fulfilled if the spectral radius $\rho(A)$ is less than one. In this case, A is called convergent, C is regular, $A^\infty = O$ and each sequence of (1.2) has the same limit which is the unique solution of (1.1). The remaining case which yields to a convergent sequence (A^k) requires $\rho(A) = 1$ with some additional conditions (cf. Theorem 3.1). In this case, C turns out to be singular, $A^\infty \neq O$, the limit of (x^k) exists, but depends on the starting vector x^0. It is one of the infinitely many solutions of the system (1.1) which we assumed to be consistent. It is this singular situation on which we focus in the present paper. To this end we start with the interval iteration

$$[x]^{k+1} = [A][x]^k + [b], \qquad k = 0, 1, \ldots, \tag{1.3}$$

where the $n \times n$ interval matrix $[A]$ and the corresponding interval vector $[b]$ can be thought to be enclosures of a given matrix $A \in \mathbb{R}^{n\times n}$ and a given vector $b \in \mathbb{R}^n$ or they are used to consider the variety of linear systems

$$(I - A)x = b, \qquad A \in [A], \ \ b \in [b] \tag{1.4}$$

simultaneously. In Section 2 we define the absolute value $|[A]| \in \mathbb{R}^{n\times n}$ of $[A]$. If $\rho(|[A]|) < 1$ each matrix $C = I - A$ in (1.4) is regular, and it is known by O. Mayer's paper [10] that each interval sequence $([x]^k)$ from (1.3) converges to the same interval vector $[x]^*$ which contains all solutions of (1.4). If $\rho(|[A]|) \geq 1$ there may be singular matrices in $[A]$, and nothing is known on the convergence of $([x]^k)$ up to now. We will address to this problem in Section 4 for the case that $|[A]|$ is irreducible extending the result of O. Mayer.

By the continuity of the interval arithmetic it is immediately clear that the limits $[x]^*$ of the convergent sequences $([x]^k)$ are the algebraic solutions of the interval system

$$[x] = [A][x] + [b]. \tag{1.5}$$

These solutions were completely studied in [9] if $|[A]|$ is irreducible. A necessary and sufficient criterion was derived there which guarantees the existence of such vectors. In addition, their shape was given. Unfortunately, simple examples already show that $([x]^k)$ does not need to converge if such a solution exists, even if $|[A]|$ is restricted to be semi-convergent. In Section 4 we show that particular classes of matrices $[A]$ with semi-convergent absolute value have to be excluded. With Theorem 4.1 we will prove a necessary and sufficient criterion for the convergence of every sequence $([x]^k)$ of (1.3). It implies $[b] \equiv b \in \mathbb{R}^n$ in the case $\rho(|[A]|) = 1$ which certainly means a restriction when considering linear systems with inexact input data. The criterion in Theorem 4.1 is nearly the same as it was stated in [6] for the convergence of the powers $[A]^k$ of interval matrices to a non-zero matrix $[A]^\infty$. A certain relation between these two problems is studied at the end of Section 4.

If our criterion of convergence is fulfilled the limit $[x]^* = [x]^*([x]^0)$ of $([x]^k)$ contains all solutions of linear systems (1.4) which are limits of (1.2) with $x^0 \in [x]^0$. (The notation $[x]^*([x]^0)$ expresses again the fact that the limit $[x]^*$ may depend on the starting vector $[x]^0$.) The element relation is a simple consequence of the inclusion isotonicity of interval arithmetic (cf., e.g., [1] or [11]). In this respect $[x]^*$ is an enclosure of the—and in the singular case: of selected—solutions of the linear systems (1.4).

In passing we note that iterative methods even for rectangular systems have already been discussed very intensively in [4].

We have organized our paper as follows: Section 2 contains the notation used throughout the paper, Section 3 presents auxiliary and known results in order to understand better the statements and conclusions of the main part of this paper contained in Section 4.

2. Notations

By $I(\mathbb{R}), I(\mathbb{R}^n), I(\mathbb{R}^{n\times n})$ we denote the set of intervals, the set of interval vectors with n components and the set of $n \times n$ interval matrices, respectively. By "interval" we always mean a real compact interval. We write interval quantities in brackets with the exception of point quantities (i.e., degenerate interval quantities) which we identify with the element which they contain. Examples are the null matrix O and the identity matrix I. We use the notation $[A] = [\underline{A}, \overline{A}] = ([a]_{ij}) = ([\underline{a}_{ij}, \overline{a}_{ij}]) \in I(\mathbb{R}^{n\times n})$ simultaneously without further reference, and we proceed similarly for the elements of $\mathbb{R}^n, \mathbb{R}^{n\times n}, I(\mathbb{R})$ and $I(\mathbb{R}^n)$. We call $[a] \in I(\mathbb{R})$ symmetric if $[a] = -[a]$, i.e., if $[a] = [-r, r]$ with some real number $r \geq 0$. For intervals $[a]$, $[b]$ we introduce the midpoint $\check{a} := (\underline{a} + \overline{a}) / 2$, the absolute value $|[a]| := \max\{|\underline{a}|, |\overline{a}|\}$, the radius $\mathrm{rad}([a]) := (\overline{a} - \underline{a})/2$ and the (Hausdorff) distance $q([a], [b]) := \max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\}$. For interval vectors and interval matrices these quantities are defined entrywise, for instance $|[A]| := (|[a]_{ij}|) \in \mathbb{R}^{n\times n}$. We assume some familiarity when working with these definitions and when applying the interval arithmetic

$$[a] \circ [b] := \{a \circ b \mid a \in [a],\ b \in [b]\} \in I(\mathbb{R}),$$
$$[a], [b] \in I(\mathbb{R}), \quad \circ \in \{+, -, \cdot, /\}, \quad 0 \notin [b] \text{ in case of "/"}.$$

Note that $[a] \circ [b]$ can be expressed by means of the bounds $\underline{a}, \overline{a}, \underline{b}, \overline{b}$ of the operands $[a]$ and $[b]$. For details see, e.g., the introductory chapters of [1] or [11].

For intervals $[a], [b], [c], [d]$ we mention the basic relations

$$\begin{aligned}
\mathrm{rad}([a] \pm [b]) &= \mathrm{rad}([a]) + \mathrm{rad}([b]),\\
\mathrm{rad}([a][b]) &\geq |[a]|\mathrm{rad}([b]),\\
q([a]+[c],[b]+[c]) &= q([a],[b]),\\
q([a]+[c],[b]+[d]) &\leq q([a],[b]) + q([c],[d]),\\
q([c][a],[c][b]) &\leq |[c]|q([a],[b]).
\end{aligned}$$

These relations yield at once to similar relations with vectors and matrices.

Since the multiplication between interval matrices is not associative (cf. [1, p. 1242], e.g.) we must explain what we mean by the k-th power of an interval matrix. Following [5] and [6] we define

$$[A]^0 := I, \qquad [A]^{k+1} := [A]^k \cdot [A], \qquad k = 0, 1, \ldots$$

and

$$^0[A] := I, \qquad ^{k+1}[A] := [A] \cdot {}^k[A], \qquad k = 0, 1, \ldots$$

It is shown in [5] that $[A]^3$ can differ from $^3[A]$. If $\lim_{k\to\infty} [A]^k$ exists (with respect to the Hausdorff distance q) then we write $[A]^\infty$ for this limit, and A^∞ if $[A] \equiv A \in \mathbb{R}^{n\times n}$.

As usual we call the matrix $A \in \mathbb{R}^{n\times n}$ non-negative if $a_{ij} \geq 0$ for $i, j = 1, \ldots, n$, writing $A \geq O$ in this case. By $A > O$ we denote non-negative matrices whose entries all are positive. We call them positive. For $A, B \in \mathbb{R}^{n\times n}$ the inequality $A \leq B$ means $B - A \geq O$, and $A \geq B$ is equivalent to $B \leq A$. For vectors we apply these definitions analogously.

According to the Theorem of Perron and Frobenius for irreducible non-negative matrices A the spectral radius $\rho(A)$ is a simple eigenvalue of A, and there are two positive eigenvectors v, w such that

$$Av = \rho(A)v, \qquad w^T A = \rho(A)w^T, \;\; w^T v = 1 \tag{2.1}$$

hold (see [14], e.g.). We call such vectors (right and left, respectively) Perron vectors of A. In our paper we will use v, w exclusively for such vectors. Note that we do not require the normalization $\sum_{i=1}^{n} v_i = 1$ or $\sum_{i=1}^{n} w_i = 1$ as was done in [3, p. 497] in order to make Perron vectors unique. In the sequel we denote by span$\{w\}$ the linear space spanned by w, and by $(\mathrm{span}\{w\})^\perp$ its orthogonal complement.

In matrix theory one often divides non-negative irreducible matrices A into two classes according to the number h of eigenvalues λ_j, $j = 0, \ldots, h-1$, with $|\lambda| = \rho(A)$: The elements of the first class are called primitive matrices. They are defined by $A \geq O$, A irreducible, $h = 1$. Here the theory of Perron and Frobenius yields to $\lambda_0 = \lambda_{h-1} = \rho(A)$ which is a simple eigenvalue of A. The elements of the second class are called cyclic matrices. They are defined by $A \geq O$, A irreducible, $h > 1$. Here the theory guarantees $\lambda_j = \rho(A)\, e^{\frac{j}{h} \cdot 2\pi i}$, $j = 0, 1, \ldots, h-1$ where these eigenvalues are again simple eigenvalues. The number $h > 1$ is called the index of the cyclic matrix A. Cyclic matrices A of index h can be brought into the so-called cyclic normal form

$$PAP^T = \begin{pmatrix} O & A_{12} & O & O & \dots & O \\ O & O & A_{23} & O & \dots & O \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ O & O & \dots & O & A_{h-2,h-1} & O \\ O & O & \dots & O & O & A_{h-1,h} \\ A_{h,1} & O & \dots & O & O & O \end{pmatrix} \tag{2.2}$$

by means of some appropriate permutation matrix P. This shows that $A^k > O$ can never occur for cyclic matrices in contrast to primitive ones for which there is a smallest integer k_0 (the so-called primitivity index) such that $A^k > O$ holds for all $k \geq k_0 = k_0(A)$. (See [2] or [14], e.g.)

Let $A \in \mathbb{R}^{n\times n}$, and let k_0 be the smallest integer such that $\text{rank}(A^{k+1}) = \text{rank}(A^k)$. This integer k_0 is called the index of A. It should not be confused with the index of a cyclic matrix. As in [2, Definition 4.10, p. 118], we define the Drazin inverse $A^D \in \mathbb{R}^{n\times n}$ of a matrix A of index k_0 as that generalized inverse X which satisfies the three conditions

$$XAX = X, \qquad AX = XA, \qquad A^{k_0} = XA^{k_0+1}.$$

Since the concept of Drazin inverse is not so widespread we shortly recall some of its basic properties. One can show that $\mathbb{R}^n$ is the direct sum of the column space $R(A^{k_0})$ of A^{k_0} and the null space $N(A^{k_0})$ of this matrix, i.e., $\mathbb{R}^n = R(A^{k_0}) \oplus N(A^{k_0})$. Since A^DA is the projector on $R(A^{k_0})$ along $N(A^{k_0})$ (cf. [2, p. 118]) the Drazin inverse is the unique matrix given by

$$A^Dx = \begin{cases} y_1 & \text{if } Ay = x, \quad x, y_1 \in R(A^{k_0}), \quad y - y_1 \in N(A^{k_0}) \\ 0 & \text{if } A^{k_0}x = 0, \end{cases} \tag{2.3}$$

(cf. [2, p. 197], with obvious corrections). Note that in the case $x \in R(A^{k_0})$ one can always choose the solution y in (2.3) such that $y \in R(A^{k_0})$, i.e., $y = y_1$. Let the Jordan canonical form J of $A = SJS^{-1}$ be represented by

$$J := \begin{pmatrix} \hat{J}_0 & O \\ O & \hat{J}_r \end{pmatrix},$$

where $\hat{J}_0$ and $\hat{J}_r$, respectively, are square block diagonal matrices whose diagonal blocks are just the singular Jordan blocks of J, and the non-singular ones, respectively. If A is non-singular then $\hat{J}_0$ is missing; if A has zero as the only eigenvalue then $\hat{J}_r$ is missing. The index k_0 of A is given by the number of columns of the largest Jordan block belonging to $\hat{J}_0$, and the Drazin inverse A^D of A can be expressed as

$$A^D = S\begin{pmatrix} O & O \\ O & (\hat{J}_r)^{-1} \end{pmatrix} S^{-1}. \tag{2.4}$$

Hence $A^D = A^{-1}$ if A is non-singular.

By $D = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^{n\times n}$ we denote the diagonal matrix whose diagonal entries are $\sigma_1, \ldots, \sigma_n$. If $|D| = I$ holds then we call $D \in \mathbb{R}^{n\times n}$ a signature matrix. Note that $D^{-1} = D$ for such matrices. We finally mention the vector $e = (1, \ldots, 1)^T \in \mathbb{R}^n$.

3. Known Results

In this section we cite some known results which are necessary for the understanding of Section 4. We start with some classical results on semi-convergent matrices A (cf. [2], [3]).

THEOREM 3.1 (Cf. [2], p. 152). *The matrix $A \in \mathbb{R}^{n\times n}$ is semi-convergent if and only if the following conditions hold:*

(i) $\rho(A) \leq 1$.

(ii) If $\rho(A) = 1$ and if λ is an eigenvalue of A with $|\lambda| = 1$ then $\lambda = 1$ and every Jordan block associated with $\lambda = 1$ is of size 1×1.

THEOREM 3.2 (Cf. [3], 8.2.11, p. 500 or [6], Lemma 3). *Let $A \in \mathbb{R}^{n\times n}$ be semi-convergent. Then $A^\infty = O$ if and only if $\rho(A) < 1$. If A is, in addition, irreducible and nonnegative with $\rho(A) = 1$ then*

$$A^\infty = vw^T, \qquad v, w \text{ as in } (2.1). \tag{3.1}$$

THEOREM 3.3 (Cf. [2], Lemma 6.13, p. 198, with obvious corrections). *Let (1.1) be consistent. Then each sequence (x^k) of iterates defined by (1.2) is convergent if and only if A is semi-convergent. The limit is independent of x^0 if and only if $\rho(A) < 1$. In any case this limit x^* is a solution of (1.1). By means of Drazin inverses it can be expressed as*

$$x^* = (I - A)^D b + \{I - (I - A)(I - A)^D\}x^0. \tag{3.2}$$

Now we restate O. Mayer's result mentioned in Section 1.

THEOREM 3.4 (Cf. [10] or [1], pp. 143 ff). *For every starting vector $[x]^0 \in \mathbf{I}(\mathbb{R}^n)$ the sequence $([x]^k)$ of iterates defined by (1.3) is convergent to the same vector $[x]^* \in \mathbf{I}(\mathbb{R}^n)$ if and only if $\rho(|[A]|) < 1$. In this case $[x]^*$ contains the solution set*

$$S := \{x \in \mathbb{R}^n \mid (I - A)x = b,\ A \in [A],\ b \in [b]\} \tag{3.3}$$

and is the unique solution of (1.5).

If $\rho(|[A]|) \geq 1$ things change. This can be seen from the following simple example.

EXAMPLE 3.1. Consider the iteration

$$[x]^{k+1} = [a][x]^k, \qquad k = 0, 1, \ldots,$$

where $[a] \in \mathbf{I}(\mathbb{R})$, i.e., $[A] = [a]$ in (1.3). Note that if $[a] \neq 0$ then $|[a]| \neq 0$, hence the 1×1 matrix $|[a]|$ is irreducible by definition.

(a) If $|[a]| > 1$ each sequence of iterates is divergent for $[x]^0 \neq 0$. This can be seen from the iterates $x^{k+1} = \tilde{a}x^k$, $k = 0, 1, \ldots$, with $\tilde{a} \in [a]$, $|\tilde{a}| > 1$, $0 \neq x^0 \in [x]^0$. We obtain $x^k \in [x]^k$ and $\lim_{k\to\infty} |[x]^k| \geq \lim_{k\to\infty} |x^k| = \infty$.

(b) If $[a] = [0, 1]$ we get

$$[x]^* = [x]^k = \left\{ \begin{array}{ll} [x]^0, & \text{if } 0 \in [x]^0 \\ [0, \overline{x}^0], & \text{if } 0 \leq \underline{x}^0 \\ [\underline{x}^0, 0], & \text{if } \overline{x}^0 \leq 0 \end{array} \right\}, \qquad k = 1, 2, \ldots,$$

i.e., we obtain convergence to $[x]^* = [x]^*([x]^0)$ for each sequence of iterates.

(c) If $[a] = [-1, 0]$ and $[x]^0 = [-1, 0]$ we see at once that $[x]^{2k+1} = [0, 1]$ and $[x]^{2k} = [-1, 0]$ hold whence $([x]^k)$ cannot be convergent in this case. If one starts with $[x]^0 = [-v, v]$, $v \geq 0$, then $[x]^* = [x]^k = [x]^0$, $k = 0, 1, \ldots$, since now the starting vectors are exactly the solutions of the equation $[x] = [a][x]$.

(d) If $[a] = [-1, 1]$ we get $[x]^* = [x]^k = |[x]^0| \cdot [-1, 1]$, $k = 1, 2, \ldots$, i.e., convergence to $[x]^* = [x]^*([x]^0)$ is guaranteed for each sequence of iterates. Note that the iterates of (c) are contained in the corresponding iterates of (d) if one starts in both cases with the same interval $[x]^0$. Thus the iterates in (c) are bounded but, as we already saw, they are not necessarily convergent. The bounds depend on $[x]^0$.

It will turn out in Section 4 that Example 3.1 is typical for the situation in the case $\rho(|[A]|) \geq 1$. Theorem 3.4 lets expect that Theorem 3.3 remains true if one replaces $\rho(A)$ by $\rho(|[A]|)$ when dealing with (1.3) instead of (1.2). Example 3.1(c) shows that this is not true. Note that in (1.3) the starting vector $[x]^0$ is allowed to be an interval vector. This initiates the transition from $\rho(A)$ to $\rho(|[A]|)$ even if $[A]$, $[b]$ are degenerate.

By the continuity of the interval arithmetic the limit of each convergent sequence $([x]^k)$ of iterates of (1.3) is a solution of (1.5). Therefore it is natural to study all solutions of (1.5) first. For irreducible absolute values $|[A]|$ a complete characterization was given in [9]. In view of Section 4 we repeat the main result.

THEOREM 3.5. *Let $|[A]|$ be irreducible with $\rho(|[A]|) = 1$, choose any Perron vector $v > 0$ of $|[A]|$. Denote by M_{sym} the set of all indices for which the columns of $[A]$ contain at least one non-degenerate symmetric entry. Construct $[B] \in \mathbf{I}(\mathbb{R}^{n\times n})$ from $[A]$ by replacing the j-th column of $[A]$ by the j-th column of the identity matrix I for all $j \in M_{\text{sym}}$ and let $\overset{\circ}{A} \in [B]$ be the unique matrix which satisfies $|\overset{\circ}{A}| = |[B]|$.*

(a) *The interval system (1.5) has a solution if and only if* $[b]$ *is degenerate, i.e.,* $[b] \equiv b \in \mathbb{R}^n$*, and*

$$x = \overset{\circ}{A}x + b \tag{3.4}$$

is solvable. In this case, there is at least one solution $\check{z}$ *of (3.4) which satisfies*

$$\check{z}_i = 0 \qquad \text{for all } i \in M_{\text{sym}}. \tag{3.5}$$

(b) *If* $[b]$ *is degenerate, i.e.,* $[b] \equiv b \in \mathbb{R}^n$*, then for any solution* $\check{z}$ *of (3.4) satisfying (3.5) and for any real number* $t \geq t_{\min}$ *with*

$$t_{\min} := \max \left\{ 0, \frac{\text{rad}([a]_{ij})}{|\check{a}_{ij}|} \cdot \frac{|\check{z}_j|}{v_j}, \frac{|\check{z}_j|}{v_j} \;\middle|\; 1 \leq i, j \leq n, \check{a}_{ij} \neq 0, \text{rad}([a]_{ij}) \neq 0 \right\} \tag{3.6}$$

the interval vector $[z]_t^* := \check{z} + tv[-1, 1]$ *is a solution of (1.5).*

Conversely, if $[z]^*$ *is any solution of (1.5) then* $[b]$ *is degenerate, i.e.,* $[b] \equiv b \in \mathbb{R}^n$*, and* $[z]^*$ *can be written in the form* $[z]^* = \check{z}^* + tv[-1, 1]$ *where* $\check{z}^*$ *solves (3.4), (3.5) and* t *satisfies (3.6) with* $\check{z} := \check{z}^*$.

(c) *If* $M_{\text{sym}} \neq \emptyset$*, i.e., if there are at least two different matrices* $\dot{A}, \ddot{A} \in [A]$ *with* $|\dot{A}| = |\ddot{A}| = |[A]|$*, then (3.4) has at most one solution which satisfies (3.5).*

(d) *If* $M_{\text{sym}} = \emptyset$*, i.e., if there is exactly one matrix* $\dot{A} \in [A]$ *with* $|\dot{A}| = |[A]|$*, then* $\dot{A} = \overset{\circ}{A}$*, (3.5) is trivially true and one of the following mutually excluding cases occurs:*

(i) $\rho(\dot{A}) < 1$*, whence (3.4) has a unique solution.*

(ii) $\rho(\dot{A}) = 1$ *and* $\dot{A} \neq D|[A]|D$ *for every signature matrix* D*, whence (3.4) has a unique solution.*

(iii) $\rho(\dot{A}) = 1$ *and* $\dot{A} = D|[A]|D$ *for some signature matrix* D*. Here, (3.4) has no solution if and only if* b *is not in the range of* $I - \dot{A}$*, i.e., if and only if* b *cannot be represented as linear combination of the column vectors of* $I - \dot{A}$*. Otherwise it has infinitely many solutions. They are given by*

$$\check{z} = \check{z}^* + sDv, \tag{3.7}$$

where $\check{z}^*$ *is any fixed particular solution of (3.4) and* s *is any real number.*

(e) *If (1.5) has a solution* $[z]^*$ *then for any linear system (1.4) there is at least one solution which is contained in* $[z]^*$*. In particular, each such system is consistent.*

The following result was proved in [5] (case $\rho(|[A]|) < 1$, see also [12]) and [6] (case $\rho(|[A]|) = 1$). Its crucial assumptions are the same as in our main result in Section 4.

THEOREM 3.6. *Let* $|[A]|$ *be irreducible with* $\text{rad}([A]) \neq O$*. Then the powers* $[A]^k$ *are convergent to a matrix* $[A]^\infty$ *if and only if the following two conditions hold:*

(i) *The matrix* $|[A]|$ *is semi-convergent.*

(ii) *If* $\rho(|[A]|) = 1$ *and if* $[A]$ *contains only one matrix* $\dot{A}$ *with* $|\dot{A}| = |[A]|$ *then* $\dot{A} \neq -D|[A]|D$ *for any signature matrix D.*

We end this section with an auxiliary result which can also be found in [6].

LEMMA 3.1. *Let* $[A] \in \mathrm{I}(\mathbb{R}^{n\times n})$ *and let* $D \in \mathbb{R}^{n\times n}$ *be a regular diagonal matrix or a permutation matrix. Then* $D[A]^k D^{-1} = (D[A]D^{-1})^k$.

4. New Results

We start our results with two lemmas which will be needed in the proof of our main result, Theorem 4.1.

LEMMA 4.1. *Let* $([x]^k)$ *be a sequence of iterates defined by (1.3) with limit* $[x]^*$.

(a) *If* $[y]^*$ *is a solution of (1.5) with* $[y]^* \subseteq [x]^0$ *then* $[y]^* \subseteq [x]^*$. *If one replaces "*$\subseteq$*" by "*$\supseteq$*" one gets an analogous result.*

(b) *If* $([y]^k)$ *is another sequence of iterates defined by (1.3) and if* $[y]^0 \subseteq [x]^0$ *then* $[y]^k \subseteq [x]^k$ *for* $k = 0, 1, \ldots$ *If, in addition,* $[y]^* := \lim_{k\to\infty} [y]^k$ *exists then* $[y]^* \subseteq [x]^*$. *If one replaces "*$\subseteq$*" by "*$\supseteq$*" one gets an analogous result.*

(c) *If* $x^0 \in [x]^0$ *and if* $x^* = x^*(x^0)$ *is the limit of* $x^{k+1} = Ax^k + b$ *for some fixed* $A \in [A]$, $b \in [b]$, *then* $x^* \in [x]^*$.

Proof. The proof of this lemma is immediate using the inclusion monotonicity and the continuity of the interval arithmetic. □

LEMMA 4.2. *Let* $|[A]|$ *be irreducible and semi-convergent with* $\rho(|[A]|) = 1$, *let* $v > 0$ *be a fixed Perron vector of* $|[A]|$ *and let (1.5) have a solution* $[z]^*$ *which, according to Theorem 3.5, can be represented as* $[z]^* = \check{z}^* + t^* v[-1, 1]$, $0 \leq t^* \in \mathbb{R}$. *Then* $[b] \equiv b \in \mathbb{R}^n$, *and for any sequence* $([x]^k)$ *of (1.3) the following assertions hold.*

(a) *There exists a real number* $\alpha = \alpha([x]^0) \geq 0$ *with* $\lim_{k\to\infty} \mathrm{rad}([x]^k) = \alpha v$.

(b) *There exists a real number* $\beta = \beta([x]^0) \geq 0$ *with* $\lim_{k\to\infty} q([x]^k, [z]^*) = \beta v$.

(c) *There exists a real number* $\gamma = \gamma([x]^0) \geq 0$ *with* $\lim_{k\to\infty} |\check{x}^k - \check{z}^*| = \gamma v$.

(d) *There is a convergent subsequence of* $([x]^k)$. *If* $[y]^*$ *is its limit then*

$$\check{y}^* = \check{z}^* + \gamma D v, \qquad D \in \mathbb{R}^{n\times n},\ |D| = I,$$

with γ *as in (c), independent of the particular convergent subsequence.*

Proof. Let w be a left Perron vector of $|[A]|$ satisfying $w^T v = 1$ and let $\{w, w^2, \ldots, w^n\}$ be an orthogonal basis of $\mathbb{R}^n$. Then $(\mathrm{span}\{w\})^\perp$ is the space spanned by $\{w^2, \ldots, w^n\}$ and, since $w^T v = 1$, the set $\{v, w^2, \ldots, w^n\}$ is also a basis of $\mathbb{R}^n$. By virtue of the existence of $[z]^*$ Theorem 3.5 implies $[b] \equiv b \in \mathbb{R}^n$.

(a) Define $\alpha_k \in \mathbb{R}$ and $s^k \in \mathbb{R}^n$ by $\mathrm{rad}([x]^k) = \alpha_k v + s^k$ with $s^k \in (\mathrm{span}\{w\})^\perp$. Using

$$\mathrm{rad}([x]^{k+1}) = \mathrm{rad}([A][x]^k + b) = \mathrm{rad}([A][x]^k) \geq |[A]|\mathrm{rad}([x]^k) \tag{4.1}$$

results in

$$\alpha_{k+1} v + s^{k+1} \geq \alpha_k v + |[A]|s^k.$$

Multiplying from the left by w^T yields to

$$\alpha_{k+1} \geq \alpha_k.$$

Since we assumed $[z]^*$ to be a solution of (1.5) its midpoint $\check{z}^*$ satisfies (3.4), (3.5). Therefore, according to Theorem 3.5, the vectors $[z]^*_t := \check{z}^* + tv[-1, 1]$ are also solutions of (1.5) for all sufficiently large $t \geq 0$. Choose t so large that $[z]^*_t \supseteq [x]^0$ holds. By Lemma 4.1(b) with $[y]^0 := [z]^*_t$ and "$\supseteq$" instead of "$\subseteq$" we have

$$[x]^k \subseteq [z]^*_t, \qquad k = 0, 1, \ldots$$

whence

$$\alpha_k v + s^k = \mathrm{rad}([x]^k) \leq \mathrm{rad}([z]^*_t) = tv.$$

Multiplying by w^T as above results in

$$\alpha_k \leq t, \qquad k = 0, 1, \ldots$$

Hence (α_k) is a monotonously increasing sequence which is bounded from above and which therefore has a limit α. Now we show

$$\lim_{k \to \infty} s^k = 0.$$

We have

$$\begin{aligned} tv &\geq \alpha_{k+m} v + s^{k+m} = \mathrm{rad}([x]^{k+m}) \geq |[A]|^k \mathrm{rad}([x]^m) \\ &= \alpha_m v + |[A]|^k s^m. \end{aligned} \tag{4.2}$$

Let k tend to infinity in (4.2). Then it turns out that the components s_i^k are bounded so that

$$\bar{s} := \left(\limsup_{k \to \infty} s_i^k \right), \qquad \underline{s} := \left(\liminf_{k \to \infty} s_i^k \right)$$

are vectors in $\mathbb{R}^n$. Moreover,

$$tv \geq \alpha v + \bar{s} \geq \alpha v + \underline{s} \geq \alpha_m v + |[A]|^\infty s^m = \alpha_m v + v w^T s^m = \alpha_m v,$$

holds, where we used (3.1). With $m \to \infty$ we finally get

$$tv \geq \alpha v + \bar{s} \geq \alpha v + \underline{s} \geq \alpha v,$$

hence

$$(t - \alpha)v \geq \bar{s} \geq \underline{s} \geq 0.$$

Assume $\overline{s}_{i_0} > 0$ for some index i_0. Then there is a subsequence (s^{k_j}) such that $\lim_{j\to\infty} s_{i_0}^{k_j} = \overline{s}_{i_0}$. By virtue of $\underline{s} \geq 0$, $\overline{s}_{i_0} > 0$ and $w > 0$ we obtain the contradiction

$$0 = w^T s^{k_j} = w_{i_0} s_{i_0}^{k_j} + \sum_{\substack{l=1 \\ l \neq i_0}}^{n} w_l s_l^{k_j} \geq w_{i_0} s_{i_0}^{k_j} - \frac{1}{2} w_{i_0} s_{i_0}^{k_j} > 0$$

for sufficiently large k_j. Therefore, $0 \geq \overline{s} \geq \underline{s} \geq 0$, i.e., $\overline{s} = \underline{s} = 0$ holds, whence $\lim_{k\to\infty} s^k = 0$ follows. This proves (a).

(b) Define $\beta_k \in \mathbb{R}$ and $t^k \in \mathbb{R}^n$ by $q([x]^k, [z]^*) = \beta_k v + t^k$ with $t^k \in (\operatorname{span}\{w\})^\perp$. Since $[z]^* = [A][z]^* + b$ we obtain analogously to (a)

$$\begin{aligned} 0 &\leq q([x]^{k+1}, [z]^*) \leq |[A]| q([x]^k, [z]^*), \\ 0 &\leq \beta_{k+1} v + t^{k+1} \leq \beta_k v + |[A]| t^k, \\ 0 &\leq \beta_{k+1} \leq \beta_k. \end{aligned}$$

Therefore, (β_k) has a limit β. We show

$$\lim_{k\to\infty} t^k = 0.$$

The steps are analogous to those in (a). We start with

$$\begin{aligned} 0 &\leq \beta_{k+m} v + t^{k+m} = q([x]^{k+m}, [z]^*) \\ &\leq |[A]|^k q([x]^m, [z]^*) = \beta_m v + |[A]|^k t^m. \end{aligned} \tag{4.3}$$

Let k tend to infinity in (4.3) and define the vectors

$$\overline{t} := \left(\limsup_{k\to\infty} t_i^k \right), \qquad \underline{t} := \left(\liminf_{k\to\infty} t_i^k \right).$$

Then

$$0 \leq \beta v + \underline{t} \leq \beta v + \overline{t} \leq \beta_m v + |[A]|^\infty t^m = \beta_m v + v w^T t^m = \beta_m v,$$

and $m \to \infty$ leads to

$$-\beta v \leq \underline{t} \leq \overline{t} \leq 0.$$

Assume $\underline{t}_{i_0} < 0$ for some index i_0 and consider a subsequence (t^{k_j}) such that $\lim_{j\to\infty} t_{i_0}^{k_j} = \underline{t}_{i_0}$. Then we obtain the contradiction

$$0 = w^T t^{k_j} = w_{i_0} t_{i_0}^{k_j} + \sum_{\substack{l=1 \\ l \neq i_0}}^{n} w_l t_l^{k_j} \leq w_{i_0} t_{i_0}^{k_j} + \frac{1}{2} w_{i_0} |t_{i_0}^{k_j}| < 0$$

for sufficiently large k_j, and (b) follows analogously to (a).

(c) From the representation

$$q([a], [b]) = |\check{a} - \check{b}| + |\operatorname{rad}([a]) - \operatorname{rad}([b])|, \qquad [a], [b] \in I(\mathbb{R})$$

(cf. [11, 1.7.1, p. 25]) we get at once

$$q([x]^k, [z]^*) = |\check{x}^k - \check{z}^*| + |\mathrm{rad}([x]^k) - \mathrm{rad}([z]^*)|$$

which implies

$$\lim_{k\to\infty} |\check{x}^k - \check{z}^*| = \gamma v \geq 0$$

using (*a*), (*b*) and $\gamma := \beta - |\alpha - t^*|$. In particular, $\gamma \geq 0$, whence $\beta \geq |\alpha - t^*|$.

(d) The existence of a convergent subsequence follows from (*b*). If $[y]^*$ is its limit we get from (*c*)

$$|\check{y}^* - \check{z}^*| = \gamma v,$$

whence $\check{y}^* - \check{z}^* = \gamma D v$, $|D| = I$. Note that in (*c*) the limit exists for the complete sequence. Therefore, γ is independent of the particular subsequence which we used in (*d*). □

Now we present our necessary and sufficient condition for the convergence of all sequences of iterates defined by (1.3) using the notation of the Drazin inverse defined in Section 2.

THEOREM 4.1. *Let $|[A]|$ be irreducible and let (1.5) have a solution $[z]^*$ (which implies $[b] \equiv b \in \mathbb{R}^n$ in the case $\rho(|[A]|) = 1$). Then each sequence $([x]^k)$ of (1.3) is convergent if and only if the following two conditions hold:*

(i) The matrix $|[A]|$ is semi-convergent.

(ii) If $\rho(|[A]|) = 1$ and if $[A]$ contains only one matrix $\dot{A}$ with $|\dot{A}| = |[A]|$ then $\dot{A} \neq -D|[A]|D$ for any signature matrix D.

In case of convergence the limit $[x]^* = [x]^*([x]^0)$ of $([x]^k)$ is a solution of (1.5). It contains the set $S([x]^0)$ of all solutions of (1.4) which one obtains as limit of the sequences (x^k) of iterates defined by (1.2) with $x^0 \in [x]^0$, i.e.,

$$\begin{aligned} &S([x]^0) \\ &= \Big\{x^* \mid x^* = (I-A)^D b + \{I - (I-A)(I-A)^D\}x^0, \\ &\qquad\qquad\qquad\qquad A \in [A],\ x^0 \in [x]^0\Big\} \\ &\subseteq [x]^*([x]^0)\,. \end{aligned} \tag{4.4}$$

Proof. Let v, w be right and left Perron vectors of $|[A]|$ according to (2.1), respectively. In the case $\rho(|[A]|) = 1$ let $[z]^* = \check{z}^* + t^* v[-1, 1]$ as in Theorem 3.5 and notify $[b] \equiv b \in \mathbb{R}^n$ by Lemma 4.2.

'⇒'

Let each sequence of iterates from (1.3) be convergent. If $\rho(|[A]|) < 1$ the assertions follow immediately. Therefore, two cases are still to be considered.

Case 1: $\rho(|[A]|) > 1$.

Choose $[x]^0 = [-v, v]$. Repeated application of (4.1) yields to

$$\mathrm{rad}([x]^k) \geq |[A]|^k \mathrm{rad}([x]^0) = \rho(|[A]|)^k v,$$

hence the sequence $(\mathrm{rad}([x]^k))$ is divergent contradicting the convergence of $([x]^k)$. Thus Case 1 cannot occur.

Case 2: $\rho(|[A]|) = 1$.

First assume $|[A]|$ to be cyclic of index h. Without loss of generality let $|[A]|$ be in cyclic normal form (2.2). Otherwise use Lemma 3.1 and consider the iteration

$$P[x]^{k+1} = (P[A]P^T)(P[x]^k) + Pb$$

with an appropriate permutation matrix P such that $P[A]P^T$ has this form.

Choose $[x]^0 = [z]^* + e^1[-1, 1]$ where e^1 denotes the first column of I. Then we get

$$q([x]^k, [z]^*) \leq |[A]|^k q([x]^0, [z]^*) = |[A]|^k q(e^1[-1, 1], 0) = |[A]|^k e^1.$$

For $k = mh + 1$ this implies

$$0 \leq q([x]^{mh+1}, [z]^*)_1 \leq (|[A]|^{mh+1})_{11} = 0.$$

For the limit $[x]^*$ of $([x]^{mh+1})_{m\in\mathbb{N}_0}$ and therefore also for $([x]^k)_{m\in\mathbb{N}_0}$ we get

$$[x]_1^* = [z]_1^* = (\check{z}^* + t^* v[-1, 1])_1. \tag{4.5}$$

By virtue of

$$\mathrm{rad}([x]^k) \geq |[A]|^k \mathrm{rad}([x]^0) = |[A]|^k \{\mathrm{rad}([z]^*) + e^1\} = t^* v + |[A]|^k e^1$$

we obtain for $k = mh$

$$\mathrm{rad}([x]^{mh}) \geq t^* v + |[A]|^{mh} e^1. \tag{4.6}$$

According to [2], proof of Theorem (2.30) on p. 35 with $p = 1$, the power $|[A]|^h$ is a diagonal block matrix with h primitive diagonal blocks C_i, $i = 1, \ldots, h$, for which $\rho(C_i) = \rho(|[A]|)^h$ holds. By the primitivity of C_i and the Theorem of Perron and Frobenius $\lambda = 1$ is the only eigenvalue of C_i such that $|\lambda| = \rho(C_i)$. It is a simple one. The matrices C_i are therefore semi-convergent with $\lim_{m\to\infty} C_i^m > O$. Taking into account (4.5), (4.6) and $|[A]|^{mh} = \mathrm{diag}(C_1^m, \ldots, C_h^m)$ we get the contradiction

$$\mathrm{rad}([x]_1^*) = t^* v_1 \geq \lim_{m\to\infty} (t^* v + |[A]|^{mh} e^1)_1 = t^* v_1 + \lim_{m\to\infty} (C_1^m)_{11} > t^* v_1.$$

Thus $|[A]|$ cannot be cyclic of index h, it must be primitive, hence $|[A]|$ is semi-convergent, and (i) holds.

In order to prove (ii) we assume that $[A]$ contains exactly one matrix $\dot{A}$ such that $|\dot{A}| = |[A]|$ and $\dot{A} = -D|[A]|D$ with some signature matrix D. We want to derive a contradiction. Without loss of generality we may assume $\dot{A} = -|[A]| = \underline{A}$. Otherwise consider the iteration

$$D[x]^{k+1} = (D[A]D)(D[x]^k) + Db, \qquad k = 0, 1, \ldots \tag{4.7}$$

and take into account Lemma 3.1. Since $\dot{A}$ is the only matrix satisfying $|\dot{A}| = |[A]|$, there is a real number $\theta \in (0, 1)$ such that

$$[A] \subseteq [-|[A]|, \theta|[A]|] = [-1, \theta]|[A]| =: [B].$$

Consider the iteration

$$[y]^{k+1} = [B][y]^k + b, \qquad k = 0, 1, \ldots$$

Choose

$$[y]^0 = [x]^0 := [\check{z}^* - \eta tv, \check{z}^* + tv], \qquad \eta \in (\theta, 1)$$

with $t > 0$ so large that

$$\check{z}^* - \eta tv < 0 < \check{z}^* + \eta tv. \tag{4.8}$$

Using (3.4) this implies

$$\begin{aligned} \underline{y}^1 &= \underline{x}^1 = -|[A]|(\check{z}^* + tv) + b = \dot{A}\check{z}^* - tv + b = \check{z}^* - tv, \\ \overline{x}_i^1 &\leq \overline{y}_i^1 = \sup([-1, \theta]|[A]|[x]^0 + b)_i, \qquad i = 1, \ldots, n. \end{aligned}$$

Moreover,

$$\begin{aligned} \overline{s}_{ij}^1 &:= -|[a]_{ij}|(\check{z}_j^* - \eta tv_j) = -|[a]_{ij}|\check{z}_j^* + (\eta - \theta)t|[a]_{ij}|v_j + \theta t|[a]_{ij}|v_j, \\ \overline{s}_{ij}^2 &:= \theta|[a]_{ij}|(\check{z}_j^* + tv_j) = \theta|[a]_{ij}|\check{z}_j^* + \theta t|[a]_{ij}|v_j. \end{aligned}$$

By virtue of $\eta - \theta > 0$ we can choose $t > 0$ so large that, in addition to (4.8), we can fulfill

$$\overline{s}_{ij}^1 \geq \overline{s}_{ij}^2, \qquad i, j = 1, \ldots, n,$$

i.e.,

$$\begin{aligned} [y]^1 &= [B][x]^0 + b = \big[-|[A]|(\check{z}^* + tv), -|[A]|(\check{z}^* - \eta tv) \big] + b \\ &= \dot{A}\check{z}^* + b + tv[-1, \eta] = \check{z}^* + tv[-1, \eta] = [\check{z}^* - tv, \check{z}^* + \eta tv]. \end{aligned}$$

Moreover, we obtain

$$\begin{aligned} \overline{x}^2 &= \overline{y}^2 = -|[A]|(\check{z}^* - tv) + b = \check{z}^* + tv, \\ \underline{x}_i^2 &\geq \underline{y}_i^2 = \inf([-1, \theta]|[A]|[y]^1 + b)_i, \\ \underline{s}_{ij}^1 &:= -|[a]_{ij}|(\check{z}_j^* + \eta tv_j) = -|[a]_{ij}|\check{z}_j^* - (\eta - \theta)t|[a]_{ij}|v_j - \theta t|[a]_{ij}|v_j, \\ \underline{s}_{ij}^2 &:= \theta|[a]_{ij}|(\check{z}_j^* - tv_j) = \theta|[a]_{ij}|\check{z}_j^* - \theta t|[a]_{ij}|v_j. \end{aligned}$$

By increasing t once more, if necessary, we end up with

$$\underline{s}_{ij}^1 \le \underline{s}_{ij}^2, \qquad i,j = 1,\dots,n,$$

i.e.,

$$\begin{aligned}[x]^2 \subseteq [y]^2 &= [B][y]^1 + b = \left[-|[A]|(\check{z}^* + \eta tv), -|[A]|(\check{z}^* - tv)\right] + b \\ &= [\check{z}^* - \eta tv,\, \check{z}^* + tv] = [y]^0.\end{aligned}$$

One easily recognizes

$$[x]^k \subseteq [y]^k = \begin{cases} [\check{z}^* - \eta tv,\, \check{z}^* + tv], & \text{if } k \text{ is even,} \\ [\check{z}^* - tv,\, \check{z}^* + \eta tv], & \text{if } k \text{ is odd,} \end{cases}$$

with $\overline{x}^k = \overline{y}^k$ if k is even and $\underline{x}^k = \underline{y}^k$ if k is odd. This results in

$$\overline{x}^{2k} = \check{z}^* + tv > \check{z}^* + \eta tv = \overline{y}^{2k+1} \ge \overline{x}^{2k+1},$$

hence $([x]^k)$ cannot be convergent contradicting the assumption. This proves (ii).

'$\Leftarrow$'

Let (i) and (ii) hold and let $([x]^k)$ be a sequence defined by (1.3). If $\rho(|[A]|) < 1$ then convergence follows from Theorem 3.4. Now let $\rho(|[A]|) = 1$. According to Lemma 4.2(d) there is a subsequence $([x]^{k_j})$ converging to some limit $[y]^{*,0}$. By virtue of $[x]^{k_j+1} = [A][x]^{k_j} + b$ the subsequence $([x]^{k_j+1})$ converges to some limit $[y]^{*,1}$ which fulfills

$$[y]^{*,1} = [A][y]^{*,0} + b. \tag{4.9}$$

Moreover, each subsequence $([x]^{k_j+m})$ converges for fixed $m \in \mathbb{N}_0$ to some limit $[y]^{*,m}$. By Lemma 4.2a) we have

$$\begin{aligned}&\operatorname{rad}([y]^{*,m}) = \lim_{j\to\infty} \operatorname{rad}([x]^{k_j+m}) = \alpha v, \\ &0 \le \alpha \in \mathbb{R} \text{ independent of } m,\ (k_j).\end{aligned} \tag{4.10}$$

From Lemma 4.2(c) we obtain

$$\check{y}^{*,m} = \check{z}^* + \gamma D^{(m)} v, \quad |D^{(m)}| = I, \quad 0 \le \gamma \in \mathbb{R} \text{ independent of } m,\ (k_j). \tag{4.11}$$

Combining (4.10) and (4.11) results in

$$[y]^{*,m} = \check{z}^* + \gamma D^{(m)} v + \alpha v[-1,1]. \tag{4.12}$$

Choose $\dot{A} \in [A]$ with $|\dot{A}| = |[A]|$. Use (4.9) and (4.12) with $m = 1$ in order to get

$$\left.\begin{aligned}\dot{A}\check{y}^{*,0} + \alpha v[-1,1] + b &= \dot{A}\check{y}^{*,0} + \alpha\dot{A}v[-1,1] + b = \dot{A}[y]^{*,0} + b \\ &\subseteq [A][y]^{*,0} + b = [y]^{*,1} = \check{z}^* + \gamma D^{(1)} v + \alpha v[-1,1].\end{aligned}\right\} \tag{4.13}$$

Since both sides of (4.13) have the same radius the inclusion can be replaced by equality. This yields to

$$\dot{A}[y]^{*,0} = [A][y]^{*,0} \tag{4.14}$$

and

$$\dot{A}y^{*,0} + b = \check{z}^* + \gamma D^{(1)}v. \tag{4.15}$$

By virtue of

$$\dot{A}y^{*,0} + b = \dot{A}\check{z}^* + \gamma \dot{A}D^{(0)}v + b = \check{z}^* + \gamma \dot{A}D^{(0)}v,$$

which is a consequence of (4.11), (3.4), and (3.5), the equality (4.15) leads to

$$\gamma \dot{A}D^{(0)}v = \gamma D^{(1)}v. \tag{4.16}$$

Case 1: $\gamma = 0$.

Here, Lemma 4.2(c) guarantees $\lim_{k\to\infty} \check{x}^k = \check{z}^k$ and together with Lemma 4.2(a) we obtain $\lim_{k\to\infty} [x]^k = \check{z}^* + \alpha v[-1, 1] =: [x]^*$. Taking this limit in (1.3) reveals that $[x]^*$ is a solution of (1.5).

Case 2: $\gamma > 0$.

From (4.16) we get

$$\dot{A}D^{(0)}v = D^{(1)}v. \tag{4.17}$$

Analogously to (4.16) we can prove

$$\gamma \dot{A}D^{(m-1)}v = \gamma D^{(m)}v$$

for arbitrary $m \in \mathbb{N}$. Thus we finally obtain

$$\dot{A}^m D^{(0)}v = D^{(m)}v. \tag{4.18}$$

Case 2.1: $\rho(\dot{A}) < 1$.

Here, (4.18) implies the contradiction

$$0 < v = |D^{(m)}v| = |\dot{A}^m D^{(0)}v| \to 0 \qquad \text{for } m \to \infty.$$

Therefore, Case 2.1 cannot occur for $\gamma > 0$. Hence $\rho(\dot{A}) < 1$ implies $\gamma = 0$.

Case 2.2: $\rho(\dot{A}) = 1$ and $M_{\text{sym}} \neq \emptyset$, where M_{sym} denotes the same set of column indices as in Theorem 3.5.

Since $M_{\text{sym}} \neq \emptyset$ there exist at least two different matrices $\dot{A}, \ddot{A} \in [A]$ satisfying $|\dot{A}| = |\ddot{A}| = |[A]|$. According to Theorem 3.5 we have $\check{z}_i^* = 0$ for $i \in M_{\text{sym}}$ and $\dot{A}\check{z}^* + b = \ddot{A}\check{z}^* + b = \check{z}^*$, hence (4.17) can also be proved with $\ddot{A}$ instead of $\dot{A}$.

Therefore, $(\dot{A} - \ddot{A})D^{(0)}v = D^{(1)}v - D^{(1)}v = 0$. Let $D^{(0)} := \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$, $\sigma_i = \pm 1$. Choose $\dot{A}, \ddot{A}$ such that $|\dot{A}| = |\ddot{A}| = |[A]|$ and

$$\dot{a}_{ij} = \begin{cases} \overline{a}_{ij}, & \text{if } \sigma_j = 1 \quad \text{and } [a]_{ij} \text{ symmetric}, \\ \underline{a}_{ij}, & \text{if } \sigma_j = -1 \text{ and } [a]_{ij} \text{ symmetric}, \end{cases}$$

$$\ddot{a}_{ij} = \begin{cases} \underline{a}_{ij}, & \text{if } \sigma_j = 1 \quad \text{and } [a]_{ij} \text{ symmetric}, \\ \overline{a}_{ij}, & \text{if } \sigma_j = -1 \text{ and } [a]_{ij} \text{ symmetric}. \end{cases}$$

Then

$$\dot{a}_{ij} - \ddot{a}_{ij} = \begin{cases} 0, & \text{if } [a]_{ij} \text{ not symmetric}, \\ (\overline{a}_{ij} - \underline{a}_{ij}), & \text{if } \sigma_j = 1 \quad \text{and } [a]_{ij} \text{ symmetric}, \\ -(\overline{a}_{ij} - \underline{a}_{ij}), & \text{if } \sigma_j = -1 \text{ and } [a]_{ij} \text{ symmetric}. \end{cases}$$

Since $\dot{A} \neq \ddot{A}$ there is at least one symmetric entry $[a]_{i_0,j_0} \neq 0$. This leads to the contradiction

$$0 = \{(\dot{A} - \ddot{A})D^{(0)}v\}_{i_0} \geq (\overline{a}_{i_0,j_0} - \underline{a}_{i_0,j_0})v_{j_0} = 2\overline{a}_{i_0,j_0}v_{j_0} > 0.$$

Hence Case 2.2 cannot occur for $\gamma > 0$.

Case 2.3: $\rho(\dot{A}) = 1$ and $M_{\text{sym}} = \emptyset$, i.e., $\dot{A}$ is the only matrix in $[A]$ with $|\dot{A}| = |[A]|$.

Lemma 5 in [6] and the assumption (ii) yield to $\dot{A} = D|[A]|D$, $D \in \mathbb{R}^{n\times n}$, $|D| = I$. Without loss of generality we may assume

$$\dot{A} = |[A]|, \tag{4.19}$$

otherwise consider (4.7). Since the matrix $|[A]|$ is irreducible and semi-convergent by assumption, $\lambda = 1$ is its only eigenvalue satisfying $|\lambda| = \rho(|[A]|)$. According to the Theorem of Perron and Frobenius $\lambda = 1$ is a simple eigenvalue, hence $|[A]|$ is primitive and there is an integer p such that

$$\dot{A}^p = (\dot{a}_{ij}^{(p)}) > O. \tag{4.20}$$

With $D^{(0)} = \operatorname{diag}(\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^{n\times n}$, $D^{(p)} = \operatorname{diag}(\tau_1, \ldots, \tau_n) \in \mathbb{R}^{n\times n}$, $|\sigma_i| = |\tau_i| = 1$ we get from (4.18) with $m = p$

$$v = D^{(p)}\dot{A}^p D^{(0)}v = (\tau_i\sigma_j\dot{a}_{ij}^{(p)})v.$$

Together with (4.20) this implies $\tau_i\sigma_j = 1$ for $i, j = 1, \ldots, n$, whence $D^{(p)} = D^{(0)} = \sigma I$, $|\sigma| = 1$. Again from (4.18) we obtain $D^{(m)}v = \sigma v$, i.e., $D^{(m)} = \sigma I$, $m = 0, 1, \ldots$, and from (4.12) the equation

$$[y]^{*,m} = \check{z}^* + \sigma\gamma v + \alpha v[-1, 1] =: [y]^* \tag{4.21}$$

follows. This vector is a solution of (1.5), i.e., it satisfies $[A][y]^* + b = [y]^*$ as can be seen from the next to last equality in (4.13).

It remains to show that the *whole* sequence $([x]^k)$ converges to $[y]^*$. To this end we start with

$$\begin{aligned} q([x]^{k_j+l},[y]^*) &= q([A][x]^{k_j+l-1}+b,[A][y]^*+b) \\ &\le |[A]|q([x]^{k_j+l-1},[y]^*) \le \cdots \le |[A]|^l q([x]^{k_j},[y]^*). \end{aligned}$$

Since $|[A]|$ is semi-convergent according to (i), there is a matrix $K \ge O$ with $|[A]|^l \le K$, $l = 0, 1, \ldots$, hence

$$q([x]^{k_j+l},[y]^*) \le Kq([x]^{k_j},[y]^*) \quad l = 0, 1, \ldots$$

Choosing j sufficiently large makes the right-hand side arbitrarily small independently of l. Hence

$$\lim_{k\to\infty} q([x]^k,[y]^*) = 0.$$

The last assertion of Theorem 4.1 follows immediately from the inclusion isotonicity of the interval arithmetic and from Theorem 3.3. □

Remark 4.1.

(a) The cases $\gamma = 0$ and $\gamma > 0$ yield to the cases in Theorem 3.5 in which (3.4), (3.5) have exactly one solution (cf. Theorem 3.5(*c*) and (*d*)(*i*)), and infinitely many ones (cf. Theorem 3.5(*d*)(*iii*)), respectively. The case mentioned in Theorem 3.5(*d*)(*ii*) is excluded by the condition (ii) in Theorem 4.1.

   By virtue of the assumption (4.19) the equation (3.7) occurs only with $D = I$ in the proof above. In order to recover this equation let $s = \sigma\gamma$ in (4.21). See also Theorem 4.2 in this respect.

(b) If $|[A]|$ is not semi-convergent, then according to Theorem 4.1 not every sequence of iterates can be convergent. Nevertheless (1.5) can have solutions. This can be seen from Example 3.1(*c*) or from the iteration

$$[x]^{k+1} = [A][x]^k, \qquad k = 0, 1, \ldots, \tag{4.22}$$

   with the matrix $[A] = \begin{pmatrix} 0 & [0,1] \\ [0,1] & 0 \end{pmatrix}$ whose absolute value is cyclic of index 2. The eigenvalues of $|[A]|$ are therefore ± 1, hence this matrix is not semi-convergent, but the system $[x] = [A][x]$ has the solutions $[x]^* = se + te[-1,1]$, $s \in \mathbb{R}$, $|s| \le t \in \mathbb{R}$, as can be seen from Theorem 3.5. Starting (1.3) with $[x]^0 = [x]^*$ yields trivially to convergent sequences.

THEOREM 4.2. *Let the assumptions of Theorem 4.1 hold including (i) and (ii), let $\rho(|[A]|) = 1$ and let $([x]^k)$ be a sequence of (1.3) with limit $[x]^*$. Choose any Perron vectors v, w of $|[A]|$ according to (2.1) and let $[z]^*$ be defined as in Theorem 4.1. Denote by $[z]^*_{\min} = \check{z}^* + t_{\min}v[-1,1]$ the solution of (1.5) with the same midpoint as $[z]^*$ and with $t_{\min} \ge 0$ from (3.6) applied to $\check{z} = \check{z}^*$. Denote by $[z]^*_{\max} = \check{z}^* + t_{\max}v[-1,1]$, $t_{\max} \ge 0$, the smallest solution with midpoint $\check{z}^*$ which contains $[x]^0$.*

(a) *If $[A]$ contains exactly one matrix $\dot{A}$ with $|\dot{A}| = |[A]|$ and if $\dot{A} = D|[A]|D$ for some signature matrix D then $[x]^*$ can be represented as*

$$[x]^* = sDv + \check{z}^* + tv[-1, 1] \tag{4.23}$$

with $s = s(v, [x]^0) \in \mathbb{R}$, $t = t(s, v, [x]^0) \geq 0$, and we obtain $[x]^* \subseteq [z]^*_{\max}$.

(b) *If $[A]$ does not fulfill the two additional assumptions of (a) then $[x]^*$ can be represented as*

$$[x]^* = \check{z}^* + tv[-1, 1] \tag{4.24}$$

with $t = t(v, [x]^0) \geq 0$, and we obtain

$$[z]^*_{\min} \subseteq [x]^* \subseteq [z]^*_{\max}. \tag{4.25}$$

In particular,

$$[x]^* = [z]^*_{\min} \qquad \text{whenever } [x]^0 \subseteq [z]^*_{\min}.$$

(c) *We have $|s| + t \leq t_{\max}$ in case (a) and $t_{\min} \leq t \leq t_{\max}$ in case (b).*

(d) *If*

$$[A][x]^k = \dot{A}[x]^k, \qquad k = k_0, k_0 + 1, \ldots \tag{4.26}$$

for some $\dot{A} \in [A]$ with $|\dot{A}| = |[A]|$ and if $\mathrm{rad}([x]^k) = \alpha_k v + s^k$, $s^k \in (\mathrm{span}\{w\})^\perp$, as in the proof of Lemma 4.2(a) then

$$t = \alpha_k, \qquad k = k_0, k_0 + 1, \ldots \tag{4.27}$$

In particular, $[x]^* = \check{z}^* + \alpha_{k_0} v[-1, 1]$ in case (b). (Note that $\check{z}^*$ is the unique solution of (3.4) and (3.5) in this case.)

Proof. (*a*)–(*c*) are immediate consequences of Lemma 4.1, Theorem 3.5, and Theorem 4.1. In order to prove (*d*) use

$$\begin{aligned} \alpha_{k+1} v + s^{k+1} &= \mathrm{rad}([x]^{k+1}) = \mathrm{rad}([A][x]^k) = \mathrm{rad}(\dot{A}[x]^k) \\ &= |\dot{A}|\mathrm{rad}([x]^k) = \alpha_k v + |\dot{A}|s^k \end{aligned} \tag{4.28}$$

with $s^k, s^{k+1} \in (\mathrm{span}\{w\})^\perp$ and multiply (4.28) from the left by w^T. This results in $\alpha_{k+1} = \alpha_k$, $k \geq k_0$, and (4.27) follows from $t = \lim_{k\to\infty} \alpha_k$ which is contained in the proof of Lemma 4.2(*a*) with α instead of the present t. □

Remark 4.2. The assumption (4.26) certainly holds if one replaces $[x]^k$ by the limit $[x]^*$. This follows from (4.14) with $[y]^{*,0} = [x]^*$. By virtue of the continuity of the interval arithmetic one can hope that (4.26) also holds if k is sufficiently large.

If $[A][x]^k = \dot{A}[x]^k$ holds for *some* $k = k_0$ this relation (4.26) does not necessarily hold, however, for *all* $k \geq k_0$. Therefore, (4.26) cannot be weakened in this respect. This can be seen from the subsequent Example 4.1 starting with $[x]^0 = 0$. Here, $[A][x]^0 = 0 = \dot{A}[x]^0$, but $[A][x]^1 \neq \dot{A}[x]^1 = \dot{A}b \in \mathbb{R}^2$ since $\mathrm{rad}([A][x]^1) = \mathrm{rad}([A]b) \neq 0$.

It is an open question, how the (unique) limit $[x]^*$ of the sequences $([x]^k)$ defined by (1.3) looks like if $\rho(|[A]|) < 1$. Some progress was made in [7] and [8] for particular classes of matrices $[A]$ but the general solution is still missing.

It is also open how s and t in (4.23), (4.24) must be chosen for arbitrarily given $[x]^0$.

The following example illustrates this problem.

EXAMPLE 4.1. Let

$$[A] = \begin{pmatrix} \frac{1}{2} & \left[-\frac{1}{2}, 0\right] \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad A_\alpha = \begin{pmatrix} \frac{1}{2} & \alpha \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad -\tfrac{1}{2} \le \alpha \le 0, \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then $\dot{A} := A_{-1/2}$ is the only matrix in $[A]$ with $|\dot{A}| = |[A]|$. Since $\rho(|[A]|) = \rho(|\dot{A}|) = 1$, $\rho(\dot{A}) = 1/\sqrt{2} < 1$, every solution of (1.5) has the representation (4.24). With the notation of Theorem 4.2 we have

$$t_{\min} = 2, \quad \check{z}^* = \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \dot{A}\check{z}^* + b, \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad [z]^*_{\min} = \begin{pmatrix} [-2, 2] \\ [0, 4] \end{pmatrix}.$$

By virtue of

$$(I - A_\alpha)^{-1} = \frac{2}{1 - 2\alpha} \begin{pmatrix} 1 & 2\alpha \\ 1 & 1 \end{pmatrix}, \qquad -\frac{1}{2} \le \alpha \le 0,$$

and

$$\frac{2(1 + 2\alpha)}{1 - 2\alpha} = -2 + \frac{4}{1 - 2\alpha} =: -2 + \beta$$

the solution set S in (3.3) reads

$$S = \left\{ (I - A_\alpha)^{-1} b \;\middle|\; -\frac{1}{2} \le \alpha \le 0 \right\} = \left\{ \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix} \;\middle|\; 2 \le \beta \le 4 \right\},$$

hence its interval hull $\square S$, i.e., its smallest enclosure by an interval vector, is given by

$$\square S = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + [2, 4] \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} [0, 2] \\ [2, 4] \end{pmatrix} \subseteq [z]^*_{\min}.$$

The subset property was already predicted by Theorem 3.5(*e*). From (4.25) we see that in our example each limit of (1.3) overestimates S at least as much as $[z]^*_{\min}$.

For (1.3) we chose different starting vectors $[x]^0$ and computed several iterates using Rump's MATLAB toolbox INTLAB (see the INTLAB home page

`http://www.ti3.tu-harburg.de/~rump/intlab/index.html`

Table 1. Starting vector vs. computed "limit."

starting vector $([x]^0)^T$	t in (4.24)	computed "limit" $([\tilde{x}]^{k_0})^T$	k_0
$([-2,2],[0,4])$	$2 = t_{\min}$	$([-2,2],[0,4])$	1
$(6,6)$	2.6407	$([-2.6407, 2.6407], [-0.6407, 4.6407])$	111
$([4,6],[4,6])$	2.7501	$([-2.7501, 2.7501], [-0.7501, 4.7501])$	110
$([4,6],[6,8])$	3.0626	$([-3.0626, 3.0626], [-1.0626, 5.0626])$	111
$([-6,6],[-6,6])$	6.0001	$([-6.0001, 6.0001], [-4.0001, 8.0001])$	106
$([-6,6],[-4,8])$	6	$([-6,6],[-4,8])$	1

or [13]) with the multiplication mode "SharpIVMult". Actually we used MATLAB Version 6.5 and INTLAB Version 4.1.2. We stopped the iteration (1.3) whenever the criterion

$$[\tilde{x}]^k = [\tilde{x}]^{k-1}$$

was fulfilled for some $k = k_0$, where here and in the sequel the tilde denotes computed, i.e., rounded quantities. Since by the outward rounding of the machine interval arithmetic (cf., e.g., [1]) we have $[x]^k \subseteq [\tilde{x}]^k$, $k = 0, 1, \ldots$, we get

$$[x]^k \subseteq [\tilde{x}]^{k_0}, \qquad k = k_0, k_0 + 1, \ldots,$$

whence

$$[x]^* \subseteq [\tilde{x}]^{k_0}. \tag{4.29}$$

Without further knowledge on a relation between $[x]^*$ and $[x]^0$ we cannot, of course, assess the quality of the approximation $[\tilde{x}]^{k_0}$ with respect to the true limit $[x]^*$. Assuming that $[\tilde{x}]^{k_0}$ differs from $[x]^*$ only slightly—say at most one unit in the last place (here 0.0001) printed out for an interval bound—our numerical experiments in Table 1 can provide a starting point for further research on the dependency of t in (4.24) on $[x]^0$.

The first line below the header of Table 1 shows $[x]^0 = [z]^*_{\min}$ which must be contained in every limit $[x]^*$. In the next line we start with the degenerate interval vector $(6,6)^T$, which is contained in the starting vectors of all subsequent lines. The smallest solution $[z]^*_{\max}$ of (1.5) enclosing $(6,6)^T$ is given by the starting vector of the last line with $t = t_{\max} = 6$. Starting with $[x]^0 = (6,6)^T$ yields to $[\tilde{x}]^6 = ([-3.0000, 2.2813], [-0.8750, 4.4063])^T$ with $q([x]^6, [\tilde{x}]^6) < 10^{-4}$. I.e., $[\tilde{x}]^6$ coincides with the exact iterate

$$[x]^6 = \left(\left[-3, \frac{73}{32}\right], \left[-\frac{7}{8}, \frac{141}{32}\right]\right)^T = ([-3, 2.28125], [-0.875, 4.40625])^T$$

within rounding when rounding as described above. Since $0 \in [x]^6_2$ we obtain $[A][x]^6 = \check{A}[x]^6$ and by virtue of $[z]^*_{\min} \subseteq [x]^6$ one gets $[z]^*_{\min} \subseteq [x]^k$ and $0 \in [x]^k_2$ for $k = 6, 7, \ldots$ Therefore, $[A][x]^k = \check{A}[x]^k$, $k = 6, 7, \ldots$, and Theorem 4.2 guarantees

Table 2. Starting vector vs. computed "limit."

starting vector $([x]^0)^T$	s in (4.23)	t in (4.23)	computed "limit" $([\tilde{x}]^{k_0})^T$	k_0
$([-2,2],[0,4])$	1	2	$([0,4],[-2,2])$	2
$(6,6)$	3.5	2.5	$([2,7],[0,5])$	61
$([4,6],[4,6])$	3.5	2.5	$([2,7],[0,5])$	60
$([4,6],[6,8])$	4	3	$([2,8],[0,6])$	60
$([-6,6],[-6,6])$	0	6	$([-5,7],[-7,5])$	2
$([-6,6],[-4,8])$	1	6	$([-4,8],[-6,6])$	2

$[x]^* = \check{z}^* + \alpha_6 v[-1,1]$ with $[x]^6 = \check{x}^6 + (\alpha_6 v + s^6)[-1,1]$, $\alpha_6 \in 2.6407 + [-1,0]\cdot 10^{-4}$, $s^6 = 0 \in (\text{span}\{w\})^\perp = (\text{span}\{(0.5,0.5)^T\})^\perp$.

While Example 4.1 deals with an interval matrix for which Theorem 3.4 does not apply and for which $[C] = I - [A]$ is *regular* our next Example 4.2 is based on an interval matrix $[A]$ for which $[C] = I - [A]$ is *singular* whence Theorem 3.4 does not apply either.

EXAMPLE 4.2. Let

$$[A] = \begin{pmatrix} \frac{1}{2} & [0, \frac{1}{2}] \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad A_\alpha = \begin{pmatrix} \frac{1}{2} & \alpha \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad 0 \le \alpha \le \frac{1}{2}, \quad b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Then $\dot{A} = A_{1/2}$ is the only matrix in $[A]$ with $|\dot{A}| = |[A]|$. Moreover, $\dot{A} = |[A]|$, hence every solution of (1.5) has the representation (4.23), i.e.,

$$[x]^* = sv + \check{z}^* + tv[-1,1], \qquad s \in \mathbb{R}, \ t \ge 0,$$

where $\check{z}^*$ satisfies $\check{z}^* = \dot{A}\check{z}^* + b$ and where v is any Perron vector of $|[A]|$. Hence

$$\check{z}^* = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

are possible choices, and

$$[x]^* = s\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}[-1,1]$$

with appropriate $s \in \mathbb{R}$, $t \ge 0$. The matrix $[C] = I - [A]$ contains the singular matrix $\tilde{C} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ whose columns are multiples of b. With the same starting vectors $[x]^0$ as in Table 1 and the same stopping criterion as in Example 4.1 we got the results in Table 2.

In order to illustrate (4.4) we first remark that the Drazin inverse $(I - A)^D$ coincides with the inverse $(I - A)^{-1}$ whenever this latter inverse exists—see (2.4). Iterating by

$$x^{k+1} = A_\alpha x^k + b, \qquad k = 0, 1, \ldots,$$

thus yields to the unique limit

$$\begin{aligned} x^* &= (I - A_\alpha)^{-1} b = \begin{pmatrix} \frac{1}{2} & -\alpha \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \frac{2}{1 - 2\alpha} \begin{pmatrix} 1 & 2\alpha \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \end{aligned} \tag{4.30}$$

for any $\alpha \in \left[0, \frac{1}{2}\right)$. It is a particularity of our example that this limit does not depend on α. By virtue of the Theorems 3.4 and 4.1 it must be contained in the interval limit $[x]^* = [x]^*([x]^0)$ of (1.3) for every starting vector $[x]^0$.

If $\alpha = \frac{1}{2}$ then

$$A_\alpha = A_{1/2} = \dot{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = S \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} S^{-1}$$

with $S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $S^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Hence

$$\begin{aligned} (I - A_{1/2})^D &= S \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S^{-1} = I - A_{1/2} \qquad \text{and} \\ (I - A_{1/2})^D b &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} = b. \end{aligned} \tag{4.31}$$

By virtue of Theorem 3.3 and (3.2) the iteration

$$x^{k+1} = A_{1/2} x^k + b, \qquad k = 0, 1, \ldots, \tag{4.32}$$

is convergent to

$$\begin{aligned} x^*(x^0) &= (I - A_{1/2})^D b + \{I - (I - A_{1/2})(I - A_{1/2})^D\} x^0 = b + A_{1/2} x^0 \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} x^0. \end{aligned}$$

as can be seen, e.g., from (4.31) and $A_{1/2}^2 = A_{1/2}$. From $x^*(x^0) \in [x]^*([x]^0)$ for any $x^0 \in [x]^0$ we get

$$[y]^* = b + A_{1/2}[x]^0 \subseteq [x]^*([x]^0).$$

Denote by $\square([y]^*, x^*)$ the interval hull of the two operands, i.e., the smallest interval vector which contains $[y]^*$ and the limit $x^* = (2, 0)^T$ from (4.30). It is obvious that this interval hull is a subset of $[x]^* = [x]^*([x]^0)$. An easy calculation shows that $\square([y]^*, x^*)$ coincides with the computed "limit" $[\tilde{x}]^{k_0}$ in all cases of Table 2. Therefore, by virtue of (4.29), we have

$$[x]^* = [\tilde{x}]^{k_0} = \square([y]^*, x^*)$$

in Table 2. In particular, this table reports the correct limits $[x]^*$.

Note that $[y]^*$ is a solution of $[x] = A_{1/2}[x] + b$ and hence a limit of (4.32). This follows from $A_{1/2}b = 0$ and from $A_{1/2}(A_{1/2}[x]^0) = A_{1/2}^2[x]^0$ which holds since $A_{1/2}$ is degenerate and non-negative. But $[y]^*$ is generally not a solution of (1.5) with $[A], b$ as given in the present example.

The final part of this section is devoted to some connection between the Theorems 3.6 and 4.1 for which the conditions (*i*) and (*ii*) coincide. We show how the convergence of $([A]^k)$ in Theorem 3.6 can be derived from Theorem 4.1 if the remaining assumptions of Theorem 3.6 hold. To this end we first remark that these assumptions for $[A]$ hold if and only if they hold for $[A]^T$. Since $[A]^k = ({}^k([A]^T))^T$ and ${}^k[A] = (([A]^T)^k)^T$ it can be seen at once that

$$[A]^\infty := \lim_{k\to\infty} [A]^k \quad \text{exists if and only if} \quad {}^\infty[A] := \lim_{k\to\infty} {}^k[A] \quad \text{exists.} \tag{4.33}$$

Trivially,

$$[A]^\infty = \left({}^\infty([A]^T)\right)^T \quad \text{and} \quad {}^\infty[A] = \left(([A]^T)^\infty\right)^T \tag{4.34}$$

hold. Denote by $[a]^l$ the l-th column of $[A]$ and let $[x]^*([a]^l)$ be the limit of the iteration

$$\begin{aligned} [x]^0 &= [a]^l \\ [x]^{k+1} &= [A][x]^k, \qquad k = 0, 1, \ldots \end{aligned} \tag{4.35}$$

which exists according to Theorem 4.1. (Note that $[z]^* = 0$ is a solution of $[x] = [A][x]$.) Then $[x]_i^k = ({}^{k+1}[A])_{il}$, $i = 1, \ldots, n$, hence

$$\lim_{k\to\infty} {}^k[A] = \left([x]^*([a]^1), \ldots, [x]^*([a]^n)\right) \tag{4.36}$$

exists and (4.33) proves the corresponding part of Theorem 3.6.

Remark 4.3. Let the assumptions of Theorem 3.6 hold, in particular, let $\mathrm{rad}([A]) \neq O$.

(a) If $\rho(|[A]|) < 1$ then ${}^\infty[A] = [A]^\infty = O$. This follows directly from (4.34), (4.36) since Theorem 3.4 yields to $[x]^*([a]^l) = 0$.

(b) If $\rho(|[A]|) = 1$ then ${}^\infty[A] \neq O$, $[A]^\infty \neq O$. With v, w as in (2.1) this can be seen from $\mathrm{rad}({}^k[A]) \geq |[A]|^{k-1}\mathrm{rad}([A])$ which becomes $\mathrm{rad}({}^\infty[A]) \geq vw^T\mathrm{rad}([A]) \neq O$ if k tends to infinity.

(c) Let $\rho(|[A]|) = 1$, let M_{sym} as in Theorem 3.5 and choose $\dot{A} \in [A]$ such that $|\dot{A}| = |[A]|$. The representation (4.36) shows that the columns of ${}^{\infty}[A]$ are limits of (4.35). Therefore, they are solutions of $[x] = [A][x]$. Taking into account $[z]^* = 0$ and combining the cases in Theorem 4.2 we get

$$[x]^*([a]^l) = s_l\tilde{D}v + t_l v[-1,1], \qquad s_l, t_l \in \mathbb{R},\ |\tilde{D}| = I, \tag{4.37}$$

$t_l \geq 0$ sufficiently large, bounded from below by $t_{\min} = t_{\min,l}$ in (3.6) with $\check{z} = s_l\tilde{D}v$. The case $t_l = 0$ is possible.

If $M_{\text{sym}} \neq \emptyset$ or if $M_{\text{sym}} = \emptyset$ and $\rho(\dot{A}) < 1$ then according to Theorem 3.5 the midpoint of a solution of $[x] = [A][x]$ is unique. Therefore it coincides with $\check{z}^* = 0$ of $[z]^* = 0$. Hence $s_l = 0$ in (4.37) for $l = 1, \ldots, n$, and ${}^{\infty}\check{A} = O$, i.e., ${}^{\infty}[A] = -({}^{\infty}[A])$. This proves parts of the Corollaries 3–5 in [6].

If $M_{\text{sym}} = \emptyset$ and $\dot{A} = D|[A]|D$, $|D| = I$ then the first summand in (4.37) cannot be skipped, $\tilde{D} = D$, and (3.6) reads

$$\begin{aligned} t_l &\geq t_{\min,l} \\ &= \max\left\{0, \frac{\text{rad}([a]_{ij})}{|\check{a}_{ij}|}\cdot s_l, s_l \,\middle|\, 1 \leq i,j \leq n,\ \check{a}_{ij} \neq 0,\ \text{rad}([a]_{ij}) \neq 0\right\}. \end{aligned}$$

If $s_l \neq 0$ for some l then (4.37) implies ${}^{\infty}[A] \neq -({}^{\infty}[A])$, and $t_{\min,l} > 0$ follows from the assumption $\text{rad}([A]) \neq O$ and $M_{\text{sym}} = \emptyset$ which guarantee the existence of at least one non-symmetric, non-degenerate entry of $[A]$.

If $s_l = 0$ for all l then (4.37) yields to ${}^{\infty}[A] = vt^T[-1,1] = -({}^{\infty}[A])$ with $0 \leq t = (t_l) \in \mathbb{R}^n$. Without loss of generality we may assume $\dot{A} = |[A]|$. From $\dot{A}^{\infty} = |[A]|^{\infty} = vw^T \in {}^{\infty}[A]$ we get $w \leq t$. From $[A] \subseteq [-\theta,1]|[A]|$ for some $\theta \in (0,1)$ we obtain ${}^{\infty}[A] \subseteq [-\theta,1]|[A]|^{\infty} = [-\theta,1]vw^T$, whence $-\theta w \leq -t$. This results in the contradiction $w \leq t \leq \theta w < w$ and shows that this case cannot occur.

Thus we have proved the subsequent theorem whose first part can already be found as Theorem 4 in [6] for $[A]^{\infty}$.

THEOREM 4.3. *Let $|[A]|$ be irreducible and semi-convergent with $\rho(|[A]|) = 1$, and let* $\text{rad}([A]) \neq O$. *Then ${}^{\infty}[A] \neq -({}^{\infty}[A])$ and $[A]^{\infty} \neq -[A]^{\infty}$, respectively, if and only if the following two conditions are fulfilled:*

(i) $[A]$ contains exactly one matrix $\dot{A}$ with $|\dot{A}| = |[A]|$.

(ii) The matrix $\dot{A}$ in (i) has the representation $\dot{A} = D|[A]|D$, where D is some signature matrix.

In this case ${}^{\infty}[A] = Dvs^T + vt^T[-1,1]$ and $[A]^{\infty} = \tilde{s}w^TD + \tilde{t}w^T[-1,1]$, respectively, where v is any fixed right Perron vector, w is any fixed left Perron vector and $s, t, \tilde{s}, \tilde{t} \in \mathbb{R}^n$ are appropriate vectors with $t, \tilde{t} \geq 0$. Otherwise ${}^{\infty}[A] = vt^T[-1,1]$ and $[A]^{\infty} = \tilde{t}w^T[-1,1]$, respectively.

Final remark. In the present paper we have considered the convergence of the interval iteration $[x]^{k+1} = [A][x]^k + [b]$, $k = 0, 1, \ldots$ We have completely discussed the case where $|[A]|$ is irreducible with $\rho(|[A]|) = 1$. For reducible matrices $|[A]|$ with $\rho(|[A]|) = 1$ things are more complicated. This case is studied in H.–R. Arndt's thesis which will be published separately. For $\rho(|[A]|) < 1$ convergence can be guaranteed by O. Mayer's result [10] mentioned in Section 1. For $\rho(|[A]|) > 1$ Case 1 in the proof of Theorem 4.1 shows that there are starting vectors $[x]^0$ for which the sequence $([x]^k)$ is divergent.

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