Computing 70, 235–259 (2003)
Digital Object Identifier (DOI) 10.1007/s00607-003-0014-6

Computing
Printed in Austria

# Iterative Methods for Linear Complementarity Problems with Interval Data*

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Received October 7, 2002; revised April 15, 2003
Published online: June 23, 2003


**Abstract**

In this paper we introduce the total step method, the single step method and the symmetric single step method for linear complementarity problems with interval data. They are applied to an interval matrix $[A]$ and an interval vector $[b]$. If all $A \in [A]$ are H-matrices with positive diagonal elements, these methods are all convergent to the same interval vector $[x^*]$. This interval vector includes the solution $x$ of the linear complementarity problem defined by any fixed $A \in [A]$ and any fixed $b \in [b]$. If all $A \in [A]$ are M-matrices, then we will show, that $[x^*]$ is optimal in a precisely defined sense. We also consider modifications of those methods, which under certain assumptions on the starting vector deliver nested sequences converging to $[x^*]$. We close our paper with some examples which illustrate our theoretical results.

*AMS Subject Classifications:* 90C33, 65G30.

*Keywords:* Linear complementarity problem, total step method, single step method, symmetric single step method, interval computation.

## 1. Introduction

Let $A$ be a real $n \times n$ matrix and $b$ an $n$-dimensional vector. Then the linear complementarity problem, denoted by LCP, is defined as follows: Determine a real vector $x$ such that

$$b + Ax \geq o, \quad x \geq o, \quad (b + Ax)^T x = 0, \tag{1}$$

or conclude that there is no such $x$. The inequalities appearing in (1) are understood componentwise.

The article [9] gives an extensive documentation of applications of complementarity problems in engineering and equilibrium modeling. Additional applications one can find in [5], [6] and [13], respectively.

It is well-known and easy to see that (1) is equivalent to solving the non-smooth nonlinear system

*Dedicated to U. Kulisch on the occasion of his 70th birthday.

$$\min\{x, b + Ax\} = o, \tag{2}$$

where the minimum is taken componentwise. This equivalence has been used in [1] to verify the existence of a solution of (2) and of (1), respectively, if a somehow computed approximate solution is available.

In [17] the case was considered in which the input data $A$ and $b$ are not precisely known, but can only be enclosed elementwise in intervals. An important application of this problem is the discretization of a free boundary problem without neglecting the discretization error. For details, see [17].

The present paper is organized as follows. After introducing the notation and some preliminaries in Section 2, we consider several different iteration methods, which allow to enclose the solution set

$$\Sigma([A],[b]) := \{x : \min\{x, b + Ax\} = o, A \in [A], b \in [b]\} \tag{3}$$

of real vectors by an interval vector: The total step method $(T)$, the single step method $(S)$, which is a special case of the successive overrelaxation method $(SOR)$, and the symmetric single step method $(SS)$. We show that under equal assumptions $(T)$, $(S)$ and $(SS)$, respectively, are convergent to the same interval vector enclosing the solution set $\Sigma([A],[b])$ defined in (3). For $(SOR)$ this is generally not the case. This method is convergent to a limit depending on the relaxation parameter $\omega$ used in this method.

With respect to inclusion the limit of $(T)$ $((S)$ and $(SS))$ is in general not the smallest interval vector enclosing $\Sigma([A],[b])$. However, under additional assumptions on $[A]$ we can show that this limit is optimal. This is proved in Theorem 3.2.

After having proved these results in Section 3, we consider modifications of $(T)$, $(S)$ and $(SS)$ which are based on the fact that if for any of these methods one is starting with an interval vector containing the limit, then all iterates contain the limit. Therefore, the enclosure of the limit might be improved by forming intersections after each iteration step. We prove that the corresponding modifications are convergent to the same limit as the unmodified methods. Furthermore we show that the modified symmetric single step method is in a precisely defined sense optimal.

We close this paper with some numerical examples illustrating the theoretical results.

In passing we note that for non-interval data generalizations of the total step method etc. have already been applied to the problem (1). See [4], [6] and [7], for example.

## 2. Preliminaries and notation

This section contains a summary of well-known or easy to prove properties and results which are used subsequently.

### 2.1. Interval arithmetic

By $\mathbf{R}, \mathbf{R}^n, \mathbf{R}^{n\times n}, \mathbf{IR}, \mathbf{IR}^n, \mathbf{IR}^{n\times n}$, we denote the set of real numbers, the set of real vectors with $n$ components, the set of real $n \times n$ matrices, the set of intervals, the set of interval vectors with $n$ components and the set of $n \times n$ interval matrices, respectively. An interval always means a real compact interval. Interval vectors and interval matrices are vectors and matrices, respectively, with interval entries. We write intervals in brackets with the exception of degenerate intervals (so-called *point intervals*) which we identify with the element being contained. Similarly we proceed with interval vectors and matrices. We denote by $I$ the identity, by $O$ the zero matrix and by $o$ the zero vector. We use the notation $[a] = [\underline{a}, \overline{a}]$ for $[a] \in \mathbf{IR}$. Analogously we write $[x] = [\underline{x}, \overline{x}] = ([x_i]) = ([\underline{x}_i, \overline{x}_i]) \in \mathbf{IR}^n$ and $[A] = [\underline{A}, \overline{A}] = ([a_{ij}]) = ([\underline{a}_{ij}, \overline{a}_{ij}]) \in \mathbf{IR}^{n\times n}$. For $[a], [b] \in \mathbf{IR}$ we define

- the *diameter* $d([a]) := \overline{a} - \underline{a}$,
- the *absolute value* $|[a]| := \max\{|\underline{a}|, |\overline{a}|\}$,
- the *distance* $q([a], [b]) := \max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\}$.

For interval vectors and interval matrices, these quantities are defined elementwise. For example, if $[a] = ([a_i])$, then $d([a]) = (d([a_i])) \in \mathbf{R}^n$. We equip $\mathbf{R}^n$ and also $\mathbf{R}^{n\times n}$ with the elementwise defined relations $\leq, <, >, \geq$. If $[a] \in \mathbf{IR}$, we define

$$\langle [a] \rangle := \min\{|a| : a \in [a]\} = \left\{ \begin{matrix} \min\{|\underline{a}|, |\overline{a}|\} & \text{if } 0 \notin [a], \\ 0 & \text{else.} \end{matrix} \right\} \tag{4}$$

If for two interval vectors $[x], [y] \in \mathbf{IR}^n$ we have $[x_i] \cap [y_i] \neq \emptyset$, $i = 1, 2, \ldots, n$, then $[x] \cap [y] = ([x_i] \cap [y_i])$, otherwise $[x] \cap [y] = \emptyset$. In addition, for $[x], [y] \in \mathbf{IR}^n$ we define $[x] \subseteq [y]$ iff $[x_i] \subseteq [y_i]$, $i = 1, \ldots, n$.

Furthermore, we repeat some relations concerning the distance:

$$q([x], [z]) \leq q([x], [y]) + q([y], [z]), \tag{5}$$
$$q([x] + [z], [y] + [z]) = q([x], [y]), \tag{6}$$
$$q([A] \cdot [x], [A] \cdot [y]) \leq |[A]| \cdot q([x], [y]), \tag{7}$$
$$q([x] + [y], [u] + [z]) \leq q([x], [u]) + q([y], [z]), \tag{8}$$

if $[u], [x], [y], [z] \in \mathbf{IR}^n$ and $[A] \in \mathbf{IR}^{n\times n}$. The so-called Minty map of $a \in \mathbf{R}$ was defined in [10] as

$$a^+ := \max\{0, a\}.$$

This definition is generalized as follows.

**Definition 2.1.** *a) Let* $[a] \in \mathbf{IR}$. *Then*

$$\max\{0, [a]\} := [\max\{0, \underline{a}\}, \max\{0, \overline{a}\}].$$

*b) Let* $[a] = ([a_i]) \in \mathbf{IR}^n$. *Then*

$$\max\{o, [a]\} := (\max\{0, [a_i]\}).$$

**Lemma 2.1.** *Let* $[a], [b] \in \mathbf{IR}$ *satisfying* $[a] \subseteq [b]$. *Then*

*a)* $\max\{0, [a]\} \subseteq \max\{0, [b]\}$.

*b)* $d(\max\{0, [a]\}) \leq d([a])$.

**Lemma 2.2.** *Let* $[a], [b] \in \mathbf{IR}$. *Then*

$$q(\max\{0, [a]\}, \max\{0, [b]\}) \leq q([a], [b]).$$

*Proof.* In [10], Lemma 2 it was shown that the Minty map is Lipschitz continuous with Lipschitz constant equal to one: For $x, y \in \mathbf{R}$ we have

$$|\max\{0, x\} - \max\{0, y\}| \leq |x - y|.$$

Therefore $q(\max\{0,[a]\},\max\{0,[b]\}) = \max\{|\max\{0,\underline{a}\} - \max\{0,\underline{b}\}|, |\max\{0,\overline{a}\} - \max\{0,\overline{b}\}|\} \leq q([a],[b])$. □

## 2.2. M- and H-matrices

Let $Z^{n\times n}$ denote the set of real matrices with nonpositive off-diagonal entries: $A = (a_{ij}) \in Z^{n\times n} \Leftrightarrow a_{ij} \leq 0$ if $i \neq j$.

**Definition 2.2.** $A \in Z^{n\times n}$ *is an M-matrix if* $A^{-1}$ *exists satisfying* $A^{-1} \geq O$.

The diagonal elements of an M-matrix $A = (a_{ij})$ are necessarily positive: $a_{ii} > 0$, $i = 1, \ldots, n$.

**Theorem 2.1** *Let* $A \in Z^{n\times n}$. *Then the following two statements are equivalent:*

*a)* $A^{-1}$ *exists and* $A^{-1} \geq O$.

*b) There exists a vector* $u > o$ *such that* $Au > o$.

For the proof we refer to [8]. A useful corollary is the following one.

**Corollary 2.1.** *Let* $A \leq B \in Z^{n\times n}$. *If* $A$ *is an M-matrix, then* $B$ *is an M-matrix, too.*

The proof can be easily performed by using part b) of Theorem 2.1.

**Definition 2.3.** *For* $A = (a_{ij}) \in \mathbf{R}^{n\times n}$ *we define the comparison matrix* $\langle A\rangle = (c_{ij}) \in \mathbf{R}^{n\times n}$ *by setting*

$$c_{ij} := \begin{cases} -|a_{ij}| & \text{if } i \neq j, \\ |a_{ij}| & \text{if } i = j. \end{cases}$$

*If $\langle A \rangle$ is an M-matrix, then $A$ is called an H-matrix.*

Note that the diagonal elements of an H-matrix are different from zero. However, they can be positive or negative. It is obvious that every M-matrix is an H-matrix, but not vice versa.

**Definition 2.4.** *An interval matrix $[A] \in \mathbf{IR}^{n\times n}$ is called*

*a) regular, if all $A \in [A]$ are nonsingular;*

*b) an M-matrix, if all $A \in [A]$ are M-matrices;*

*c) an H-matrix, if all $A \in [A]$ are H-matrices.*

Interval M- and H-matrices have been introduced in [2] and [14], respectively.

**Definition 2.5.** For $[A] = ([a_{ij}]) \in \mathbf{IR}^{n\times n}$ *we define the comparison matrix* $\langle [A] \rangle = (c_{ij}) \in \mathbf{R}^{n\times n}$ *using* (4) *by setting*

$$c_{ij} := \begin{cases} -|[a_{ij}]| & \text{if } i \neq j, \\ \langle [a_{ij}] \rangle & \text{if } i = j. \end{cases}$$

Using the comparison matrix and Corollary 2.1 we get the following lemma.

**Lemma 2.3.** *Let $[A] \in \mathbf{IR}^{n\times n}$.*

*a) If $\underline{A}$ is an M-matrix and if $\overline{A} \in Z^{n\times n}$, then $[A]$ is an M-matrix.*

*b) If $\langle [A] \rangle$ is an M-matrix, then $[A]$ is an H-matrix.*

## *2.3. Regular splittings*

**Definition 2.6.** *Let $A, B, C \in \mathbf{R}^{n\times n}$. Then $A = B - C$ is a regular splitting of $A$ if $C \geq O$ and $B$ is nonsingular with $B^{-1} \geq O$.*

**Theorem 2.2.** *Assume that $A \in \mathbf{R}^{n\times n}$ is nonsingular, that $A^{-1} \geq O$ and that $A = B - C$ is a regular splitting of $A$. Then $\rho(B^{-1}C) < 1$, where $\rho(\cdot)$ denotes the spectral radius of a matrix.*

Regular splittings were introduced in [18], where one can also find the proof of Theorem 2.2.

### 3. Iterative methods for the enclosure of the solution set $\Sigma([A],[b])$

In this section we assume that the reader is familiar with the concept of $P$ contractions for proving the convergence of a fixed point iteration to a unique fixed point for an arbitrary starting interval vector. For the details we refer to Chapter 11 in [2], especially to Theorem 11.4, Theorem 11.5 and Corollary 11.6.

#### *3.1. The total step method (T)*

**Theorem 3.1.** *Let $[b] \in \mathbf{IR}^n$ and $[A] \in \mathbf{IR}^{n\times n}$. Assume that $[A]$ is an H-matrix satisfying $\underline{a}_{ii} > 0$, $i = 1, \ldots, n$. We define the interval matrices*

$$[D]^{-1} := \begin{pmatrix} \frac{1}{[a_{11}]} & & O \\ & \ddots & \\ O & & \frac{1}{[a_{nn}]} \end{pmatrix}, \quad [R] := -\begin{pmatrix} 0 & [a_{12}] & \cdots & [a_{1n}] \\ [a_{21}] & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & [a_{n-1n}] \\ [a_{n1}] & \cdots & [a_{nn-1}] & 0 \end{pmatrix},$$

*the function*

$$f(x; D^{-1}, R, b) := \max\{o, D^{-1}(Rx - b)\}, \quad x \in \mathbf{R}^n$$

*for arbitrary but fixed $D^{-1} \in [D]^{-1}$, $R \in [R]$, $b \in [b]$ and its interval extension*

$$f([x]; [D]^{-1}, [R], [b]) := \max\{o, [D]^{-1}([R][x] - [b])\}, \quad [x] \in \mathbf{IR}^n.$$

*(Since $[D]^{-1}, [R], [b]$ are fixed, we simply write $f([x])$ instead of $f([x]; [D]^{-1}, [R], [b])$ in the following).*

*Then the following holds:*

*a) The iteration (total step method)*

$$(T) \quad \begin{cases} [x^0] \in \mathbf{IR}^n, & \underline{x}^0 \geq o \text{ arbitrary}, \\ [x^{k+1}] := f([x^k]), & k = 0, 1, 2, \ldots, \end{cases}$$

*converges to a unique interval vector $[x^*]$ satisfying $[x^*] = f([x^*])$.*

*b) $\Sigma([A], [b]) \subseteq [x^*]$.*

*Proof. a)* Let $[x], [y] \in \mathbf{IR}^n$. Then we have by Lemma 2.2

$$q(f([x]), f([y])) \leq q\Big([D]^{-1}([R][x] - [b]), [D]^{-1}([R][y] - [b])\Big).$$

Using $|[D]^{-1}| = \langle [D] \rangle^{-1}$ we get

$$q(f([x]), f([y])) \leq \langle [D] \rangle^{-1} |[R]| q([x], [y]).$$

Define $B := \langle [D] \rangle$, $C := |[R]|$ and $\tilde{A} := B - C$. Obviously $B^{-1} \geq O$ and $C \geq O$. Since $\tilde{A} = \langle [A] \rangle$ and since $[A]$ is an H-matrix, it follows $\tilde{A}^{-1} \geq O$. According to Theorem 2.2 (applied to $\tilde{A}$) we can conclude that $\rho(B^{-1}C) < 1$. Hence, $f$ is a $P$ contraction with $P = B^{-1}C$ and assertion $a$) follows by Theorem 11.4 in [2].

Note that we have in the proof of $a$) not yet used $\underline{a}_{ii} > 0$, $i = 1, \ldots, n$. Furthermore we do not use $\underline{x}^0 \geq o$. However, if $\underline{x}^0 \not\geq o$ then $\underline{x}^1 \geq o$.

$b$) Let $x \in \Sigma([A], [b])$. Then there is an $A = (a_{ij}) \in [A]$ and a $b \in [b]$ satisfying (2). We define

$$D := \begin{pmatrix} a_{11} & & O \\ & \ddots & \\ O & & a_{nn} \end{pmatrix}, \quad R := -\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1n} \\ a_{n1} & \cdots & a_{nn-1} & 0 \end{pmatrix}.$$

Then we have

$$o = \min\{x, b + Ax\} = \min\{x, D(D^{-1}b + x - D^{-1}Rx)\}.$$

Since $a_{ii} \geq \underline{a}_{ii} > 0$, $i = 1, \ldots, n$, this equation is equivalent to

$$o = \min\{x, D^{-1}b + x - D^{-1}Rx\} = \min\{x, x + D^{-1}(b - Rx)\}$$

and to

$$o = x + \min\{o, D^{-1}(b - Rx)\} = x - \max\{o, D^{-1}(Rx - b)\},$$

which leads to

$$x = \max\{o, D^{-1}(Rx - b)\} = f(x; D^{-1}, R, b).$$

Using Corollary 11.6 in [2], we get $x \in [x^*]$. □

**Remark 3.1.** $a$) As was already mentioned above the assumption $\underline{a}_{ii} > 0$, $i = 1, \ldots, n$ was not used in the proof of part $a$) in the preceding theorem. In fact, without this assumption the total step method $(T)$ can be convergent to a fixed point $[x^*]$, although the set (3) is empty. Consider, for example, the case $[A] = -I$ and $[b] = -e$, where all components of $e$ are equal to one. The corresponding LCP has no solution. On the other hand even for arbitrary $[x^0]$ we obtain the fixed point $o$.

$b$) Consider the special case that $[A] = A \in \mathbf{R}^{n \times n}$, that $A$ is an H-matrix with $a_{ii} > 0$, $i = 1, \ldots, n$ and that $[b] = b \in \mathbf{R}^n$. Then, as a special case of Theorem 3.1, it follows that the LCP defined by $A$ and $b$ has a unique solution $x$. This fact is well-known and was proved in [6] (Theorem 3.3.15) by different means. See also Lemma 3.5 in [4].

*c*) Let $[x^*] = ([x_i^*]) \in \mathbf{IR}^n$ be the limit of the total step method $(T)$ from the preceding theorem. Then, in general, there is an interval vector $[x] = ([x_i]) \in \mathbf{IR}^n$ with the property that it also holds $\Sigma([A],[b]) \subseteq [x]$ and furthermore $[x_i] \subseteq [x_i^*]$, $i = 1, \dots, n$, where at least for one $i$ strict inclusion holds. In this sense $[x^*]$ is not an optimal enclosure of $\Sigma([A],[b])$ in this case.

*Example 3.1.* Let

$$[A] = \begin{pmatrix} [4,5] & [1,2] \\ [-1,0] & [2,3] \end{pmatrix}, \quad [b] = \begin{pmatrix} [-2,-1] \\ [-1,1] \end{pmatrix}.$$

It is easy to verify that $[A]$ is an H-matrix and that the unique fixed point of the total step method $(T)$ applied to $[A]$ and $[b]$ is

$$[x^*] = \begin{pmatrix} [0,\frac{1}{2}] \\ [0,\frac{3}{4}] \end{pmatrix}.$$

However, we will show that for any $t \in [0,\frac{1}{2}]$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} := \begin{pmatrix} t \\ \frac{3}{4} \end{pmatrix} \notin \Sigma([A],[b]).$$

Let $b \in [b]$ and $A \in [A]$. Then for the second component of $b + A \cdot x(t)$ we have

$$b_2 + (A \cdot x(t))_2 \geq -1 - t + \frac{3}{2}.$$

Since $x_2(t) = \frac{3}{4} > 0$, by complementarity $-1 - t + \frac{3}{2} = 0$ has to hold, and therefore $t = \frac{1}{2}$. However, for $t = \frac{1}{2}$ we have $x_1(t) > 0$ and

$$b_1 + (A \cdot x(t))_1 \geq -2 + 2 + \frac{3}{4} > 0,$$

which contradicts the complementarity.

Under additional assumptions on the matrix $[A]$ we now show that the case described in part *c*) of Remark 3.1 and illustrated in the preceding example cannot occur.

**Theorem 3.2.** *Let $[A] \in \mathbf{IR}^{n \times n}$ be an M-matrix and $[b] \in \mathbf{IR}^n$. Let $u \in \mathbf{R}^n$ be the unique solution of the LCP defined by $\overline{b}$ and $\overline{A}$ and let correspondingly $v \in \mathbf{R}^n$ be the unique solution of the LCP defined by $\underline{b}$ and $\underline{A}$. Then it holds:*

*a*) $inf(\Sigma([A],[b])) = u$ *and* $sup(\Sigma([A],[b])) = v$.

*b*) $[x^*] = [u,v]$ *for the fixed point* $[x^*]$ *of* $(T)$.

*Proof.* Note that by part *b*) of Remark 3.1 the LCPs under consideration have unique solutions $u \geq o$ and $v \geq o$, respectively.

*a*) Let $A = (a_{ij}) \in [A]$, $b = (b_i) \in [b]$ and let $x \geq o$ be the unique solution of the LCP defined by $A$ and $b$. Setting $\Im := \{i : u_i = 0\}$ we define $\tilde{A} = (\tilde{a}_{ij})$ and $\tilde{b} = (\tilde{b}_i)$ by

$$\tilde{a}_{ij} := \begin{cases} \overline{a}_{ij} & \text{if } i \notin \Im, \\ 0 & \text{if } i \in \Im \text{ and } j \neq i, \\ \overline{a}_{ii} & \text{if } i \in \Im \text{ and } j = i, \end{cases} \quad \tilde{b}_i := \begin{cases} \overline{b}_i & \text{if } i \notin \Im, \\ 0 & \text{if } i \in \Im, \end{cases}$$

$i, j = 1, \ldots, n$. For example, let $n = 5$ and $\Im = \{2, 4\}$ then

$$\tilde{A} = \begin{pmatrix} \overline{a}_{11} & \overline{a}_{12} & \overline{a}_{13} & \overline{a}_{14} & \overline{a}_{15} \\ 0 & \overline{a}_{22} & 0 & 0 & 0 \\ \overline{a}_{31} & \overline{a}_{32} & \overline{a}_{33} & \overline{a}_{34} & \overline{a}_{35} \\ 0 & 0 & 0 & \overline{a}_{44} & 0 \\ \overline{a}_{51} & \overline{a}_{52} & \overline{a}_{53} & \overline{a}_{54} & \overline{a}_{55} \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} \overline{b}_1 \\ 0 \\ \overline{b}_3 \\ 0 \\ \overline{b}_5 \end{pmatrix}.$$

We will show

$$\tilde{b} + \tilde{A}u = o \leq \tilde{b} + \tilde{A}x. \tag{9}$$

Case 1: If $i \in \Im$, then

$$(\tilde{b} + \tilde{A}u)_i = 0 \leq \overline{a}_{ii}x_i = (\tilde{b} + \tilde{A}x)_i.$$

Case 2: If $i \notin \Im$, then

$$\overline{b}_i + \sum_{j=1}^{n} \overline{a}_{ij}u_j = 0 \leq b_i + \sum_{j=1}^{n} a_{ij}x_j \leq \overline{b}_i + \sum_{j=1}^{n} \overline{a}_{ij}x_j.$$

Because of $\overline{A} \leq \tilde{A} \in Z^{n \times n}$ we can conclude according to Corollary 2.1 that $\tilde{A}$ is an M-matrix. Hence, it follows from (9) that $u \leq x$.

Setting $\Re := \{i : x_i = 0\}$ we define $\hat{A} = (\hat{a}_{ij})$ and $\hat{b} = (\hat{b}_i)$ by

$$\hat{a}_{ij} := \begin{cases} \underline{a}_{ij} & \text{if } i \notin \Re, \\ 0 & \text{if } i \in \Re \text{ and } j \neq i, \\ \underline{a}_{ii} & \text{if } i \in \Re \text{ and } j = i, \end{cases} \quad \hat{b}_i := \begin{cases} \underline{b}_i & \text{if } i \notin \Re, \\ 0 & \text{if } i \in \Re, \end{cases}$$

$i, j = 1, \ldots, n$. We will show

$$\hat{b} + \hat{A}x \leq o \leq \hat{b} + \hat{A}v. \tag{10}$$

Case 1: If $i \in \Re$, then

$$(\hat{b} + \hat{A}x)_i = 0 \le \underline{a}_{ii} v_i = (\hat{b} + \hat{A}v)_i.$$

Case 2: If $i \notin \Re$, then

$$\underline{b}_i + \sum_{j=1}^{n} \underline{a}_{ij} x_j \le b_i + \sum_{j=1}^{n} a_{ij} x_j = 0 \le \underline{b}_i + \sum_{j=1}^{n} \underline{a}_{ij} v_j.$$

Because of $\underline{A} \le \hat{A} \in Z^{n\times n}$ we can again conclude according to Corollary 2.1 that $\hat{A}$ is an M-matrix. Hence, it follows from (10) that $x \le v$.

*b)* Let $[D]^{-1}$, $[R]$ and $f([x])$ be defined as in Theorem 3.1. Since $[A]$ is an M-matrix we have $\underline{R} \ge O$ and

$$f([u,v]) = \max\{o, [D]^{-1}[\underline{R}u - \overline{b}, \overline{R}v - \underline{b}]\} =: [c,d].$$

For fixed $i \in \{1,\ldots,n\}$ we have to consider three cases and we will show that in every case $[u_i, v_i] = [c_i, d_i]$ holds. Note in the following that $[\underline{r}_{ij}, \overline{r}_{ij}] = [r_{ij}] = -[a_{ij}] = [-\overline{a}_{ij}, -\underline{a}_{ij}]$ if $i \ne j$.

Case 1: $(\overline{R}v - \underline{b})_i \le 0$.

Then $c_i = d_i = 0$ and we have to show $v_i = 0$. Assume that $v_i > 0$. Then by the complementarity one would have $\underline{b}_i + (\underline{A}v)_i = 0$. But this is not true, since $(\overline{R}v - \underline{b})_i \le 0$ is equivalent to

$$\underline{b}_i + \sum_{j=1, j\ne i}^{n} \underline{a}_{ij} v_j \ge 0,$$

and if $v_i > 0$ one has

$$\underline{b}_i + \sum_{j=1}^{n} \underline{a}_{ij} v_j > \underline{b}_i + \sum_{j=1, j\ne i}^{n} \underline{a}_{ij} v_j \ge 0.$$

Case 2: $(\underline{R}u - \overline{b})_i > 0$.

Then we have to show that $u_i = c_i$ and $v_i = d_i$ where

$$c_i = \frac{1}{\overline{a}_{ii}} \left(\underline{R}u - \overline{b}\right)_i \text{ and } d_i = \frac{1}{\underline{a}_{ii}} \left(\overline{R}v - \underline{b}\right)_i.$$

Assume that $u_i = 0$. Then by the complementarity one would have $\overline{b}_i + (\overline{A}u)_i \ge 0$. However, since $(\underline{R}u - \overline{b})_i > 0$ is equivalent to

$$\overline{b}_i + \sum_{j=1, j\ne i}^{n} \overline{a}_{ij} u_j < 0$$

one has

$$\overline{b}_i + \sum_{j=1}^{n} \overline{a}_{ij} u_j = \overline{b}_i + \sum_{j=1, j\neq i}^{n} \overline{a}_{ij} u_j < 0$$

if $u_i = 0$. Therefore, $u_i > 0$ and $\overline{b}_i + \sum_{j=1}^{n} \overline{a}_{ij} u_j = 0$. Hence, $u_i = c_i$. Since $0 < u_i \leq v_i$, we have $\underline{b}_i + \sum_{j=1}^{n} \underline{a}_{ij} v_j = 0$. Hence, $v_i = d_i$.

Case 3: $(\underline{R}u - \overline{b})_i \leq 0$ and $(\overline{R}v - \underline{b})_i > 0$.

Then one can show as in Case 1 that $c_i = 0 = u_i$ and

$$d_i = \frac{1}{\underline{a}_{ii}} (\overline{R}v - \underline{b}_i) = v_i$$

as in Case 2. □

*3.2. Successive overrelaxation (SOR)*

We now consider a method for enclosing $\Sigma([A],[b])$ which can be considered as a generalization of the well-known $(SOR)$ method. For $\omega = 1$ this method specializes to an iterative method which may be considered as a generalization of the Gauss-Seidel- or single step method $(S)$.

**Theorem 3.3.** *Let $[b] \in \mathbf{IR}^n$ and $[A] \in \mathbf{IR}^{n\times n}$. Assume that $[A]$ is an H-matrix satisfying $\underline{a}_{ii} > 0$, $i = 1, \ldots, n$. We define*

$$[D]^{-1} := \begin{pmatrix} \frac{1}{[a_{11}]} & & O \\ & \ddots & \\ O & & \frac{1}{[a_{nn}]} \end{pmatrix}, \quad [L] := -\begin{pmatrix} 0 & \cdots & \cdots & 0 \\ [a_{21}] & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ [a_{n1}] & \cdots & [a_{nn-1}] & 0 \end{pmatrix},$$

$$[U] := -\begin{pmatrix} 0 & [a_{12}] & \cdots & [a_{1n}] \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \cdots & \ddots & [a_{n-1n}] \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \text{ and } [R] := [L] + [U].$$

*Let $[x^*]$ be the unique fixed point of the equation*

$$[x] = \max\{o, [D]^{-1}([R][x] - [b])\}$$

*(see Theorem 3.1). Then the following holds:*

*a) For*

$$0 < \omega < \frac{2}{1 + \rho(\langle [D] \rangle^{-1} |[R]|)} =: \omega_0$$

*the iteration*

$$(SOR) \begin{cases} [x^0] \in \mathbf{IR}^n, \underline{x}^0 \geq o \text{ arbitrary}, \\ [x^{k+1}] := \max\{o, (1-\omega)[x^k] \; + \omega[D]^{-1}\big([U][x^k] + [L][x^{k+1}] - [b]\big)\}, \\ k = 0, 1, 2, \ldots, \end{cases}$$

*converges to a unique interval vector* $[\tilde{x}]$ *satisfying*

$$[\tilde{x}] = \max\{o, (1-\omega)[\tilde{x}] + \omega[D]^{-1}([U][\tilde{x}] + [L][\tilde{x}] - [b])\}.$$

*b)* $\Sigma([A], [b]) \subseteq [\tilde{x}]$.

*c)* $[\tilde{x}] = [x^*]$ if $0 < \omega \leq 1$.

*d)* $[\tilde{x}] \supseteq [x^*]$ if $1 < \omega < \omega_0$.

*Proof. a)* For fixed $D \in [D]$, $L \in [L]$, $U \in [U]$, $b \in [b]$ we define the function

$$f(x; D^{-1}, L, U, b) := \max\{o, (1-\omega)x + \omega D^{-1}(Ux + Lx - b)\}, x \in \mathbf{R}^n,$$

its interval extension

$$f([x]) := \max\{o, (1-\omega)[x] + \omega[D]^{-1}([U][x] + [L][x] - [b])\}, \quad [x] \in \mathbf{IR}^n$$

and the function

$$g([x], [y]) := \max\{o, (1-\omega)[y] + \omega[D]^{-1}([U][y] + [L][x] - [b])\},$$

$[x], [y] \in \mathbf{IR}^n$. Obviously

$$g([x], [x]) = f([x])$$

holds. Using Lemma 2.2 and $|[D]^{-1}| = \langle [D] \rangle^{-1}$, we can conclude that

$$q(g([x], [z]), g([y], [z])) \leq Jq([x], [y])$$

where $J = \omega \langle [D] \rangle^{-1} |[L]|$ and

$$q(g([z], [x]), g([z], [y])) \leq Gq([x], [y])$$

with $G = |1-\omega|I + \omega \langle [D] \rangle^{-1} |[U]|$. We show that

$$\rho(J) < 1 \quad \text{and} \quad \rho((I-J)^{-1}G) < 1$$

hold. Then using Theorem 11.5 in [2] we can conclude assertion $a$).

To verify $\rho(J) < 1$, we set $B := \frac{1}{\omega}\langle[D]\rangle$, $C := |[L]|$ and $\tilde{A} := B - C$. Since $\tilde{A} \in Z^{n\times n}$ is a triangular matrix with positive diagonal entries, $\tilde{A}$ is an M-matrix. Using Theorem 2.2 we get $\rho(J) < 1$. Setting $\tilde{H}_\omega := (I-J)^{-1}G$ we have

$$\tilde{H}_\omega = \left(I - \omega\langle[D]\rangle^{-1}|[L]|\right)^{-1}\left(|1-\omega|I + \omega\langle[D]\rangle^{-1}|[U]|\right).$$

Using Lemma 2 in [12] with $B := \langle[D]\rangle^{-1}|[R]|$ and $\tilde{L}_\omega := \tilde{H}_\omega$ it holds $\rho(\tilde{H}_\omega) < 1$.

$b$) Let $x \in \Sigma([A],[b])$. Then there are $D \in [D]$, $L \in [L]$, $U \in [U]$ and $b \in [b]$ satisfying

$$\begin{aligned} o &= \min\{x, b + (D - L - U)x\} \\ &= \min\{x, b + \left(\omega^{-1}D - L - U + (1-\omega^{-1})D\right)x\} \\ &= \min\{x, \omega^{-1}D\omega D^{-1}\left(\omega^{-1}Dx + b - Lx - Ux + (1-\omega^{-1})Dx\right)\}. \end{aligned}$$

Because of $\underline{a}_{ii} > 0$, $i = 1, \ldots, n$, this is equivalent to

$$o = \min\{x, x + \omega D^{-1}\left(b - Lx - Ux + (1-\omega^{-1})Dx\right)\}.$$

This equation can be rewritten as

$$\begin{aligned} o &= x + \min\{o, \omega D^{-1}\left(b - Lx - Ux + (1-\omega^{-1})Dx\right)\} \\ &= x - \max\{o, \omega D^{-1}\left(Lx + Ux - b - (1-\omega^{-1})Dx\right)\}, \end{aligned}$$

which leads to

$$x = \max\{o, (1-\omega)x + \omega D^{-1}(Ux + Lx - b)\} = f(x; D^{-1}, L, U, b).$$

Using Corollary 11.6 in [2] we get $x \in [\tilde{x}]$.

$c$) We have

$$\begin{aligned} [x^*] &= \max\{o, [D]^{-1}([R][x^*] - [b])\} \\ &= \max\{o, [D]^{-1}([L][x^*] + [U][x^*] - [b])\}. \end{aligned}$$

Let $i \in \{1, \ldots, n\}$. Then we use the following notation

$$[c_i] := \frac{1}{[a_{ii}]}\left(-\sum_{j=1, j\neq i}^{n} [a_{ij}][x_j^*] - [b_i]\right). \tag{11}$$

Case 1: Suppose $[x_i^*] = 0$ then $\overline{c}_i \leq 0$ and we can conclude that

$$[x_i^*] = 0 = \max\{0, (1-\omega)[x_i^*] + \omega[c_i]\}.$$

Case 2: Suppose $\underline{x}_i^* = 0$ and $\overline{x}_i^* > 0$. Then $\underline{c}_i \leq 0$ and $\overline{c}_i > 0$. Since $0 < \omega \leq 1$ we have the following equalities

$$\begin{aligned}&\max\{0, (1-\omega)[c_i] + \omega[c_i]\} = [0, \overline{c}_i]\\ &\quad = \max\{0, (1-\omega)\max\{0, [c_i]\} + \omega[c_i]\}\end{aligned}$$

and

$$\begin{aligned}[x_i^*] &= \max\{0, [c_i]\} = \max\{0, (1-\omega)[c_i] + \omega[c_i]\}\\ &= \max\{0, (1-\omega)\max\{0, [c_i]\} + \omega[c_i]\}\\ &= \max\{0, (1-\omega)[x_i^*] + \omega[c_i]\}.\end{aligned}$$

Case 3: Suppose $\underline{x}_i^* > 0$. Then it is $\underline{c}_i > 0$ and it holds

$$\max\{0, (1-\omega)[c_i] + \omega[c_i]\} = \max\{0, (1-\omega)\max\{0, [c_i]\} + \omega[c_i]\}.$$

Then $[x_i^*] = \max\{0, (1-\omega)[x_i^*] + \omega[c_i]\}$ follows as in Case 2. Due to the uniqueness proved in *a*) we have $[x^*] = [\tilde{x}]$.

*d*) We start (*SOR*) with $[\tilde{x}^0] := [x^*]$ and show that

$$[x^*] \subseteq [\tilde{x}^k], \quad k = 0, 1, 2, \ldots.$$

Then using *a*) we can conclude that $[x^*] \subseteq [\tilde{x}] = \lim_{k\to\infty} [\tilde{x}^k]$.

Let $i \in \{1, \ldots, n\}$ and suppose that for some $k \geq 0$

$$\begin{aligned}&[x^*] \subseteq [\tilde{x}^k] \quad \text{and}\\ &[x_j^*] \subseteq [\tilde{x}_j^{k+1}] \quad \text{for } j = 1, \ldots, i-1.\end{aligned}$$

We will use the notation (11) and

$$[d_i] := \frac{1}{[a_{ii}]}\left(-\sum_{j=1}^{i-1}[a_{ij}][\tilde{x}_j^{k+1}] - \sum_{j=i+1}^{n}[a_{ij}][\tilde{x}_j^k] - [b_i]\right).$$

Case 1: Suppose $[x_i^*] = 0$. Then $0 = [x_i^*] = \max\{0, [c_i]\}$ implies

$$\begin{aligned}0 &= \max\{0, (1-\omega)\cdot 0 + \omega[c_i]\}\\ &\in \max\{0, (1-\omega)[\tilde{x}_i^k] + \omega[d_i]\} = [\tilde{x}_i^{k+1}].\end{aligned}$$

Case 2: Suppose $\underline{x}_i^* = 0$ and $\overline{x}_i^* > 0$. Since $\omega > 1$ we have

$$\begin{aligned}[x_i^*] &= \max\{0, [c_i]\} \subseteq \max\{0, \omega[c_i]\} = \max\{0, (1-\omega)\cdot[x_i^*] + \omega[c_i]\} \\ &\subseteq \max\{0, (1-\omega)\cdot[\tilde{x}_i^k] + \omega[d_i]\} = [\tilde{x}_i^{k+1}].\end{aligned}$$

Case 3: Suppose $\underline{x}_i^* > 0$. Then it is $\underline{c}_i > 0$ and it holds

$$[c_i] = \max\{0, [c_i]\}.$$

Hence, since $\omega > 1$ we get

$$\begin{aligned}[x_i^*] &= \max\{0, [c_i]\} \subseteq \max\{0, (1-\omega)[c_i] + \omega[c_i]\} \\ &= \max\{0, (1-\omega)[x_i^*] + \omega[c_i]\} \subseteq [\tilde{x}_i^{k+1}].\end{aligned}$$

□

The special case $\omega = 1$ of the $(SOR)$ method is usually called Gauss-Seidel- or single step method $(S)$. The preceding theorem shows that for all $0 < \omega \leq 1$ the $(SOR)$ method and the single step method have the same limit, whereas for $\omega > 1$ the limit of the $(SOR)$ method is with respect to inclusion in general bigger. Subsequently we only consider the case $\omega = 1$.

### *3.3. The symmetric single step method (SS)*

**Theorem 3.4.** *Let $[b] \in \mathbf{IR}^n$ and let $[A] \in \mathbf{IR}^{n\times n}$ be an H-matrix. We define $[L]$, $[U]$, $[D]^{-1}$ and $[R]$ as in Theorem 3.3. Then, the sequence $\{[u^k]\}_{k=0}^{\infty}$ calculated according to the iteration method (symmetric single step method)*

$$(SS)\begin{cases} [u^{k+\frac{1}{2}}] := \max\Big\{o, [D]^{-1}\Big([U][u^k] + [L][u^{k+\frac{1}{2}}] - [b]\Big)\Big\}, \\ [u^{k+1}] := \max\Big\{o, [D]^{-1}\Big([U][u^{k+1}] + [L][u^{k+\frac{1}{2}}] - [b]\Big)\Big\}, \\ \qquad k = 0, 1, 2, \ldots, \end{cases}$$

*converges for all interval vectors* $[u^0] \in \mathbf{IR}^n$ to $[x^*]$, *where* $[x^*]$ *is the unique fixed point of the equation*

$$[x] = \max\Big\{o, [D]^{-1}([R][x] - [b])\Big\}.$$

*Proof.* From $[R][x^*] = [U][x^*] + [L][x^*]$ it follows as in the proof of Theorem 3.3 that $q([u^{k+1}], [x^*]) \leq Pq([u^k], [x^*])$, where $P :=$

$$\Big(I - \langle[D]\rangle^{-1}|[U]|\Big)^{-1}\langle[D]\rangle^{-1}|[L]|\Big(I - \langle[D]\rangle^{-1}|[L]|\Big)^{-1}\langle[D]\rangle^{-1}|[U]|$$

is the symmetric Gauss-Seidel iteration matrix of the M-matrix $\langle[A]\rangle$ which is known to satisfy $\rho(P) < 1$ ([3]). Since

$$q\big([u^{k+1}], [x^*]\big) \leq P^{k+1} q\big([u^0], [x^*]\big),$$

it follows that $\lim_{k\to\infty} [u^k] = [x^*]$. □

## 4. Modifications of (T), (S) and (SS)

In this section we consider modifications of the preceding iterative methods which are based on the fact that if for any of these methods one is starting with an interval vector containing the limit, then all iterates contain the limit. Therefore the enclosure of the limit might be improved by forming intersections after each iteration step.

### 4.1. (T), (S) and (SS) with intersection

**Theorem 4.1.** *Let* $[b] \in \mathbf{IR}^n$ *and let* $[A] \in \mathbf{IR}^{n\times n}$ *be an H-matrix with* $\underline{a}_{ii} > 0$, $i = 1, \dots, n$. *We define* $[L]$, $[U]$, $[D]^{-1}$ *and* $[R]$ *as in Theorem 3.3. Furthermore, let* $[x^*]$ *be the (due to Theorem 3.1) unique fixed point of*

$$f([x]) = \max\left\{o, [D]^{-1}([R][x] - [b])\right\}.$$

*We assume that we have an interval vector [start]* $\in \mathbf{IR}^n$ *satisfying* $[x^*] \subseteq [start]$. *We consider the iteration methods:*

α) *Total step method with intersection.*

$$(TI)\begin{cases} [t^0] := [start] \\ [t^{k+1}] := [t^k] \cap \max\{o, [D]^{-1}([R][t^k] - [b])\}. \end{cases}$$

β) *Single step method with intersection.*

$$(SI)\begin{cases} [s^0] := [start] \\ \mathit{for}\ i = 1\ \mathit{to}\ n\ \mathit{do} \\ [s_i^{k+1}] := \\ [s_i^k] \cap \max\left\{0, \frac{1}{[a_{ii}]}\left(-\sum\limits_{j=1}^{i-1}[a_{ij}][s_j^{k+1}] - \sum\limits_{j=i+1}^{n}[a_{ij}][s_j^k] - [b_i]\right)\right\}. \end{cases}$$

γ) *Symmetric single step method with intersection.*

$$(SSI)\begin{cases} [z^0] := [start] \\ \mathit{for}\ i = 1\ \mathit{to}\ n\ \mathit{do} \\ [z_i^{k+\frac{1}{2}}] := \\ \quad [z_i^k] \cap \max\left\{0, \frac{1}{[a_{ii}]}\left(-\sum\limits_{j=1}^{i-1}[a_{ij}][z_j^{k+\frac{1}{2}}] - \sum\limits_{j=i+1}^{n}[a_{ij}][z_j^k] - [b_i]\right)\right\} \\ \mathit{for}\ i = n\ \mathit{downto}\ 1\ \mathit{do} \\ [z_i^{k+1}] := [z_i^{k+\frac{1}{2}}] \cap \\ \qquad \max\left\{0, \frac{1}{[a_{ii}]}\left(-\sum\limits_{j=1}^{i-1}[a_{ij}][z_j^{k+\frac{1}{2}}] - \sum\limits_{j=i+1}^{n}[a_{ij}][z_j^{k+1}] - [b_i]\right)\right\}. \end{cases}$$

*Then it holds:*

*a)* $[t^k] \supseteq [s^k] \supseteq [z^k], \quad k = 0, 1, 2, \ldots$ .

*b)* $\lim_{k\to\infty} [t^k] = \lim_{k\to\infty} [s^k] = \lim_{k\to\infty} [z^k] = [x^*]$.

*c) If* $[A] = A \in \mathbf{R}^{n\times n}$ *and* $[b] = b \in \mathbf{R}^n$, *then* (*TI*), (*SI*) and (*SSI*) are convergent to the unique solution of the LCP defined by A and b.

*Proof. a*) The proof is by induction. We have

$$[t^0] = [s^0] = [z^0] = [start].$$

First, we show $[t^{k+1}] \supseteq [s^{k+1}]$ assuming $[t^k] \supseteq [s^k]$. It is

$$\begin{aligned}[s_1^{k+1}] &= \max\left\{0, \frac{1}{[a_{11}]}\left(-\sum_{j=2}^{n}[a_{1j}][s_j^k] - [b_1]\right)\right\} \cap [s_1^k] \\ &\subseteq \max\left\{0, \frac{1}{[a_{11}]}\left(-\sum_{j=2}^{n}[a_{1j}][t_j^k] - [b_1]\right)\right\} \cap [t_1^k] = [t_1^{k+1}].\end{aligned}$$

Since $[s_1^{k+1}] \subseteq [s_1^k] \subseteq [t_1^k]$ we have

$$\begin{aligned}[s_2^{k+1}] &= \max\left\{0, \frac{1}{[a_{22}]}\left(-[a_{21}][s_1^{k+1}] - \sum_{j=3}^{n}[a_{2j}][s_j^k] - [b_2]\right)\right\} \cap [s_2^k] \\ &\subseteq \max\left\{0, \frac{1}{[a_{22}]}\left(-[a_{21}][t_1^k] - \sum_{j=3}^{n}[a_{2j}][t_j^k] - [b_2]\right)\right\} \cap [t_2^k] \\ &= [t_2^{k+1}].\end{aligned}$$

Continuing in this manner we can show $[t_i^{k+1}] \supseteq [s_i^{k+1}]$ for $i = 1, \ldots, n$.

Finally, we verify $[z^{k+1}] \subseteq [s^{k+1}]$ assuming $[z^k] \subseteq [s^k]$. It is easy to see that $[z^{k+\frac{1}{2}}] \subseteq [s^{k+1}]$ since both interval vectors are defined in the same way. Then using $[a] \cap [b] \subseteq [b]$ we get

$$[z^{k+1}] \subseteq [z^{k+\frac{1}{2}}] \subseteq [s^{k+1}].$$

*b*) With $[t^0] \supseteq [x^*]$ and assuming $[t^k] \supseteq [x^*]$ we have

$$[t^{k+1}] = f([t^k]) \cap [t^k] \supseteq f([x^*]) \cap [x^*] = [x^*].$$

Hence, $[g] := \lim_{k\to\infty} [t^k] \supseteq [x^*]$. It is

$$[g] = f([g]) \cap [g] \subseteq f([g]).$$

Now, we consider $(T)$ with $[x^0] = [g]$. Assuming $[x^k] \supseteq [g]$ we have

$$[x^{k+1}] = f([x^k]) \supseteq f([g]) \supseteq [g].$$

Then we get via Theorem 3.1

$$[x^*] = \lim_{k\to\infty} [x^k] \supseteq [g].$$

Using Theorem 3.3 and Theorem 3.4 one can show

$$\lim_{k\to\infty} [s^k] = [x^*] = \lim_{k\to\infty} [z^k]$$

in the same manner.

*c)* Starting $(T)$ with $[x^0] = o$ we have $d([x^k]) = o$ for all $k$. Since $\lim_{k\to\infty} [x^k] = [x^*]$ we have $d([x^*]) = o$. So, $[x^*] = x^*$, a point vector, which by Theorem 3.1 is the solution of the LCP defined by $A$ and $b$. By part b) of the present theorem we know that $(TI)$, $(SI)$ and $(SSI)$ all converge to $x^*$. □

**Remark 4.1.** *a)* Implementing $(SI)$ we can propose the columnwise procedure given in [15]. This is especially advantageous if one uses Pascal-XSC ([11]), since this language supports the multiplication of a column vector of an interval matrix by an interval.

*b)* Using the idea described in [2], p. 168, it is easy to see that $(SSI)$ can be performed with essentially the same amount of work as $(SI)$. An exception is the first step. Furthermore the division by $[a_{ii}]$ has to be performed twice in every step. This could be avoided by computing once and for all the intervals $[a_{ij}]/[a_{ii}]$, $i \neq j$. However, proceeding in this manner would increase the limit $[x^*]$ in the set theoretic sense. This is, of course, undesirable.

*c)* Statement *a)* of the preceding theorem shows that $(SSI)$ is the method of choice.

### *4.2. How to get an interval vector [start]?*

We assume that $[b] \in \mathbf{IR}^n$ and that $[A] \in \mathbf{IR}^{n\times n}$ is an H-matrix with $\underline{a}_{ii} > 0$, $i = 1, \ldots, n$. Let $[D]^{-1}$, $[R]$ and $[x^*]$ defined as in Theorem 3.1. Then we consider $(T)$ with an arbitrary $[x^0]$, $\underline{x}^0 \geq o$. We have seen in the proof of Theorem 3.1 that

$$\rho(P) < 1, \quad \text{where} \quad P := \langle [D] \rangle^{-1} |[R]|.$$

With

$$\begin{aligned} q([x^{m+1}], [x^m]) &= q(f([x^m]), f([x^{m-1}])) \\ &\leq P \cdot q([x^m], [x^{m-1}]) \leq \ldots \leq P^m \cdot q([x^1], [x^0]) \end{aligned}$$

we get for $l > k$:

$$\begin{aligned} q([x^l],[x^k]) &\le q([x^l],[x^{l-1}]) + \cdots + q([x^{k+1}],[x^k]) \\ &\le P^{l-1}\cdot q([x^1],[x^0]) + \cdots + P^k \cdot q([x^1],[x^0]) \\ &= P^k \cdot (I + P + \cdots + P^{l-k-1}) \cdot q([x^1],[x^0]) \\ &\le P^k \cdot \left(\sum_{j=0}^{\infty} P^j\right) \cdot q([x^1],[x^0]) \\ &= P^k \cdot (I-P)^{-1} \cdot q([x^1],[x^0]). \end{aligned}$$

Since $\lim_{l\to\infty}[x^l] = [x^*]$, it holds that (set $k := 1$)

$$q([x^*],[x^1]) \le P\cdot (I-P)^{-1}\cdot q([x^1],[x^0]) =: v,$$

which is equivalent to

$$|\underline{x}^* - \underline{x}^1| \le v, \quad |\overline{x}^* - \overline{x}^1| \le v$$

and which implies

$$\underline{x}^1 - v \le \underline{x}^*, \quad \overline{x}^* \le \overline{x}^1 + v.$$

Hence, we get $[x^*] \subseteq [\underline{x}^1 - v, \overline{x}^1 + v] =: [start]$.

## 5. Examples

We have implemented $(TI)$, $(SI)$ and $(SSI)$ as in Theorem 4.1 using PASCAL-XSC ([11]), where we have also taken into account the statements $a)$ and $b)$ of Remark 4.1. The iteration $(TI)$ is stopped as soon as there is an iteration step $k$ with $[t^{k+1}] = [t^k]$. The stopping criteria for $(SI)$ and $(SSI)$ are analogous.

In the sequel we present some examples where the interval matrices $[A]$ are H-matrices satisfying $\underline{a}_{ii} > 0$, $i = 1,\ldots,n$.

*Example 5.1.* Let

$$[A] = \begin{pmatrix} [\frac{1}{8},1] & [-\frac{1}{4},-\frac{1}{5}] \\ [-\frac{1}{4},-\frac{1}{10}] & 1 \end{pmatrix}, \quad [b] = \begin{pmatrix} [-3,-1] \\ [1,2] \end{pmatrix}.$$

The shape of $\Sigma([A],[b])$ has already been discussed in [16] and is depicted in Fig. 1. Note that the line from $(1,0)$ to $(4,0)$ belongs to $\Sigma([A],[b])$. For $(TI)$ we get

[ 1.000000000000000E + 000, 4.400000000000001E + 001]
[ 0.000000000000000E + 000, 1.000000000000001E + 001]

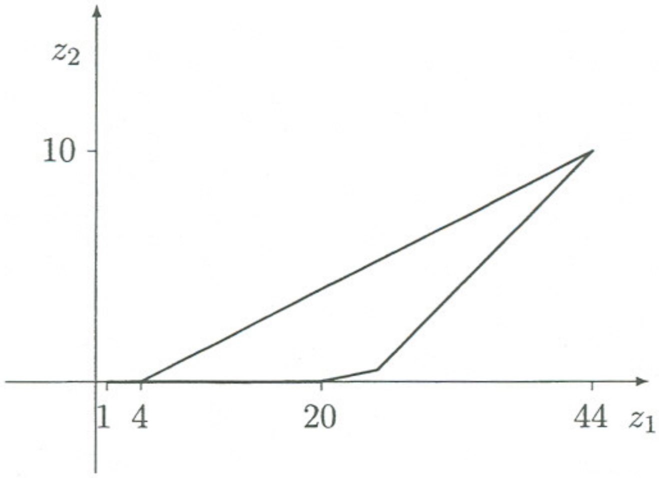


**Fig. 1.** The shape of $\Sigma([A],[b])$

after 100 iteration steps. For $(SI)$ and $(SSI)$ we get the same result after 51 iteration steps. According to Theorem 3.2 this interval vector is the smallest interval vector enclosing $\Sigma([A],[b])$, since $[A]$ is even an M-matrix. Using the $(SOR)$ method with $\omega = 1.0001$ we get

$$\begin{array}{l} [\,9.956987396218869\mathrm{E}-001,\ 4.401260378113437\mathrm{E}+001] \\ [\,0.000000000000000\mathrm{E}+000,\ 1.000415126037813\mathrm{E}+001] \end{array}$$

after 52 iteration steps which corresponds to part $d)$ of Theorem 3.3. Using the $(SOR)$ method with $\omega = 0.95$ we get

$$\begin{array}{l} [\,9.999999999999998\mathrm{E}-001,\ 4.400000000000003\mathrm{E}+001] \\ [\,0.000000000000000\mathrm{E}+000,\ 1.000000000000001\mathrm{E}+001] \end{array}$$

after 58 iteration steps.

*Example 5.2.* Let

$$[A] = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & [4,9] & -1 & 0 \\ 0 & 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad [b] = \begin{pmatrix} 2 \\ -3 \\ [-1,1] \\ [2,4] \\ 0 \end{pmatrix}.$$

We get the inclusion

$$
\begin{array}{l}
[\,0.000000000000000\mathrm{E}+000,\ 0.000000000000000\mathrm{E}+000] \\
[\,1.529411764705882\mathrm{E}+000,\ 1.857142857142858\mathrm{E}+000] \\
[\,5.882352941176469\mathrm{E}-002,\ 7.142857142857144\mathrm{E}-001] \\
[\,0.000000000000000\mathrm{E}+000,\ 0.000000000000000\mathrm{E}+000] \\
[\,0.000000000000000\mathrm{E}+000,\ 0.000000000000000\mathrm{E}+000]
\end{array}
$$

of $\Sigma([A],[b])$ after 40 steps using $(TI)$, after 21 steps using $(SI)$ and after 20 steps using $(SSI)$, respectively. Note that $[A]$ is not an M-matrix due to the (4,5)-entry of the matrix.

*Example 5.3.* Let

$$
[A] = \begin{pmatrix} [7,8] & [2,3] & [0,1] & [-1,2] \\ -2 & [5,6] & [-1,0] & [-1,2] \\ [-4,-3] & [-1,1] & [6,6.5] & [0.5,0.8] \\ [-2,2] & [-2,1] & [0,0.5] & 5 \end{pmatrix}, \quad [b] = \begin{pmatrix} [0,1] \\ [-1,0] \\ 0 \\ [1,2] \end{pmatrix}.
$$

We get the inclusion

$$
\begin{array}{l}
[\,0.000000000000000\mathrm{E}+000,\ 0.000000000000000\mathrm{E}+000] \\
[\,0.000000000000000\mathrm{E}+000,\ 2.068965517241380\mathrm{E}-001] \\
[\,0.000000000000000\mathrm{E}+000,\ 3.448275862068967\mathrm{E}-002] \\
[\,0.000000000000000\mathrm{E}+000,\ 0.000000000000000\mathrm{E}+000]
\end{array}
$$

of $\Sigma([A],[b])$ after 27 steps using $(TI)$, after 14 steps using $(SI)$ and after 14 steps using $(SSI)$, respectively.

Considering a point problem

$$
A = \begin{pmatrix} 7.5 & 2.1 & 0.7 & -0.3 \\ -2 & 5.7 & 0 & 1.8 \\ -3.3 & 1 & 6.2 & 0.7 \\ 1 & -1 & 0.25 & 5 \end{pmatrix}, \quad b = \begin{pmatrix} 0.2 \\ -0.6 \\ 0 \\ 1.3 \end{pmatrix},
$$

we get the inclusion

$$
\begin{array}{l}
[\,0.000000000000000\mathrm{E}+000,\ 0.000000000000000\mathrm{E}+000] \\
[\,1.052631578947368\mathrm{E}-001,\ 1.052631578947369\mathrm{E}-001] \\
[\,0.000000000000000\mathrm{E}+000,\ 0.000000000000000\mathrm{E}+000] \\
[\,0.000000000000000\mathrm{E}+000,\ 0.000000000000000\mathrm{E}+000]
\end{array}
$$

of the unique solution of the LCP defined by $A$ and $b$ after 3 steps using $(TI)$, after 3 steps using $(SI)$ and after 2 steps using $(SSI)$, respectively.

*Example 5.4.* Let

$$[A] = \begin{pmatrix} [20,21] & [-1,-0.5] & [0,0.1] & [0.1,0.2] & [-2,-1] \\ [-0.3,-0.2] & [20,21] & [0.8,1] & [0.2,0.4] & [-0.2,0] \\ [0,0.1] & [0.1,0.2] & [20,21] & [0.1,0.2] & [-2,-1] \\ [-0.3,-0.2] & [0,0.1] & [0.8,1] & [30,31] & [-0.2,0] \\ [-1,-0.5] & [0,0.1] & [0.1,0.2] & [-2,-1] & [30,31] \end{pmatrix},$$

$$[b] = \begin{pmatrix} [0.2,0.4] \\ [-1,-0.8] \\ [-0.2,0.2] \\ [0.2,0.4] \\ [-1,-0.8] \end{pmatrix}.$$

We get the inclusion

$$\begin{matrix} [\,0.000000000000000E+000, & 0.000000000000000E+000] \\ [\,3.746923870762879E-002, & 5.033333333333335E-002] \\ [\,0.000000000000000E+000, & 1.314598713979520E-002] \\ [\,0.000000000000000E+000, & 0.000000000000000E+000] \\ [\,2.555927320124862E-002, & 3.333333333333334E-002] \end{matrix}$$

of $\Sigma([A],[b])$ after 11 steps using $(TI)$, after 7 steps using $(SI)$ and after 6 steps using $(SSI)$, respectively.

*Example 5.5.* Let

$$[A] = \begin{pmatrix} [1,1.5] & -0.5 & \cdots & \cdots & -0.5 \\ 0 & \ddots & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & -0.5 & -0.5 \\ \vdots & \cdots & 0 & [1,1.5] & -0.5 \\ 0 & \cdots & \cdots & 0 & [1,1.5] \end{pmatrix} \in \mathbf{IR}^{10\times 10}$$

and

$$[b_i] = \left\{ \begin{array}{ll} [0.2,0.3] & \text{if } i = 2k+1, \\ [-1,-0.9] & \text{if } i = 2k, \end{array} \right\} \quad i = 1,..,10.$$

We get the inclusion

$$\begin{array}{l}
[\,3.081847279378140E+000,\ 1.951054687500001E+001] \\
[\,2.911385459533605E+000,\ 1.380703125000001E+001] \\
[\,1.583539094650204E+000,\ 8.404687500000004E+000] \\
[\,1.787654320987653E+000,\ 6.403125000000002E+000] \\
[\,7.407407407407402E-001,\ 3.468750000000001E+000] \\
[\,1.155555555555555E+000,\ 3.112500000000001E+000] \\
[\,2.666666666666664E-001,\ 1.275000000000001E+000] \\
[\,7.999999999999998E-001,\ 1.650000000000001E+000] \\
[\,0.000000000000000E+000,\ 3.000000000000001E-001] \\
[\,5.999999999999998E-001,\ 1.000000000000000E+000]
\end{array}$$

of $\Sigma([A],[b])$ after 10 steps using $(TI)$, after 10 steps using $(SI)$ and after 2 steps using $(SSI)$, respectively.

*Example 5.6.* We consider Example 5.1 in [17]. There, a free boundary problem was discretized taking into account the discretization error. This leads to an LCP with the matrix

$$A = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & \cdots & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \cdots & 0 & -\frac{1}{2} & 1 \end{pmatrix} \in \mathbf{R}^{n\times n}$$

and an interval vector $[b]$. Since $A$ is an M-matrix, $(TI)$, $(SI)$ and $(SSI)$, respectively, will converge to a limit which is the smallest interval vector containing $\Sigma(A,[b])$. For $n = 10$ we get the inclusion

$$\begin{array}{l}
[8.210179430213943E-002,\ 8.276674431625078E-002] \\
[6.605772783177625E-002,\ 6.717240846882514E-002] \\
[5.182710250343162E-002,\ 5.321876250425957E-002] \\
[3.937435778530811E-002,\ 4.090743764525911E-002] \\
[2.866892713400047E-002,\ 3.023994094279071E-002] \\
[1.968505647879717E-002,\ 2.121766916763587E-002] \\
[1.240158636605719E-002,\ 1.384192182526096E-002] \\
[6.801700849114046E-003,\ 8.113913186310685E-003] \\
[2.872647508163023E-003,\ 4.034783240482190E-003] \\
[6.054339940538283E-004,\ 1.605607503113720E-003]
\end{array}$$

of $\Sigma(A, [b])$ after 884 steps using $(TI)$, after 443 steps using $(SI)$ and after 242 steps using $(SSI)$, respectively.

**Final remark.** For LCPs with interval data it is not yet precisely understood how the speed of convergence of the methods considered in this paper is dependent on the given data. However, this understanding is a prerequisite for the construction of methods which are faster convergent. This will be part of research done in the future.

## Acknowledgement

We are grateful to the referee who has given a series of valuable comments.

## References

[1] Alefeld, G., Chen, X., Potra, F.: Numerical validation of solutions of linear complementarity problems. Numer. Math. *83*, 1–23 (1999).

[2] Alefeld, G., Herzberger, J.: Introduction to interval computations. Academic Press, London 1983.

[3] Alefeld, G., Varga, R. S.: Zur Konvergenz des symmetrischen Relaxationsverfahrens. Numer. Math. *25*, 291–295 (1976).

[4] Bai, Z. Z.: The convergence of parallel iteration algorithms for linear complementarity problems. Computers Math. Applic. *32*, 1–17 (1996).

[5] Bastian, M.: Lineare Komplementärprobleme im Operations Research und in der Wirtschaftstheorie. Verlag Anton Hain, Meisenheim am Glan 1976.

[6] Cottle, R. W., Pang, J. S., Stone, R. E.: The linear complementarity problem. Academic Press, London 1992.

[7] Cryer, C. W.: The solution of a quadratic programming problem using systematic overrelaxation. SIAM J. Control *9*, 385–392 (1971).

[8] Fan, K.: Topological proofs for certain theorems on matrices with nonnegative elements. Monatsh. Math. *62*, 219-237 (1958).

[9] Ferris, M. C., Pang, J. S.: Engineering and economic applications of complementarity problems. SIAM Rev. *39*, 669–713 (1997).

[10] Harker, P., Xiao, B.: A nonsmooth Newton method for variational inequalities, I: Theory. Mathematical Programming *65*, 151–194 (1994).

[11] Klatte, R., Kulisch, U., Neaga, M., Ratz, D., Ullrich, Ch.: PASCAL-XSC – Language reference with examples. Springer Verlag, Berlin 1992.

[12] Kulisch, U.: Über reguläre Zerlegungen von Matrizen und einige Anwendungen. Numer. Math. *11*, 444–449 (1968).

[13] Murty, K. G.: Linear complementarity, linear and nonlinear programming. Heldermann Verlag, Berlin 1988.

[14] Neumaier, A.: Interval methods for systems of equations. University Press, Cambridge 1990.

[15] Niethammer, W.: A note on the implementation of the successive overrelaxation method for linear complementarity problems. Numer. Algorithms *4*, 197–200 (1993).

[16] Schäfer,U.: Das lineare Komplementaritätsproblem mit Intervalleinträgen. Dissertation, Universität Karlsruhe 1999.

[17] Schäfer, U.: An enclosure method for free boundary problems based on a linear complementarity problem with interval data. Numer. Funct. Anal. and Optimiz. *22*, 991–1011 (2001).

[18] Varga, R. S.: Matrix iterative analysis. Prentice Hall, London 1962.

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