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Numerical validation of solutions of linear complementarity problems

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Summary. This paper proposes a validation method for solutions of linear complementarity problems. The validation procedure consists of two sufficient conditions that can be tested on a digital computer. If the first condition is satisfied then a given multidimensional interval centered at an approximate solution of the problem is guaranteed to contain an exact solution. If the second condition is satisfied then the multidimensional interval is guaranteed to contain no exact solution. This study is based on the mean value theorem for absolutely continuous functions and the reformulation of linear complementarity problems as nonsmooth nonlinear systems of equations.

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1. Introduction

Linear Complementarity Problems (LCP) model many important problems in engineering, management and economics. Furthermore linear and quadratic programming problems can be written as LCP.

Several algorithms have been developed for solving LCP [11,21,22,25, 26,31], but few validation methods have been studied to give guaranteed bounds on the distance between the numerical solution and the exact solution. One likely reason for this omission is that the traditional interval

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methods were developed for continuously differentiable equations. The aim of this paper is to give an efficient numerical validation method for solutions of LCP by using the mean value theorem for absolutely continuous functions.

Primal-dual interior-point algorithms are the most efficient methods to date for solving linear complementarity problems. For an excellent description of theoretical results and software development we refer the reader to the recent monograph of Steve Wright [31]. Typically a primal-dual interior-point method produces a point with primal-dual gap less than a given tolerance ε. It is then important to know if this approximate solution is close to an exact solution of the problem. If the point is produced by an infeasible-interior-point method (see [31]) then the problem may not have a solution in spite of the fact that a point with small primal-dual gap has been computed (see Example 4.5).

The goal of the present paper is to give a sufficient condition that can be tested on a digital computer and which guarantees that a multidimensional interval centered at an approximate solution of the linear complementarity problem contains an exact solution. In other words our paper will present an algorithm for enclosing the solution of the problem. We also give another sufficient condition which guarantees that a given multidimensional interval contains no exact solution of the problem. Our algorithm uses tools from interval analysis. In particular it uses an efficient interval extension of the slope of a semi-smooth nonlinear operator associated with linear complementarity problems. For applications of the notion of interval extensions of the slope of an operator in numerical optimisation see the recent book of Kearfott [15].

The remaining part of this paper is as follow. In Sect. 2 we discuss three reformulations for LCP as nonlinear equations and present a new interval operator for the numerical validation of the solution of LCP. In Sect. 3 we propose algorithms for testing the existence of solutions. In Sect. 4 we report numerical results to illustrate the robustness of the new method.

In this paper we denote an interval by $[x] = \{x \in \mathbb{R}^n, \underline{x} \leq x \leq \overline{x}\}$.

2. Verification of solutions of nonlinear equations

2.1. The Krawczyk operator

In the last decade a lot of effort has been spent on validation of solutions of nonlinear equations $H(x) = 0$. Most validation methods are enclosing methods that compute an n-dimensional interval $[x] \subset \mathbb{R}^n$ that is guaranteed to contain an exact solution x^*. In what follows we will construct such a validation method for LCP.

The Krawczyk operator and the validation method proposed by Alefeld, Gienger and Potra [2] are applicable for validation of solutions of nonlinear equations with continuously differentiable functions. Chen [4] studied a generalization of the Krawczyk operator and the Alefeld-Gienger-Potra method to nondifferentiable equations.

The method in [2] is based on the mean value theorem for differentiable functions and an interval extension of the derivative, stated quantitatively in the form

$$H(x) - H(y) \in H'([x])(x - y), \quad \text{for all } x, y \in [x]. \tag{2.1}$$

The Krawczyk operator is defined by

$$K(x, A, [x]) = x - A^{-1}H(x) + (I - A^{-1}H'([x]))([x] - x),$$

where A is an $n \times n$ nonsingular matrix.

The method in [4] is based on the mean value theorem for local Lipschitzan functions

$$H(x) - H(y) \in \text{co}\partial H([x])(x - y), \quad \text{for all } x, y \in [x], \tag{2.2}$$

where "co" denotes the covex hull, ∂F denotes the generalized Jacobian in Clarke's sense [8] and

$$\text{co}\partial H([x]) = \text{co}\{V \in \partial H(x), x \in [x]\}.$$

An interval operator for nonsmooth equations is defined by

$$B(x, A, [x]) = x - A^{-1}H(x) + (I - A^{-1}L_{[x]})([x] - x),$$

where $L_{[x]}$ is an interval matrix satisfying $\text{co}\partial H([x]) \subseteq L_{[x]}$. See [4].

It has been observed repeatedly that the interval extension of the derivative of a differentiable function can be replaced by a smaller interval. For example, the slope function [1, 14, 15, 17, 28, 29].

In this paper we give smaller intervals for both differentiable and nondifferentiable functions. This study is based on the mean value theorem

$$f(x) - f(y) = \int_0^1 df(x + t(y - x); (x - y))dt, \tag{2.3}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is absolutely continuous and $df(u; v)$ is the directional derivative of f in the direction v. According to [27], the directional derivative exists almost everywhere and (2.3) holds. If the Fréchet derivative of f exists at almost every point in $\text{co}\{x, y\}$ then we can define

$$g(x, y) = \int_0^1 f'(x + t(y - x))dt,$$

and we have

$$f(x) - f(y) = g(x, y)(x - y). \tag{2.4}$$

If we assume that we can define g so that (2.4) holds for all x, y, then g is a slope for f [14]. If we also assume that for any $x \in \mathbb{R}^n$ and for any interval $[x] \subset \mathbb{R}^n$, we have an interval extension $g(x, [x])$ of $g(x, y)$. Then for any $y \in [x]$,

$$f(x) - f(y) \in g(x, [x])(x - y).$$

Let $H : \mathbb{R}^n \to \mathbb{R}^n$ be an absolutely continuous mapping. For given $x, y \in \mathbb{R}^n$ we can find a slope $G_i(x, y)$ for each component H_i of H. Let $G_i(x, [x])$ be the interval extension of the slope for $G_i(x, y), i = 1, 2, \ldots, n$, and let

$$G(x, [x]) = \begin{pmatrix} G_1(x, [x]) \\ \vdots \\ G_n(x, [x]) \end{pmatrix}.$$

Then we have

$$H(x) - H(y) \in G(x, [x])(x - y), \quad \text{for any } y \in [x].$$

Replacing the interval extension of the derivative in the Krawczyk operator by $G(x, [x])$, we obtain a new interval operator

$$L(x, A, [x]) = x - A^{-1}H(x) + (I - A^{-1}G(x, [x]))([x] - x).$$

This operator has the same properties as the Krawczyk operator for the purpose of validation. In particular,

if $L(x, A, [x]) \subseteq [x]$, then there exists a solution of $H(x) = 0$;

if $L(x, A, [x]) \cap [x] = \emptyset$, then there is no zero of H in $[x]$.

For completeness we repeat the well known simple proofs of these two important properties:

Consider the mapping $R : [x] \subset \mathbb{R}^n \to \mathbb{R}^n$ where

$$R(y) = y - A^{-1}H(y)$$

and where A is a nonsingular matrix. R is continuous since H is absolutely continuous by assumption. For arbitrary $y \in [x]$ and a fixed $x \in [x]$ we have

$$\begin{aligned} R(y) &= y - A^{-1}H(y) \\ &= x - A^{-1}H(x) + y - x + A^{-1}(H(x) - H(y)) \\ &= x - A^{-1}H(x) + (y - x) + A^{-1}G(x, y)(x - y) \\ &= x - A^{-1}H(x) + (I - A^{-1}G(x, y))(y - x) \\ &\in x - A^{-1}H(x) + (I - A^{-1}G(x, [x]))([x] - x) =: L(x, A, [x]). \end{aligned}$$

Therefore, if $L(x, A, [x]) \subseteq [x]$, then $R(y) \in [x]$ for all $y \in [x]$, and by the Brouwer fixed point theorem there exists a fixed point y^* of R in $[x]$ which is also a solution of $H(y) = 0$.
To prove the second part, assume that $H(y^*) = 0$ for some $y^* \in [x]$. Then, as before,

$$y^* = R(y^*) \in L(x, A, [x])$$

for a fixed $x \in [x]$. This contradicts the condition $L(x, A, [x]) \cap [x] = \emptyset$.

Since $G(x, [x]) \subseteq H'([x])$ for $x \in [x]$ if H is differentiable on $[x]$, $L(x, A, [x])$ is smaller than the Krawczyk operator $K(x, A, [x])$. Notice that $G(x, [x])$ is not only dependent on $[x]$ but also on x. A good choice of x can make $G(x, [x])$ much smaller than $H'([x])$. The other advantage of $L(x, A, [x])$ is that it is applicable even if H is nondifferentiable.

2.2. *The linear complementarity problem(LCP)*

Let $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. The linear complementarity problem (LCP) is the problem of finding an $x \in \mathbb{R}^n$ such that

$$Mx + q \geq 0, \quad x \geq 0, \quad (Mx + q)^{\mathrm{T}} x = 0. \tag{2.5}$$

Many algorithms for solving (2.5) are designed via an equivalent system of nonlinear equations

$$H(x) = 0. \tag{2.6}$$

The equivalence means that x^* solves (2.6) if and only if x^* solves (2.5). See for example [10].

2.3. *The LCP as a system of nonlinear equations*

In the study of LCP, the following three equivalent formulas of nonlinear equations are often used.

Mangasarian's formula [21]

$$\tilde{F}(x) = \begin{pmatrix} (m_1^{\mathrm{T}}x + q_1)|m_1^{\mathrm{T}}x + q_1| + x_1|x_1| - (m_1^{\mathrm{T}}x + q_1 - x_1)^2 \\ \vdots \\ (m_n^{\mathrm{T}}x + q_n)|m_n^{\mathrm{T}}x + q_n| + x_n|x_n| - (m_n^{\mathrm{T}}x + q_n - x_n)^2 \end{pmatrix}, \tag{2.7}$$

where $m_i \in \mathbb{R}^n$ is the i-th row of M.

Pang's formula [24]

$$F(x) = \min(Mx + q, x), \tag{2.8}$$

where the "min" operator denotes the componentwise minimum of two vectors.

The Fischer-Burmeister formula [11]

$$\hat{F}(x) = \begin{pmatrix} x_1 + m_1^{\mathrm{T}} x + q_1 - \sqrt{x_1^2 + (m_1^{\mathrm{T}} x + q_1)^2} \\ \vdots \\ x_n + m_n^{\mathrm{T}} x + q_n - \sqrt{x_n^2 + (m_n^{\mathrm{T}} x + q_n)^2} \end{pmatrix}. \tag{2.9}$$

The Mangasarian function is differentiable in $\mathbb{R}^n$ and its derivative at x is

$$\begin{aligned} \tilde{F}_i'(x) = {} & 2(m_i^{\mathrm{T}} x + q_i - x_i(1 - \mathrm{sgn}(x_i)))e_i^{\mathrm{T}} \\ & +2(x_i - (m_i^{\mathrm{T}} x + q_i)(1 - \mathrm{sgn}(m_i^{\mathrm{T}} x + q_i)))m_i^{\mathrm{T}}, \end{aligned} \tag{2.10}$$

where

$$\mathrm{sgn}(\alpha) = \begin{cases} 1, & \alpha \geq 0, \\ -1, & \alpha < 0 \end{cases}$$

and $e_i \in \mathbb{R}^n$ denotes the vector with all elements equal to 0 except the i-th element equal to 1.

The Pang function is piecewise linear. Each component F_i of F is differentiable at x if $m_i^{\mathrm{T}} x + q_i \neq x_i$, and

$$F_i'(x) = \begin{cases} e_i^{\mathrm{T}}, & \text{if } (m_i - e_i)^{\mathrm{T}} x + q_i > 0 \\ m_i^{\mathrm{T}}, & \text{if } (m_i - e_i)^{\mathrm{T}} x + q_i < 0. \end{cases}$$

The Fischer-Burmeister function is semismooth. Each component $\hat{F}_i$ of $\hat{F}$ is differentiable at x if $(m_i^{\mathrm{T}} x + q_i)^2 + x_i^2 > 0$, and

$$\hat{F}_i'(x) = \Big(1 - \frac{x_i}{\sqrt{x_i^2 + (m_i^{\mathrm{T}} x + q_i)^2}}\Big)e_i^{\mathrm{T}} + \Big(1 - \frac{m_i^{\mathrm{T}} x + q_i}{\sqrt{x_i^2 + (m_i^{\mathrm{T}} x + q_i)^2}}\Big)m_i^{\mathrm{T}}.$$

Each component of $\tilde{F}, F$ and $\hat{F}$ is absolutely continuous and we can define their slopes. However, $\tilde{F}'$ and $\hat{F}'$ are not easily used in the verification methods. Although the Mangasarian function and the Fischer-Burmeister function have nicer properties for global convergence analysis than the Pang function, they are not linear in any domain. The Pang function is piecewise linear and keeps the original linear form in each piece. This property is advantageous for the interval extension. Therefore we study the slope and the interval extension for the Pang function.

2.4. The slope for the Pang function

Let us denote

$$S_i^+ = \{x \mid x \in [x], (m_i - e_i)^{\mathrm{T}} x + q_i > 0\}$$

and

$$S_i^- = \{x \mid x \in [x], (m_i - e_i)^{\mathrm{T}} x + q_i < 0\}.$$

For given x, y, if $x \in S_i^+ \cup S_i^-$ or $y \in S_i^+ \cup S_i^-$, then

$$\int_0^1 F_i'(x + t(y-x))dt = \begin{cases} e_i^{\mathrm{T}}, & x, y \notin S_i^- \\ m_i^{\mathrm{T}}, & x, y \notin S_i^+ \\ m_i^{\mathrm{T}} + t_i(e_i - m_i)^{\mathrm{T}}, & x \notin S_i^-, y \in S_i^- \\ e_i^{\mathrm{T}} + t_i(m_i - e_i)^{\mathrm{T}}, & x \notin S_i^+, y \in S_i^+, \end{cases}$$

where

$$t_i = \frac{(m_i - e_i)^{\mathrm{T}} x + q_i}{(m_i - e_i)^{\mathrm{T}}(x - y)}.$$

If $x \notin S_i^+ \cup S_i^-$ and $y \notin S_i^+ \cup S_i^-$, then for any $t \in [0,1]$,

$$(m_i - e_i)^{\mathrm{T}}(x + t(y - x)) + q_i = 0.$$

In this case,

$$F_i(x) - F_i(x + t(y-x)) = t m_i^{\mathrm{T}}(x - y) = t e_i^{\mathrm{T}}(x - y).$$

This means that F is nondifferentiable on the line segment between x and y. Nevertheless, it does not affect to define the interval extension $G(x, [x])$.

To define the interval extension, we fix x and consider the following linear programming problems

$$\begin{array}{lll} & \inf\ (m_i - e_i)^{\mathrm{T}} y + q_i & \quad (\text{if } (m_i - e_i)^{\mathrm{T}} x + q_i \geq 0) \\ (2.11) & \text{s.t.} \quad y \in S_i^- & \end{array}$$

and

$$\begin{array}{lll} & \sup\ (m_i - e_i)^{\mathrm{T}} y + q_i & \quad (\text{if } (m_i - e_i)^{\mathrm{T}} x + q_i \leq 0) \\ (2.12) & \text{s.t.} \quad y \in S_i^+. & \end{array}$$

Let y^i be a solution of the linear programming problem, and let

$$t_i^* = \frac{(m_i - e_i)^{\mathrm{T}} x + q_i}{(m_i - e_i)^{\mathrm{T}}(x - y^i)}.$$

Then we can define the interval extension by

$$G_i(x,[x]) = \begin{cases} e_i^{\mathrm{T}}, & S_i^- = \emptyset \\ m_i^{\mathrm{T}}, & S_i^+ = \emptyset \\ m_i^{\mathrm{T}} + [t_i^*,1](e_i - m_i)^{\mathrm{T}}, & x \notin S_i^-, S_i^- \neq \emptyset \\ e_i^{\mathrm{T}} + [t_i^*,1](m_i - e_i)^{\mathrm{T}}, & x \notin S_i^+, S_i^+ \neq \emptyset \end{cases}$$

Lemma 2.1 *For any fixed $x \in \mathbb{R}^n$, $y \in [x]$, we have*

$$F(x) - F(y) \in G(x,[x])(x-y).$$

Proof. Suppose $(m_i - e_i)^{\mathrm{T}} x + q_i \geq 0$. If $y \notin S_i^-$ and $S_i^- = \emptyset$, then

$$F_i(x) - F_i(y) = x_i - y_i = e_i^{\mathrm{T}}(x-y) = G_i(x,[x])(x-y).$$

If $y \notin S_i^-$ and $S_i^- \neq \emptyset$, then

$$\begin{aligned} F_i(x) - F_i(y) &= e_i^{\mathrm{T}}(x-y) \\ &= (m_i^{\mathrm{T}} + (e_i - m_i)^{\mathrm{T}})(x-y) \\ &\in (m_i^{\mathrm{T}} + [t_i^*,1](e_i - m_i)^{\mathrm{T}})(x-y) \\ &= G_i(x,[x])(x-y). \end{aligned}$$

If $y \in S_i^-$, then $S_i^- \neq \emptyset$ and we have

$$\begin{aligned} F_i(x) - F_i(y) &= \left(\int_0^{t_i} e_i^{\mathrm{T}} dt + \int_{t_i}^1 m_i^{\mathrm{T}} dt\right)(x-y) \\ &= (m_i^{\mathrm{T}} + t_i(e_i - m_i)^{\mathrm{T}})(x-y), \end{aligned}$$

where

$$t_i = \frac{(m_i - e_i)^{\mathrm{T}} x + q_i}{(m_i - e_i)^{\mathrm{T}}(x-y)}.$$

Since $0 > (m_i - e_i)^{\mathrm{T}} y + q_i \geq (m_i - e_i)^{\mathrm{T}} y^i + q_i$,

$$1 > t_i \geq \frac{(m_i - e_i)^{\mathrm{T}} x + q_i}{(m_i - e_i)^{\mathrm{T}}(x - y^i)} = t_i^*.$$

Hence

$$m_i^{\mathrm{T}} + t_i(e_i - m_i)^{\mathrm{T}} \in m_i^{\mathrm{T}} + [t_i^*,1](e_i - m_i)^{\mathrm{T}}$$

and

$$F_i(x) - F_i(y) \in G_i(x,[x])(x-y).$$

Similarly, we can prove this lemma for the case $(m_i - e_i)^{\mathrm{T}} x + q_i \leq 0$. □

3. Algorithm

In this section we first give an algorithm to define the interval extension $G(x, [x])$ based on Pang's formula. Next we give a verification method for the LCP by using $G(x, [x])$.

Notice that

$$\begin{aligned} m_i^{\mathrm{T}} + [0,1](e_i - m_i)^{\mathrm{T}} &= e_i^{\mathrm{T}} + [0,1](m_i - e_i)^{\mathrm{T}} \\ &= [\min(e_i^{\mathrm{T}}, m_i^{\mathrm{T}}), \max(e_i^{\mathrm{T}}, m_i^{\mathrm{T}})]. \end{aligned}$$

By the analysis in the last section, if $(m_i - e_i)^{\mathrm{T}} x + q_i = 0$, and both S_i^- and S_i^+ are nonempty, then

$$G_i(x, [x]) = [\min(e_i^{\mathrm{T}}, m_i^{\mathrm{T}}), \max(e_i^{\mathrm{T}}, m_i^{\mathrm{T}})].$$

The following algorithm defines $G_i(x, [x])$ for a given interval $[x]$.

Algorithm 3.1 *Given $x \in \mathbb{R}^n$ and $[x] \subset \mathbb{R}^n$.*

1. *Solve the linear program*

$$\begin{aligned} &\min\ (m_i - e_i)^{\mathrm{T}} x + q_i \\ &\text{s.t.} \quad \underline{x} \le x \le \overline{x}. \end{aligned} \tag{3.1}$$

   Let y^i be a solution of (3.1). If $(m_i - e_i)^{\mathrm{T}} y^i + q_i \ge 0$, (i.e. $S_i^- = \emptyset$), let $G_i(x, [x]) = e_i^{\mathrm{T}}$. Otherwise perform step 2.

2. *Solve the linear program*

$$\begin{aligned} &\max\ (m_i - e_i)^{\mathrm{T}} x + q_i \\ &\text{s.t.} \quad \underline{x} \le x \le \overline{x}. \end{aligned} \tag{3.2}$$

   Let z^i be a solution of (3.2). If $(m_i - e_i)^{\mathrm{T}} z^i + q_i \le 0$, (i.e. $S_i^+ = \emptyset$), let $G_i(x, [x]) = m_i^{\mathrm{T}}$. Otherwise perform step 3.

3. *(In this case, $(m_i - e_i)^{\mathrm{T}} y^i + q_i < 0$ and $(m_i - e_i)^{\mathrm{T}} z^i + q_i > 0$, i.e. $S_i^- \ne \emptyset$, $S_i^+ \ne \emptyset$.) We perform the following steps.*

   3.1 If $(m_i - e_i)^{\mathrm{T}} x + q_i = 0$, let

$$G_i(x, [x]) = [\min(e_i^{\mathrm{T}}, m_i^{\mathrm{T}}), \max(e_i^{\mathrm{T}}, m_i^{\mathrm{T}})].$$

   3.2 If $(m_i - e_i)^{\mathrm{T}} x + q_i > 0$, let

$$t_i = \frac{(m_i - e_i)^{\mathrm{T}} x + q_i}{(m_i - e_i)^{\mathrm{T}} (x - y^i)}$$

   and

$$G_i(x, [x]) = m_i^{\mathrm{T}} + [t_i, 1](e_i - m_i)^{\mathrm{T}}.$$

3.3 If $(m_i - e_i)^{\mathrm{T}} x + q_i < 0$, *let*

$$t_i = \frac{(m_i - e_i)^{\mathrm{T}} x + q_i}{(m_i - e_i)^{\mathrm{T}}(x - z^i)}$$

and

$$G_i(x, [x]) = e_i^{\mathrm{T}} + [t_i, 1](m_i - e_i)^{\mathrm{T}}.$$

Optimal solutions of linear programming problems (3.1) and (3.2) are given by the formulae

$$y_j^i = \begin{cases} \underline{x}_j & (m_i - e_i)_j \geq 0 \\ \overline{x}_j & \text{otherwise}, \qquad j = 1, 2..., n \end{cases}$$

and

$$z_j^i = \begin{cases} \underline{x}_j & (m_i - e_i)_j \leq 0 \\ \overline{x}_j & \text{otherwise}, \qquad j = 1, 2, ..., n. \end{cases}$$

Based on the results in [2,4], we propose the following verification method.

Algorithm 3.2 *Let* $r > 0$ *be a given tolerance and let* x *be an approximate solution of*

$$F(z) = \min(Mz + q, z) = 0. \tag{3.3}$$

Calculate

$$[x] = x + r[-e, e] \tag{3.4}$$

where $e = [1, \ldots, 1]^{\mathrm{T}}$ *and choose a nonsingular matrix* A. *Compute*

$$L(x, A, [x]) = x - A^{-1}F(x) + (I - A^{-1}G(x, [x]))([x] - x). \tag{3.5}$$

- *If*

$$L(x, A, [x]) \subseteq [x], \tag{3.6}$$

then there is a solution $x^* \in [x]$ *of (3.3).*

- *If*

$$L(x, A, [x]) \cap [x] = \emptyset, \tag{3.7}$$

then the interval $[x]$ *contains no solution of (3.3).*

4. Numerical results

In this section we first give an example in $\mathbb{R}$ to illustrate application of our interval operator $L(x, A, [x])$ and compare with former operators. Next we report numerical results by using the programming language PASCAL-XSC [16] on an HP-9000 workstation.

Example 4.1 Let $M = -1$ and $q = 1$. Then the problem

$$F(x) = \min(Mx + q, x) = 0$$

has two solutions $x^* = 0$ and $x^* = 1$. The function F is not differentiable at $x = 1/2$.

We choose $[x] = [\frac{1}{2} - a, \frac{1}{2} + b]$, where $a, b > 0, a \le \frac{1}{2}$.

First we apply the Krawczyk operator to the differentiable equation $\tilde{F}(x) = 0$. By (2.10), we have

$$\begin{aligned}
\tilde{F}'([x]) &= 2(-[x]+1) - 2([x] - (-[x]+1)(1-[-1,1])) \\
&= 2[\frac{1}{2} - b, \frac{1}{2} + a] - 2([\frac{1}{2} - a, \frac{1}{2} + b] - [\frac{1}{2} - b, \frac{1}{2} + a][0,2]) \\
&= [-4b, 4a] + [\frac{1}{2} - b, \frac{1}{2} + a][0,4] \\
&= [-4b, 4a] + [\min(0, 2-4b), 2+4a] \\
&= [-4b + \min(0, 2-4b), 8a].
\end{aligned}$$

Let $m[x]$ be the midpoint of $[x]$. Then $[x] - m[x] = \frac{1}{2}[-(a+b), a+b]$ and

$$\begin{aligned}
\mathrm{W}((I - A^{-1}\tilde{F}'([x]))([x] - \mathrm{m}[x])) &\ge (1 + 4|A^{-1}|\min(a,b))(a+b) \\
&> W([x]) = (a+b),
\end{aligned}$$

where $W([x])$ denotes the diameter of the interval $[x]$.

Hence if $\frac{1}{2} \in [x]$, then for any A, $K(m[x], A, [x]) \not\subseteq [x]$.

Next we consider the operator for nonsmooth equations in [4]. By the definition of (3.3),

$$F(x) = \min(Mx + q, x) = \begin{cases} x, & x \le \frac{1}{2} \\ -x+1, & x > \frac{1}{2}. \end{cases}$$

By the definition of $\partial F(x)$,

$$\mathrm{co}\partial F([x]) = [-1, 1].$$

This implies that for any $a, b > 0$ and A,

$$\begin{aligned}
&(I - A^{-1}\mathrm{co}\partial F([x]))([x] - \mathrm{m}[x]) \\
&= [1 - |A^{-1}|, 1 + |A^{-1}|][-\frac{a+b}{2}, \frac{a+b}{2}]
\end{aligned}$$

and

$$\mathrm{W}((I - A^{-1}\mathrm{co}\partial F([x]))([x] - \mathrm{m}[x])) = (1 + |A^{-1}|)(a + b) > W([x]).$$

Hence if $\frac{1}{2} \in [x]$, then for any A, $B(m[x], A, [x]) \not\subseteq [x]$.

Now we consider $L(m[x], A, [x])$. Since $(m_i - e_i)^{\mathrm{T}}x + q_i = -2x + 1$, $\overline{x} = \frac{1}{2} + b$ is the optimal solution of

$$\min_{x \in [x]} (m_i - e_i)^{\mathrm{T}} x + q$$

and $\underline{x} = \frac{1}{2} - a$ is the optimal solution of

$$\max_{x \in [x]} (m_i - e_i)^{\mathrm{T}} x + q.$$

From $(m-e)^{\mathrm{T}}\overline{x} + q = -2b$ and $(m-e)^{\mathrm{T}}\underline{x} + q = 2a$, $S^- \neq \emptyset$ and $S^+ \neq \emptyset$. Let $x = \mathrm{m}[x] = \frac{1}{2} + \frac{b-a}{2}$ and $b \geq \max\{\frac{1}{2}, 3a\}$. Then $(m-e)^{\mathrm{T}}x + q < 0$,

$$F(x) = \frac{1}{2}(1 - b + a),$$

$$t_i = \frac{b - a}{b + a}$$

and

$$G(x, [x]) = [-1, \frac{3a - b}{b + a}].$$

Let $A = -1$. Then

$$L(m[x], A, [x]) = 1 + [0, 1 - \frac{b - 3a}{b + a}][-\frac{a + b}{2}, \frac{a + b}{2}].$$

Hence for any $b \geq \max\{\frac{1}{2}, 3a\}$, $L(m[x], A, [x]) \subseteq [x]$. The point 1 is a zero of $F(x)$ and $1 \in L(m[x], A, [x])$.

We also can choose other x such that $L(x, A, [x]) \subseteq [x]$. For instance, we consider $x = \underline{x}$. Using the analysis above,

$$t = \frac{(m - e)\underline{x} + q}{(m - e)(\underline{x} - \overline{x})} = \frac{a}{a + b},$$

$$m + (e - m)[t, 1] = [\frac{a - b}{a + b}, 1]$$

and

$$G(\underline{x}, [x]) = [\frac{a - b}{a + b}, 1].$$

Let $A \geq 1$. Then

$$L(\underline{x}, A, [x]) = (1 - A^{-1})(\frac{1}{2} - a) + [1 - A^{-1}, 1 - A^{-1}\frac{a-b}{a+b}][0, a+b]$$

$$= (1 - A^{-1})(\frac{1}{2} - a) + [0, (a+b)(1 - A^{-1}\frac{a-b}{a+b})].$$

If $a = \frac{1}{2}$ and $b \leq \frac{1}{2}$, then

$$[x] = [0, \frac{1}{2} + b]$$

and

$$L(\underline{x}, A, [x]) = [0, (\frac{1}{2} + b)(1 - A^{-1}\frac{1-2b}{1+2b})] \subseteq [x].$$

In the remainder of this paper we present some numerical results obtained by an evaluation of $Mx + q$ and a corresponding modification of algorithm (3.1) using a floating point system:

Performing algorithm (3.1) on a computer using a floating point system we have to take into account rounding errors. For example, in step 1 of this algorithm we have to compute $(m_i - e_i)^{\mathrm{T}} y^i + q_i$ and to check whether it is not less zero. However, in a floating point system there is no guarantee that the true value is also nonnegative if the computed result is nonnegative. Similar remarks hold for the sign tests in steps 2 and 3, respectively. In order to include these possibilities we first compute floating point intervals $[hy]^i$, $[hz]^i$ and $[hx]^i$ satisfying

$$(m_i - e_i)^{\mathrm{T}} y^i + q_i \in [hy]^i,$$
$$(m_i - e_i)^{\mathrm{T}} z^i + q_i \in [hz]^i,$$
$$(m_i - e_i)^{\mathrm{T}} x + q_i \in [hx]^i.$$

where y^i and z^i are the optimal solution of problems (3.1) and (3.2) . Then $G_i(x, [x])$ is defined as follows:

1. If $\inf [hy]^i \geq 0$ (i. e. $S_i^- = \emptyset$), $G_i(x, [x]) = e_i^{\mathrm{T}}$. Otherwise perform step 2.
2. If $\sup [hz]^i \leq 0$ (i. e. $S_i^+ = \emptyset$), $G_i(x, [x]) = m_i^{\mathrm{T}}$. Otherwise perform step 3.
3. If $\inf [hy]^i < 0 < \sup [hz]^i$ then we have one of the following cases
   3.1 If $0 \in [hx]^i$, let $G_i(x, [x]) = [\min(e_i^{\mathrm{T}}, m_i^{\mathrm{T}}), \max(e_i^{\mathrm{T}}, m_i^{\mathrm{T}})]$.

   3.2 If $\inf [hx]^i > 0$ (which implies $S_i^+ \neq \emptyset$), let

   $$[T]_i \text{ be an enclosure of } \left(1 - \frac{\inf [hy]^i}{\inf [hx]^i}\right)^{-1}, \quad t_i = \inf [T]_i,$$

   and

   $$G_i(x, [x]) = m_i^{\mathrm{T}} + [t_i, 1](e_i - m_i)^{\mathrm{T}}.$$

3.3 If $\sup{[hx]^i} < 0$ (which implies $S_i^- \neq \emptyset$), let

$$[T]_i \text{ be an enclosure of } \left(1 - \frac{\sup{[hz]^i}}{\sup{[hx]^i}}\right)^{-1}, \quad t_i = \inf{[T]_i},$$

and

$$G_i(x, [x]) = e_i^{\mathrm{T}} + [t_i, 1](m_i - e_i)^{\mathrm{T}}.$$

□

With the slope $G(x, [x])$ computed by using the above modification of algorithm (3.1), we choose

$$A = \begin{cases} \mathrm{mid}G(x,[x]) & \text{if } \mathrm{mid}G(x,[x]) \text{ is nonsingular} \\ \mathrm{mid}G(x,[x]) + 10^{-6}I & \text{if } \mathrm{mid}G(x,[x]) \text{ is singular.} \end{cases}$$

Then we calculate $L(x, A, [x])$ where we use an approximation $\mathrm{inv}A$ of A^{-1}, which is computed by the module matinv in Hammer, Hocks, Kulisch and Ratz [13]. With the exception of Example 4.5 we know an exact solution x^* of the problem. Then we take an approximate solution of the form

$$x = x^* - r\alpha e, \tag{4.8}$$

where α is a given parameter in the interval $(-1, 1)$ and we consider the interval $[x]$ given by (3.4). Therefore the starting interval $[x]$ is

$$[x] = x^* + r\,[-1-\alpha, 1-\alpha]\,e$$

and it always contains x^*. All examples that have exact solutions are computed using several values of the shifting coefficient α and the dimension n. For each pair (α, n), we examine the range of r for which the validation (3.6) is successful. We have performed extensive testing of the method. In the following tables we give only two values r_a, r_b for which the validation was performed successfully. It is likely that the verification will be successful for any radius $r \in [r_a, r_b]$. We have verified this first for three values $r_i = r_a + i\,\frac{r_b - r_a}{4}$, $i = 1, 2, 3$. Furthermore we have chosen three values between r_a and r_b in a geometric progression by defining $q := (r_b/r_a)^{\frac{1}{4}}$ and choosing $r_i = q^i r_a$, $i = 1, 2, 3$. (This choice was suggested by one of the referees since for the arithmetic progression considered before, the three values are all of the same order as the right end point r_b). Also in this case the verification was successful for all r_i and for all examples 4.2, 4.3, 4.4.

In many cases verification was possible for much smaller values of r, but we decided to test with 10^{-16} which is sufficiently small for all practical purposes.

The LCP presented in Example 4.5 has no solution although it has an ϵ-approximate solution x with $\epsilon = 6 \cdot 10^{-6}$. In this case we show that condition (3.7) is satisfied for $r = 0.25$ which guarantees that the problem has no exact solution in $[x]$.

Example 4.2 (Murty [23])

$$M = \begin{pmatrix} 1 & 2 & 2 & \ldots & 2 \\ 0 & 1 & 2 & \ldots & 2 \\ 0 & 0 & 1 & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}, \quad q = -(1, \ldots, 1)^{\mathrm{T}}.$$

The solution of the LCP is

$$x^* = (0, \ldots, 0, 1).$$

The radii for successful verification are:

α		$n = 5$	$n = 10$	$n = 20$	$n = 50$	$n = 100$
-0.75	r_a	10^{-16}	10^{-16}	10^{-16}	10^{-16}	10^{-16}
	r_b	1	$3 \cdot 10^{-1}$	10^{-1}	$4 \cdot 10^{-2}$	$2 \cdot 10^{-2}$
-0.5	r_a	10^{-16}	10^{-16}	10^{-16}	10^{-16}	10^{-16}
	r_b	$4 \cdot 10^{-1}$	10^{-1}	$5 \cdot 10^{-2}$	$2 \cdot 10^{-2}$	10^{-2}
0	r_a	10^{-16}	10^{-16}	10^{-16}	10^{-16}	10^{-16}
	r_b	10^{-1}	$6 \cdot 10^{-2}$	$2 \cdot 10^{-2}$	10^{-2}	$5 \cdot 10^{-3}$
0.5	r_a	10^{-16}	10^{-16}	10^{-16}	10^{-16}	10^{-16}
	r_b	$9 \cdot 10^{-2}$	$3 \cdot 10^{-2}$	10^{-2}	$6 \cdot 10^{-3}$	$3 \cdot 10^{-3}$
0.75	r_a	10^{-16}	10^{-16}	10^{-16}	10^{-16}	10^{-16}
	r_b	$8 \cdot 10^{-2}$	$3 \cdot 10^{-2}$	10^{-2}	$5 \cdot 10^{-3}$	$2 \cdot 10^{-3}$

Example 4.3 (Fathi [9])

$$M = \begin{pmatrix} 1 & 2 & 2 & \ldots & 2 \\ 2 & 5 & 6 & \ldots & 6 \\ 2 & 6 & 9 & \ldots & 10 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 6 & 10 & \ldots & 4(n-1)+1 \end{pmatrix}, \quad q = -(1, \ldots, 1)^{\mathrm{T}}.$$

The solution of the LCP is

$$x^* = (1, 0, \ldots, 0).$$

The radii for successful verification are:

α		$n=5$	$n=10$	$n=20$	$n=50$	$n=100$
-0.75	r_a	10^{-16}	10^{-16}	10^{-16}	10^{-16}	10^{-16}
	r_b	$7\cdot10^{-2}$	$2\cdot10^{-2}$	$5\cdot10^{-3}$	$8\cdot10^{-4}$	$2\cdot10^{-4}$
-0.5	r_a	10^{-16}	10^{-16}	10^{-16}	10^{-16}	10^{-16}
	r_b	$4\cdot10^{-4}$	10^{-2}	$3\cdot10^{-3}$	$4\cdot10^{-4}$	10^{-4}
0	r_a	10^{-16}	10^{-16}	10^{-16}	10^{-16}	10^{-16}
	r_b	$2\cdot10^{-2}$	$5\cdot10^{-3}$	10^{-3}	$2\cdot10^{-4}$	$5\cdot10^{-5}$
0.5	r_a	10^{-16}	10^{-16}	10^{-16}	10^{-16}	10^{-16}
	r_b	10^{-2}	$3\cdot10^{-3}$	$8\cdot10^{-4}$	10^{-4}	$3\cdot10^{-5}$
0.75	r_a	10^{-16}	10^{-16}	10^{-16}	10^{-16}	10^{-16}
	r_b	10^{-2}	$2\cdot10^{-3}$	$7\cdot10^{-4}$	10^{-4}	$2\cdot10^{-5}$

Example 4.4 The dual of the linear program

$$\begin{array}{ll} \min & c^{\mathrm{T}}u \\ & \text{s.t. } Au \geq b\,,\ u \geq 0, \end{array} \tag{4.9}$$

can be written under the form

$$\begin{array}{ll} \max & b^{\mathrm{T}}y \\ & \text{s.t. } A^{\mathrm{T}}y \leq c\,,\ y \geq 0. \end{array} \tag{4.10}$$

By introducing the slack variables $v \geq 0$ and $z \geq 0$, the programs (4.9) and (4.10) can be rewritten as

$$\begin{array}{ll} \min & c^{\mathrm{T}}u \\ & \text{s.t. } Au - v = b\,,\ u \geq 0\,,\ v \geq 0, \end{array} \tag{4.11}$$

and

$$\begin{array}{ll} \max & b^{\mathrm{T}}y \\ & \text{s.t. } A^{\mathrm{T}}y + z = c\,,\ y \geq 0\,,\ z \geq 0. \end{array} \tag{4.12}$$

Since for any feasible u, v, y, z we have

$$\begin{aligned} c^{\mathrm{T}}u - b^{\mathrm{T}}y &= u^{\mathrm{T}}(A^{\mathrm{T}}y + z) - y^{\mathrm{T}}(Au - v) \\ &= u^{\mathrm{T}}z + y^{\mathrm{T}}v \geq 0, \end{aligned}$$

it is easily seen that u, v, y, z is an optimal solution of (4.11) and (4.12) if and only if $x = (u, y)$ and $s = (z, v)$ is a solution of an LCP of the form

$$s = Mx + q\,,\; s \geq 0\,,\; x \geq 0\,,\; x^{\mathrm{T}} s = 0 \tag{4.13}$$

with

$$M = \begin{pmatrix} 0 & -A^{\mathrm{T}} \\ A & 0 \end{pmatrix}, \quad q = (c, -b).$$

For example in [20, p. 46] a problem of the form (4.10) with

$$A^{\mathrm{T}} = \begin{pmatrix} 2\,1\,1 \\ 1\,2\,3 \\ 2\,2\,1 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}, \quad c = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$$

is solved by the simplex method and the exact solution

$$y = \begin{pmatrix} 1/5 \\ 0 \\ 8/5 \end{pmatrix}, \quad z = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$

is found. The corresponding primal problem (4.12) has the exact solution

$$u = \begin{pmatrix} 6/5 \\ 3/5 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 1/5 \\ 0 \end{pmatrix}.$$

Therefore the solution of the LCP (4.13) is

$$\begin{aligned} x^* &= (6/5, 3/5, 0, 1/5, 0, 8/5) \\ s^* &= (0, 0, 4, 0, 1/5, 0). \end{aligned}$$

The radii for successful verification are:

α		$n=6$
-0.75	r_a	10^{-15}
	r_b	$2 \cdot 10^{-2}$
-0.5	r_a	10^{-15}
	r_b	$2 \cdot 10^{-2}$
0	r_a	10^{-15}
	r_b	$5 \cdot 10^{-2}$
0.5	r_a	10^{-15}
	r_b	$6 \cdot 10^{-2}$
0.75	r_a	10^{-15}
	r_b	$5 \cdot 10^{-2}$

Example 4.5 Let us consider now an LCP of the form (4.13) with

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 2 \\ 1 \\ -10^{-6} \end{pmatrix}$$

The primal-dual pair

$$x = \begin{pmatrix} 10^{-6} \\ 10^{-6} \\ 1 \end{pmatrix}, \quad s = \begin{pmatrix} 3 \\ 2 \\ 10^{-6} \end{pmatrix}$$

is an approximate solution of this LCP in the sense that

$$x^{\mathrm{T}} s = 6 \cdot 10^{-6}, \; \|s - Mx - q\|_\infty = 4 \cdot 10^{-6}. \tag{4.14}$$

However the LCP has no solution since this LCP corresponds to a linear programming problem whose primal (4.9) is infeasible (here $A = (-1, -1)$, $b = 10^{-6}$, $c = (2, 1)^{\mathrm{T}}$). By taking $[x] = x + r[-e, e]$ with $r = 0.25$ we obtain

$$[x] \cap L(x, A, [x]) = \emptyset, \tag{4.15}$$

which shows that there is no solution of the LCP in $[x]$. We note that in infeasible interior point methods [31] one often uses the stopping criterion

$$\max\{x^{\mathrm{T}} s, \; \|s - Mx - q\|_\infty\} \leq \epsilon. \tag{4.16}$$

A primal-dual pair satisfying (4.16) is called an ϵ-approximate solution. Relation (4.15) guarantees that no exact solution exists within an l_∞ distance of 0.25 from an ϵ-approximate solution with $\epsilon = 6 \cdot 10^{-6}$!

Random test problems

The ideas of this paper have also been extensively tested on a number of randomly generated problems with known solution characteristics, so that different features of the algorithms can be tested. (cf.[4,6]). The procedure for generating test problems allows the user to specify

- the size of the problem: n,
- the condition number of the matrix M: τ
- the structure of a solution x^*,
  the number of components: n_1, $(Mx^* + q)_i = 0, x_i^* = 0, i \leq n_1$
  the number of components: n_2, $(Mx^* + q)_i > 0, x_i^* = 0, i \leq n_2$
  the number of components: n_3, $(Mx^* + q)_i = 0, x_i^* > 0, i \leq n_3$
  $n_1 + n_2 + n_3 = n$.
- the range of $(Mx^* + q)_i \in [0, m_1]$
- the range of $x_i^* \in [0, m_2]$.

Method for generating an LCP

1. Generate $M \in \mathbb{R}^{n\times n}$.

Randomly generate two orthogonal matrices $U \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ [18,19]. Define a diagonal matrix $\Sigma \in \mathbb{R}^{n\times n}$ whose diagonal elements are

$$\begin{aligned}
&\Sigma_{1,1} = 1/\tau \\
&\Sigma_{i,i} = \tau^{v_i}, \quad i = 2, \ldots, \gamma - 1 \\
&\Sigma_{\gamma,\gamma} = \tau, \\
&\Sigma_{\gamma+1,\gamma+1} = 0, \ldots, \Sigma_{n,n} = 0
\end{aligned}$$

where $\gamma \leq n$ is the rank of Σ and $v_i, i = 2, \ldots, \gamma - 1$ are uniformly distributed in the interval $(-1, 1)$. Let

$$M = U\Sigma Q.$$

The matrix M has the smallest non-zero singular value $1/\tau$ and the largest singular value τ. If the rank γ of M is equal to n, then the condition number of M is τ^2. If $U = Q^{\mathrm{T}}$, then M is a symmetric positive semi-definite matrix.

2. Generate $q \in \mathbb{R}^n$. First set

$$J = (\underbrace{0, \ldots, 0}_{n_1}, \underbrace{1, \ldots, 1}_{n_2}, 2, \ldots, 2)^{\mathrm{T}} \in \mathbb{R}^n.$$

Next make a "perfect shuffle" in J [6,18] such that the numbers 0,1 and 2 are randomly distributed in J.

Randomly generate two positive vectors $\hat{q}, \hat{x} \in \mathbb{R}^n$ with elements in the range $(0, m_1)$ and $(0, m_2)$ respectively. Let

$$x_i^* = \begin{cases} \hat{x}_i \text{ if } J_i = 2 \\ 0 \ \text{ otherwise. } i = 1, 2, ..., n \end{cases}$$

and let

$$q_i = \begin{cases} -(Mx^*)_i + \hat{b}_i \text{ if } J_i = 1 \\ -(Mx^*)_i \qquad\quad \text{otherwise } i = 1, 2, ..., n. \end{cases}$$

The tests on the randomly generated problems show that our enclosing method is very robust. The validation is obtained in one step, but the enclosure can be improved by iterating with the Krawczyk operator L. In case of problems with strictly complementary solutions (when $n_1 = 0$) an exact (up to machine precision) enclosure is obtained in one or two iterations. In the degenerate case (when $n_1 \geq 0$) tens of iterations are needed in order to obtain an exact enclosure, since the convergence of the iterative procedure is linear. We only give the first iteration for two four-dimensional examples, one with $n_1 = 0$ and one with $n_1 = 1$.

Examples

$n_1 = 0$, $n_2 = 2$, $n_3 = 2$, $\tau = 2.000000000000000$,
$r = 2.000000000000000 \cdot 10^{-1}$, $\alpha = 0$, $M = (M_1|M_2|M_3|M_4)$

$$M_1 = \begin{pmatrix} 1.388713122168711 \\ -4.699766249426920 \cdot 10^{-1} \\ 7.370559770214220 \cdot 10^{-2} \\ -4.110090461033111 \cdot 10^{-1} \end{pmatrix},$$

$$M_2 = \begin{pmatrix} -4.699766249426920 \cdot 10^{-1} \\ 1.453401598450949 \\ 3.334909523505895 \cdot 10^{-2} \\ -5.175564143615730 \cdot 10^{-1} \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 7.370559770214220 \cdot 10^{-2} \\ 3.334909523505895 \cdot 10^{-2} \\ 6.604515405730874 \cdot 10^{-1} \\ -1.651162344083680 \cdot 10^{-1} \end{pmatrix},$$

$$M_4 = \begin{pmatrix} -4.110090461033111 \cdot 10^{-1} \\ -5.175564143615730 \cdot 10^{-1} \\ -1.651162344083680 \cdot 10^{-1} \\ 1.477373564900058 \end{pmatrix}$$

$$x_{\text{sol}} = \begin{pmatrix} 0.000000000000000 \\ 0.000000000000000 \\ 2.908386450683878 \\ 2.251076643937769 \end{pmatrix}, \quad q = \begin{pmatrix} 8.679035675427925 \cdot 10^{-1} \\ 2.692546385763099 \\ -1.549159013124430 \\ -2.845459307376360 \end{pmatrix}$$

$$\begin{aligned} [x] = & [\,-2.000000000001 \cdot 10^{-1}, 2.000000000001 \cdot 10^{-1}] \\ & [\,-2.000000000001 \cdot 10^{-1}, 2.000000000001 \cdot 10^{-1}] \\ & [\,2.708386450683, 3.108386450684] \\ & [\,2.051076643937, 2.451076643938] \\ L = & [\,-5.281586749876 \cdot 10^{-2}, 5.281586749876 \cdot 10^{-2}] \\ & [\,0.000000000000, 0.000000000000] \\ & [\,2.906101880631, 2.910671020737] \\ & [\,2.236638467824, 2.265514820052] \end{aligned}$$

$n_1 = 1, n_2 = 2, n_3 = 1, \tau = 2.000000000000000$,
$r = 2.000000000000000 \cdot 10^{-1}, \alpha = 0, M = (M_1|M_2|M_3|M_4)$

$$M_1 = \begin{pmatrix} 1.388713122168711 \\ -4.699766249426920 \cdot 10^{-1} \\ 7.370559770214220 \cdot 10^{-2} \\ -4.110090461033111 \cdot 10^{-1} \end{pmatrix},$$

$$M_2 = \begin{pmatrix} -4.699766249426920 \cdot 10^{-1} \\ 1.453401598450949 \\ 3.334909523505895 \cdot 10^{-2} \\ -5.175564143615730 \cdot 10^{-1} \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 7.370559770214220 \cdot 10^{-2} \\ 3.334909523505895 \cdot 10^{-2} \\ 6.604515405730874 \cdot 10^{-1} \\ -1.651162344083680 \cdot 10^{-1} \end{pmatrix},$$

$$M_4 = \begin{pmatrix} -4.110090461033111 \cdot 10^{-1} \\ -5.175564143615730 \cdot 10^{-1} \\ -1.651162344083680 \cdot 10^{-1} \\ 1.477373564900058 \end{pmatrix}$$

$$x_{\text{sol}} = \begin{pmatrix} 0.000000000000000 \\ 0.000000000000000 \\ 0.000000000000000 \\ 2.251076643937769 \end{pmatrix}, \quad q = \begin{pmatrix} 9.252128641303051 \cdot 10^{-1} \\ 2.789538442487311 \\ 9.950524251712144 \cdot 10^{-1} \\ -3.325681126317601 \end{pmatrix}$$

$$\begin{aligned} [x] = {} & [-2.000000000001 \cdot 10^{-1}, 2.000000000001 \cdot 10^{-1}] \\ & [-2.000000000001 \cdot 10^{-1}, 2.000000000001 \cdot 10^{-1}] \\ & [-2.000000000001 \cdot 10^{-1}, 2.000000000001 \cdot 10^{-1}] \\ & [2.051076643937, 2.451076643938] \\ L = {} & [-1.181342316972 \cdot 10^{-1}, 1.181342316972 \cdot 10^{-1}] \\ & [0.000000000000, 0.000000000000] \\ & [0.000000000000, 0.000000000000] \\ & [2.218211403191, 2.283941884684] \end{aligned}$$

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