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The Shape of the Solution Set for Systems of Interval Linear Equations with Dependent Coefficients

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Abstract. A standard system of interval linear equations is defined by $\mathbf{A}x = \mathbf{b}$, where $\mathbf{A}$ is an $m \times n$ coefficient matrix with (compact) intervals as entries, and $\mathbf{b}$ is an m-dimensional vector whose components are compact intervals. It is known that for systems of interval linear equations the solution set, i. e., the set of all vectors x for which $Ax = b$ for some $A \in \mathbf{A}$ and $b \in \mathbf{b}$, is a polyhedron.

In some cases, it makes sense to consider not all possible $A \in \mathbf{A}$ and $b \in \mathbf{b}$, but only those A and b that satisfy certain linear conditions describing dependencies between the coefficients. For example, if we allow only symmetric matrices A $\left(a_{ij} = a_{ji}\right)$, then the corresponding solution set becomes (in general) piecewise-quadratic.

In this paper, we show that for general dependencies, we can have arbitrary (semi)algebraic sets as projections of solution sets.

1. Informal Introduction

Many real-life problems require solution of the systems of linear equations $Ax = b$, or

$$\sum_{j=1}^{n} a_{ij}x_j = b_i ,$$

where the coefficients a_{ij} and b_i are known, and the values x_i have to be determined.

In applications, the values of the physical quantities that are denoted by these coefficients a_{ij} and b_i come from measurement and are, therefore, only approximately known. For each of these quantities c, the only information that we usually have after the measurement is that the difference $\Delta c = \tilde{c} - c$ between the actual (unknown) value

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c and the measured value $\tilde{c}$ cannot exceed the bound $\Delta(c)$ guaranteed by the manufacturer of the corresponding measuring instrument. In other words, we know that the actual value c belongs to the interval $\mathbf{c} = [\tilde{c} - \Delta(c), \tilde{c} + \Delta(c)]$. In such situations, we arrive at the following problem:

We know: the intervals $\mathbf{a}_{ij}$ and $\mathbf{b}_i$ for the coefficients.

We want to find: the solution set, i. e., the set of all the vectors $x = (x_1, \ldots, x_n)$ for which $\sum_j a_{ij} x_j = b_i$ for some $a_{ij} \in \mathbf{a}_{ij}$ and $b_i \in \mathbf{b}_i$.

In many applications, we are interested only in some of the variables $(x_1, \ldots, x_n)$. In this case, in mathematical terms, we are interested in the projection of the solution set on a subspace formed by the desired variables.

This problem is usually described as the problem of solving the system of interval linear equations $\sum_j \mathbf{a}_{ij} x_j = \mathbf{b}_i$, or $\mathbf{A}x = \mathbf{b}$, where $\mathbf{A}$ and $\mathbf{b}$ are a matrix and a vector with interval components $\mathbf{a}_{ij}$ and $\mathbf{b}_i$, respectively. For examples of applications of these systems, see, e. g., [K, KK].

It is known that the solution set for this system is a polyhedron [OP, Be, H, R, N, AM, AKM96], and therefore, its projection is also a polyhedron.

The above description of a solution set makes sense if the measurement errors in all coefficients are independent; in other words, if the values of a_{ij} and b_i are not a priori related. In real life, sometimes, they are related. For example, in some physically meaningful situations, we know that the matrix a_{ij} is symmetric ($a_{ij} = a_{ji}$ [J]). In this case, it makes sense to look only for such vectors x for which $\sum a_{ij} x_j = b_i$ for some $b_i \in \mathbf{b}_i$ and for a symmetric matrix $a_{ij} \in \mathbf{a}_{ij}$. It turns out [AKM96], that the shape of such symmetric solution set is not necessarily a polyhedron (i. e., a set with a piecewise-linear border): in general, it is a set with a piecewise-quadratic border.

A similar shape can be proven for the case when we know that the matrix a_{ij} is skew-symmetric [J, AKM95].

In some applications [J] (especially in control [Ba]), we can have even more complicated dependencies between the coefficients. How can we describe these dependencies in precise mathematical terms? The main reason for this dependency is that the errors in several different coefficients may be caused by the same factor. Let us denote all the factors that influence the coefficients by $f_1, \ldots, f_p$. Then, the coefficients a_{ij} and b_i depend on these factors: $a_{ij} = a_{ij}(f_1, \ldots, f_p)$ and $b_i = b_i(f_1, \ldots, f_p)$. For each of these independent factors f_α, we know the interval $\mathbf{f}_\alpha$ of possible values. These factors are usually small, so, we can neglect quadratic terms in the dependency of the coefficients on f_α, and thus restrict ourselves to the case when the dependency is linear, i. e., when $a_{ij} = a_{ij}^{(0)} + \sum_\alpha a_{ij\alpha} f_\alpha$ and $b_i = b_i^{(0)} + \sum_\alpha b_{i\alpha} f_\alpha$. In this case, we can define a solution set to be the set of all possible vectors x for which for some $f_\alpha \in \mathbf{f}_\alpha$, $\sum a_{ij} x_j = b_i$ for the corresponding a_{ij} and b_i.

As an example, we can consider the following dependency of the coefficients a_{ij}, $1 \le i, j \le 2$, on the factors f_α:

$$A = \begin{pmatrix} 5 & 2 + f_1 \\ f_1 + f_2 & 7 \end{pmatrix},$$

where $f_1 \in [-1,1]$ and $f_2 \in [0,1]$. This dependency can be described by

$$A = A^{(0)} + \sum_{\alpha=1}^{2} A^{(\alpha)} f_\alpha ,$$

where

$$A^{(0)} = (a_{ij}^{(0)}) = \begin{pmatrix} 5 & 2 \\ 0 & 7 \end{pmatrix}, \quad A^{(1)} = (a_{ij1}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$A^{(2)} = (a_{ij2}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Standard systems of interval linear equations $\mathbf{A}x = \mathbf{b}$ can be viewed as a particular case of this definition: namely, we can take $p = m \cdot n + m$, so that each of $m \times n$ coefficients a_{ij} and each of m coefficients b_i are equal to the corresponding f_α, with $\mathbf{f}_\alpha$ equal to $\mathbf{a}_{ij}$ or, correspondingly, to $\mathbf{b}_i$.

Similarly, symmetric systems of interval linear equations can be thus represented, if we take factors f_α that correspond to b_i and factors f_α that correspond to unordered pairs (i,j); in this case, $p = n(n+1)/2 + n$, and a_{ij} and a_{ji} are equal to one and the same factor f_α. For example, a general symmetric interval 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{pmatrix}$$

with $\mathbf{a}_{12} = \mathbf{a}_{21}$ can be represented as

$$A = A^{(0)} + A^{(1)} f_1 + A^{(2)} f_2 + A^{(1)} f_3 ,$$

where

$$A^{(0)} = 0, \quad A^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A^{(3)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$f_1 \in \mathbf{a}_{11}, \quad f_2 \in \mathbf{a}_{12} = \mathbf{a}_{21}, \quad f_3 \in \mathbf{a}_{22}.$$

What shapes can the correspondent solution sets and their projections have?

(a) For no dependencies, we get piecewise-linear shapes.

(b) For the simplest possible dependencies (i. e., for symmetric or skew-symmetric matrices), we get piecewise-quadratic shapes.

(c) It is natural to assume that in the general case of dependencies, we will get algebraic dependencies of arbitrary order.

In this paper, we prove that this assumption is correct. Thus, we get a complete description of all possible shapes.

2. Definitions and the main result

Definition 2.1. (i) Let m, n, and p be integers. By a *system of interval linear equations with dependent coefficients*, we mean a tuple

$$\left(\left\{a_{ij}^{(0)}\right\}, \left\{a_{ij\alpha}\right\}, \left\{b_i^{(0)}\right\}, \left\{b_{i\alpha}\right\}, \left\{\mathbf{f}_\alpha\right\}\right), \quad 1 \le i \le m, \quad 1 \le j \le m, \quad 1 \le \alpha \le p,$$

where $a_{ij}^{(0)}$, $a_{ij\alpha}$, $b_i^{(0)}$, and $b_{i\alpha}$ are real numbers, and $\mathbf{f}_\alpha$ are intervals.

(ii) We say that a vector $x = (x_1, \ldots, x_n)$ is a *solution* of the system of interval linear equations with dependent coefficients if for some $f_\alpha \in \mathbf{f}_\alpha$, we have

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad \text{for all} \quad i, \tag{2.1}$$

where

$$a_{ij} = a_{ij}^{(0)} + \sum_{\alpha=1}^{p} a_{ij\alpha} f_\alpha \tag{2.2}$$

and

$$b_i = b_i^{(0)} + \sum_{\alpha=1}^{p} b_{i\alpha} f_\alpha . \tag{2.3}$$

The set of all solutions of a given system is called its *solution set.*

(iii) Let $I = \{i_1, \ldots, i_q\} \subset \{1, \ldots, n\}$. By a *projection of the solution set on* I, we mean the set of all vectors $(x_{i_1}, \ldots, x_{i_q}) \in \mathbb{R}^q$ that can be extended to a solution $(x_1, \ldots, x_n)$ of a system.

To describe projections of solution sets in the general case, we need the following definition (see, e. g., [A]):

Definition 2.2. A set $S \subseteq \mathbb{R}^q$ is called *semialgebraic* if it is a finite union of subsets, each of which is defined by a finite system of polynomial equations $P_r(x_1, \ldots, x_q) = 0$ and inequalities of the types $P_s(x_1, \ldots, x_q) > 0$ and $P_t(x_1, \ldots, x_q) \ge 0$ (for some polynomials P_i).

Theorem 2.3. (i) *Each projection of the solution set of a system of interval linear equations with dependent coefficients is semialgebraic.*

(ii) *Every semialgebraic set can be represented as a projection of the solution set of some system of interval linear equations with dependent coefficients.*

3. Proof

Remark 3.1. To make this proof easier to read, we will emphasize certain important parts of the proof. We will use three different types of emphasis:

(a) The important mini-goals that we are going to achieve in the course of proving the theorem will be underlined by a single line.

(b) The important methods that we use to achieve these mini-goals will be underlined by a double line.

(c) Finally, the formulations of the important intermediate mini-results (lemmas) that are proved in the course of proving the theorem, and important displayed formulas, will be framed.

Proof of the first part. The first part of this theorem follows directly from the famous Tarski-Seidenberg theorem [T, S] (see also [A]), according to which, crudely speaking, every relation that is obtained from a semialgebraic relation by adding quantifiers $\forall x$, $\exists x$ (that run over all real numbers x), is still semialgebraic. In other words, if the set $S \subset \mathbb{R}^v$ of all tuples $(z_1, \ldots, z_v)$ that satisfy a certain relation $P(z_1, \ldots, z_v)$ is semialgebraic, then the set of all tuples that satisfy the relation $(Q_1 z_1)(Q_2 z_2) \ldots P(z_1, \ldots)$, where each Q_i is a quantifier ($\forall$ or $\exists$), is also semialgebraic.

For our problem, a vector $(x_{i_1}, \ldots, x_{i_q}) \in \mathbb{R}^q$ belongs to the projection $\pi(S)$ of the solution set S iff there exist real numbers $x_1, \ldots, x_n, f_1, \ldots, f_p, \{a_{ij}\}$, and $\{b_i\}$ that satisfy the algebraic equalities (2.1) – (2.3) and inequalities $f_\alpha^- \le f_\alpha \le f_\alpha^+$, where $f_\alpha^\pm$ denote the bounds of the interval $\mathbf{f}_\alpha$ $\left(\mathbf{f}_\alpha = [f_\alpha^-, f_\alpha^+]\right)$. Formally,

$$\pi(S) = \{(x_{i_1}, \ldots, x_{i_q}) \in \mathbb{R}^q \mid \exists x_1 \ldots \exists x_n \exists f_1 \ldots \exists f_p : (2.1) - (2.3) \text{ hold, and } f_\alpha^- \le f_\alpha \le f_\alpha^+\}.$$

Conditions (2.1) – (2.3) are polynomial equalities, and p conditions $f_\alpha^- \le f_\alpha \le f_\alpha^+$, $1 \le \alpha \le p$, are polynomial inequalities. Therefore, the set of all tuples

$$(x_1, \ldots, x_n, f_1, \ldots, f_p)$$

that satisfy (2.1) – (2.3) and these inequalities for f_α is a semialgebraic set. Hence, by Tarski-Seidenberg's theorem, the projection $\pi(S)$ is also a semialgebraic set.

Proof of the second part. Let us now prove the second part of the theorem, that every semialgebraic set $S \subseteq \mathbb{R}^q$ can be represented as a projection of an appropriate solution set. We will construct the corresponding system of interval linear equations with dependent coefficients step-by-step. Initially, we start with the variables x_1, $\ldots$, x_q. On each step, we will add new variables, new parameters f_α, and new equations.

Remark 3.2. Some constructions used in this proof were originally proposed (for a different purpose) in [KLN].

1. First, we will add the following new variables:

(a) variables $x[P_s]$ that correspond to all the polynomials P_s from the definition of the semialgebraic set S;

(b) variables $x[M_s]$ that correspond to all non-constant monomials M_s that form these polynomials;

(c) variables $x[m]$ that correspond to all the monomials that are obtained from monomials M_s by decreasing the degrees of some (or all) of the variables, and that are different from the variables themselves (for monomials m that coincide with one of the variables x_i, we will take this very variable as corresponding to m, i. e., we will take $x[x_i] = x_i$).

For example, for a polynomial $P = 2x_1^3 + x_1^2x_2^2 + 5$, we add:

(d) a variable $x[P]$ that corresponds to this polynomial;

(e) variables $x[M_1]$ and $x[M_2]$ that correspond to the two monomials $M_1 = x_1^3$ and $M_2 = x_1^2x_2^2$ that form this polynomial, and

(f) variables $x[m]$ that correspond to the monomials x_1^2, $x_1x_2^2$, $x_1^2x_2$, x_1x_2, and x_2^2.

2. We will form the system of equations in such a way that if the variables $x[m]$ (including $x_i = x[x_i]$) are taken from the solution set of this system, then for each monomial m, the value of $x[m]$ is equal to the value $m(x_1, \ldots, x_q)$ of the monomial m, and the value of $x[P]$ is equal to the value $P(x_1, \ldots, x_q)$ of the polynomial P for the given $x_1, \ldots, x_q$. The first equations from the desired interval system are as follows: for every polynomial $P_s = c_0 + c_1M_1 + \cdots + c_zM_z$, we form an equation

$$x[P_s] - c_1x[M_1] - \ldots - c_zx[M_z] = c_0 . \tag{3.1}$$

These equations do not depend on any factors f_α at all; they guarantee that if $x[M_i] = M_i(x_1, \ldots, x_q)$ for all $i = 1, \ldots, z$, then $x[P_s] = P_s(x_1, \ldots, x_q)$.

3. In order to guarantee the proper relationship between the variables that correspond to different monomials (e. g., monomials x_1, x_2, and x_1x_2), we must be sure that if a $m'' = m \cdot m'$, then $x[m''] = x[m] \cdot x[m']$. To be sure in that, we must describe this relationship in terms of a system of interval linear equations.

Actually, since with every variable $x[m]$, we have variables that correspond to all monomials of smaller degree, it is sufficient to consider the case when $m' = m \cdot x_i$. Indeed, if we can guarantee that for all monomials m, $x[m \cdot x_i] = x[m] \cdot x[x_i] = x[m] \cdot x_i$, then, starting with the variables themselves, and adding one variable at a time to the product representing the monomial, we will be able to prove that $x[m] = m(x_1, \ldots, x_q)$ for each monomial m.

Indeed, e. g., for $m = x_1^2x_2^2$, we will be able to prove this property by consequently considering:

$$\begin{aligned} x[x_1] &= x_1 , \\ x[x_1^2] &= x[x_1 \cdot x_1] = x[x_1] \cdot x_1 = x_1^2 , \\ x[x_1^2x_2] &= x[x_1^2 \cdot x_2] = x[x_1^2] \cdot x_2 = x_1^2x_2 , \\ x[x_1^2x_2^2] &= x[x_1^2x_2 \cdot x_2] = x[x_1^2x_2] \cdot x_2 = (x_1^2x_2) \cdot x_2 = x_1^2x_2^2 . \end{aligned}$$

4. So, for our purpose, it is sufficient to be able, for every pair (m, x_i) consisting of a monomial m and a variable x_i, to describe the relationship $x[m \cdot x_i] = x[m] \cdot x_i$ in terms of an appropriate system of interval linear equations.

To do that, for each such pair, we add two new auxiliary variables $x_1[m, x_i]$ and $x_2[m, x_i]$, and two new auxiliary factors $f_1[m, x_i]$ and $f_2[m, x_i]$ with

$$\mathbf{f}_1[m, x_i] = \mathbf{f}_2[m, x_i] = [-1, 1] .$$

These auxiliary factors and variables will only be used in the following equations that describe this relationship:

(3.2) $$\boxed{f_1[m,x_i]\cdot x[m\cdot x_i]+f_2[m,x_i]\cdot x[m] \;=\; 0\,,}$$

(3.3) $$\boxed{f_1[m,x_i]\cdot x_i \;=\; -f_2[m,x_i]\,,}$$

(3.4) $$\boxed{f_1[m,x_i]\cdot x_1[m,x_i]+f_2[m,x_i]\cdot x_2[m,x_i] \;=\; 1\,.}$$

Let us show that for every three variables x_i, $x[m]$, and $x[m\cdot x_i]$, the equality $x[m\cdot x_i]=x[m]\cdot x_i$ holds iff the system (3.2) – (3.4) has a solution for some

$$\boxed{x_1[m,x_i],\;\; x_2[m,x_i],\;\; f_1[m,x_i]\in \mathbf{f}_1[m,x_i],\;\text{ and }\; f_2[m,x_i]\in \mathbf{f}_2[m,x_i]}\,.$$

4.1. Let us first assume that $x[\ldots]$ and $f[\ldots]\in \mathbf{f}[\ldots]$ satisfy the equations (3.2) – (3.4), and show that in this case, $x[m\cdot x_i]=x[m]\cdot x_i$.

Let us first show that $f_1[m,x_i]\neq 0$. Indeed, if $f_1[m,x_i]=0$, then, from (3.3), we will conclude that $f_2[m,x_i]$ is also equal to 0. Therefore, the left-hand side of (3.4) is equal to 0, and it cannot be equal to 1. The contradiction shows that $f_1[m,x_i]\neq 0$.

Since $f_1[m,x_i]\neq 0$, from equation (3.2), we conclude that

(3.5) $$x[m\cdot x_i] \;=\; -\frac{f_2[m,x_i]}{f_1[m,x_i]}\,x[m]\,,$$

and from (3.3), we conclude that

(3.6) $$x_i \;=\; -\frac{f_2[m,x_i]}{f_1[m,x_i]}\,.$$

Substituting (3.6) into (3.5), we conclude that $x[m\cdot x_i]=x[m]\cdot x_i$.

4.2. Let us now show that if $x[m\cdot x_i]=x[m]\cdot x_i$, then there exist values $x[\ldots]$ and $f[\ldots]\in \mathbf{f}[\ldots]$ that satisfy (3.2) – (3.4).

To prove this statement, we will consider two possible cases:

(a) If $|x_i|\le 1$, we take $f_1[m,x_i]=1$, $f_2[m,x_i]=-x_i$, $x_1[m,x_i]=1$, and $x_2[m,x_i]=0$. It is easy to see that $f[\ldots]\in \mathbf{f}[\ldots]=[-1,1]$, and that equations (3.2) – (3.4) are satisfied.

(b) If $|x_i|>1$, we take $f_1[m,x_i]=-1/x_i$, $f_2[m,x_i]=1$, $x_1[m,x_i]=0$, and $x_2[m,x_i]=1$.

5 – 11. Let us now describe how to represent equalities and inequalities that are used in the definition of a semialgebraic set.

By definition, a semialgebraic set S is a finite union $S=S_1\cup\ldots\cup S_u$ of the sets S_k that are described by these equalities and inequalities.

5 – 7. The case of $u=1$.

5. Let us first consider the simplest case, when this union consists of only one such set (i. e., when $u=1$). In this case, an equality $P_r=0$ can be described by adding the equation $x[P_r]=0$ to our system. Let us now show how to represent the inequalities $x[P]\ge 0$ and $x[P]>0$.

6. To represent the inequality $x[P] \geq 0$, we add an auxiliary factor $f[P]$, and an equation

$$\text{(3.7)} \qquad \boxed{f[P] \cdot x[P] \;=\; 1 - f[P]\,,}$$

where $\mathbf{f}[P] = [0, 1]$. Let us show that $\boxed{x[P] \geq 0 \text{ iff this equation has a solution.}}$

Indeed, if $x[P] \geq 0$, then we can take $f[P] = 1/(1 + x[P])$. This value is in [0,1], and the direct substitution confirms that it satisfies the equation.

Vice versa, if the equation is satisfied, and $f[P] \in [0, 1]$, then $f[P] \geq 0$ and $1 - f[P] \geq 0$. Let us show that $f[P] > 0$. Indeed, if $f[P] = 0$, then from the equation, we would conclude that $1 - f[P] = 0$, i. e., that $f[P] = 1 \neq 0$. The contradiction shows that $f[P] > 0$. Hence, from the equation, we conclude that $x[P] = (1 - f[P])/f[P] \geq 0$.

7. We will represent strict inequality $x[P] > 0$ as a system consisting of the above-described representation of an inequality $x[P] \geq 0$, and another system that represents $x[P] \neq 0$.

To represent the relation $x[P] \neq 0$, we will introduce two auxiliary variables $x_1[P]$ and $x_2[P]$ and two factors $f_1[P]$ and $f_2[P]$ with $\mathbf{f}_1[P] = \mathbf{f}_2[P] = [-1, 1]$, and add three new equations:

$$\text{(3.8)} \qquad \boxed{f_1[P] \cdot x[P] \;=\; f_2[P]\,,}$$

$$\text{(3.9)} \qquad \boxed{f_1[P] \cdot x_1[P] \;=\; 1\,,}$$

$$\text{(3.10)} \qquad \boxed{f_2[P] \cdot x_2[P] \;=\; 1\,.}$$

Let us show that $x[P] \neq 0$ iff this system has a solution.

7.1. Let us assume that $x[P] \neq 0$. To show that a solution exists, we will consider two possible cases:

(a) If $|x[P]| \leq 1$, $x[P] \neq 0$, then we can take $f_1[P] = 1$, $f_2[P] = x[P]$, $x_1[P] = 1$, $x_2[P] = 1/(x[P])$.

(b) If $|x[P]| > 1$, then we can take $f_1[P] = 1/(x[P])$, $f_2[P] = 1$, $x_1[P] = x[P]$, and $x_2[P] = 1$.

7.2. Let us now assume that the system (3.8) – (3.10) has a solution. Then, from (3.9), we conclude that $f_1[P] \neq 0$; from (3.10), that $f_2[P] \neq 0$; and therefore, from (3.8), that $x[P] = -f_2[P]/f_1[P] \neq 0$.

Conclusion for $u = 1$. Since we are now able to represent equalities and inequalities in terms of linear interval equations with dependent coefficients, we have thus concluded the proof of the second part of the theorem for the case when the semialgebraic set consists of only one component.

8 – 11. The case $u \geq 2$.

8. Let us now consider the case when the given semialgebraic set S is a union of finitely many sets $S = S_1 \cup \ldots \cup S_u$, $u \geq 2$, and each of the sets $S_1, \ldots, S_u$ is described by a system of polynomial equalities and inequalities. In this case, a vector $x = (x_1, \ldots, x_q) \in \mathbb{R}^q$ belongs to S iff for one of these sets, it satisfies the corresponding equalities and inequalities.

To describe the condition $x \in S$ in terms of interval linear systems, we will do the following:

(a) First, we will introduce u new variables $s_1, \ldots, s_u$, and add interval linear equations that will guarantee that the values of each of these variables will be 0 or 1 ($s_k = 1$ will mean that the solution vector x belongs to S_k, and $s_k = 0$ will mean that it does not).

(b) Second, we will add interval linear equations whose solvability is equivalent to the fact that at least one of these variables $s_1, \ldots, s_u$ be equal to 1 (this means that it is x belongs to one of the sets $S_1, \ldots, S_u$, and thus, that it belongs to their union S).

(c) Third, for each of the sets S_k, for every of the conditions $P_r(x_1, \ldots, x_q) = 0$, $P_s(x_1, \ldots, x_q) \geq 0$, $P_t(x_1, \ldots, x_q) > 0$ that define this set S_k, we will add new interval linear equations that represent the corresponding conditional statements:

(d) "if $s_k = 1$, then $P_r = 0$";

(e) "if $s_k = 1$, then $P_s \geq 0$";

(f) "if $s_k = 1$, then $P_r > 0$".

Then, if x is a solution of the resulting system of equations, we will have one of the variables s_k equal to 1, and for this variable s_k, all equalities and inequalities that define S_k will be satisfied. Thus, we will have $x \in S_k$ and therefore, $x \in S = \bigcup_k S_k$.

Vice versa, if $x \in S$, then $x \in S_k$ for some k, so, for $s_k = 1$, we will have a solution of the combined interval linear equation system.

9. Let us first describe how to ensure that a variable s_k only takes the values 0 or 1 (i.e., in computer terms, that it is a Boolean variable).

To ensure that, for each $k = 1, \ldots, u$, we will introduce two new factors: $f_1[s_k]$ and $f_2[s_k]$ with $\mathbf{f}_1[s_k] = \mathbf{f}_2[s_k] = [-1, 1]$, and add two new equations:

$$\boxed{2f_1[s_k] \cdot s_k \;=\; 1 + f_1[s_k],} \tag{3.11}$$

$$\boxed{2s_k \;=\; 1 + f_2[s_k].} \tag{3.12}$$

Let us show that $s_k \in \{0, 1\}$ iff this system has a solution.

9.1. If $s_k = 0$, then we can take $f_1[s_k] = f_2[s_k] = -1$. If $s_k = 1$, then we can take $f_1[s_k] = f_2[s_k] = 1$.

9.2. Vice versa, if equations (3.11) – (3.12) are satisfied, then for $z = 2s_k - 1$, from (3.12), we conclude that $|z| = |f_2[s_k]| \leq 1$, and from (3.11), we conclude that $z \cdot f_1[s_k] = 1$, and therefore, $|z| \geq 1/|f_1[s_k]| \geq 1$. Hence, $|z| \leq 1$ and $|z| \geq 1$, so, $|z| = 1$, and $z = \pm 1$. From $2s_k - 1 = \pm 1$, we conclude that $s_k = 0$ or $s_k = 1$.

10. Let us now express the fact that at least one of the given u Boolean variables $s_1, \ldots, s_u$ must take the value 1.

To express it, we need u new factors $f_3[s_k]$, $1 \leq k \leq u$, with $\mathbf{f}_3[s_k] = [-1, 1]$, and a new equation

$$f_3[s_1] \cdot s_1 + \cdots + f_3[s_u] \cdot s_u = 1 . \tag{3.13}$$

Let us show that for Boolean variables s_k, this equation is satisfiable iff one of the Boolean variables is different from 0.

Indeed, if this equation is satisfied, then all variables s_k cannot be equal to 0, because then, the left-hand side of this equation would be also equal to 0, and not to 1. Vice versa, if, e. g., $s_k = 1$, we can satisfy the equation by taking $f_3[s_k] = 1$ and $f_3[s_l] = 0$ for $l \neq k$.

11. Let us now describe how conditional statements "if $s_k = 1$, then ..." described in Part 8 of the proof can be represented in terms of interval linear equations.

11.1. The conditional statement "if $s_k = 1$, then $P = 0$" will be represented as $s_k \cdot P = 0$.

Indeed, if $s_k = 0$, then this equality is always true, and if $s_k = 1$, then it is exactly $P = 0$.

The expression $s_k \cdot P$ is a polynomial in terms of the variables s_k and x_i, and we already know how to represent (unconditional) polynomial equations in terms of systems of interval linear equations.

For example, we can follow the construction from Part 4 of this proof: Namely, we introduce the new variable $x[s_k \cdot P]$ with the property that $x[s_k, P] = s_k \cdot x[P]$, and add the equation $x[s_k \cdot P] = 0$.

For that purpose, we add two auxiliary variables $x_1[P, s_k]$ and $x_2[P, s_k]$, two auxiliary variables $f_1[P, s_k] \in [-1, 1]$ and $f_2[P, s_k] \in [-1, 1]$, and four new equations (these equations are similar to (3.2) – (3.4)):

$$f_1[P, s_k] \cdot x[s_k \cdot P] + f_2[P, s_k] \cdot s_k = 0 , \tag{3.14}$$

$$f_1[P, s_k] \cdot x[P] = -f_2[P, s_k] , \tag{3.15}$$

$$f_1[P, s_k] \cdot x_1[P, s_k] + f_2[P, s_k] \cdot x_2[P, s_k] = 1 , \tag{3.16}$$

$$x[P, s_k] = 0 . \tag{3.17}$$

11.2. The conditional statement "if $s_k = 1$, then $P \geq 0$" will be represented as $s_k \cdot P \geq 0$.

Indeed, if $s_k = 0$, then $s_k \cdot P = 0 \geq 0$, so, this inequality is always true; if $s_k = 1$, then this inequality is exactly $P \geq 0$.

We already know (see Part 6) how to represent inequalities of the type $x[s_k \cdot P] \geq 0$ in terms of systems of interval linear equations.

11.3. Similarly to Part 7 of the proof, we will represent the conditional strict inequality "if $s_k = 1$ then $P > 0$" as a combination of a conditional non-strict inequality "if $s_k = 1$ then $P \geq 0$", and an auxiliary conditional inequality "if $s_k = 1$ then $P \neq 0$."

To represent an auxiliary conditional inequality in terms of an system of interval linear equations, we will first reformulate as an equivalent unconditional inequality: namely, an inequality $1 - s_k + P^2 \neq 0$.

Indeed, if $s_k = 1$, then this inequality turns into $P \neq 0$. If $s_k = 0$, then $1 - s_k + P^2 = 1 + P^2 \neq 0$ for all values of P, so this condition does not impose any restrictions on the value of P.

The left – hand side of this equivalent inequality is a polynomial in s_k and x_i, so, we know how to describe a variable $x[\dots]$ that is equal to the value of this polynomial. Another possibility is as follows:

(a) first, we describe a new variable $x[P^2]$ for P^2, by using a construction similar to the one used in Part 4;

(b) second, we describe a new variable $x[1 - s_k + P^2]$ by using a construction from Part 2; and

(c) third, we describe the condition $x[1 - s_k + P^2] \neq 0$ as in Part 7.

In other words, first, we add three new auxiliary variables $x[P^2]$, $x_1[P,P]$, and $x_2[P,P]$, and two new auxiliary factors $f_1[P,P]$ and $f_2[P,P]$ with $\mathbf{f}_1[P,P] = \mathbf{f}_2[P,P] = [-1,1]$. These auxiliary factors and variables will only be used in the following equations that describe this relationship (these equations are similar to (3.2) – (3.4)):

$$f_1[P,P] \cdot x[P^2] + f_2[P,P] \cdot x[P] = 0, \tag{3.18}$$

$$f_1[P,P] \cdot x[P] = -f_2[P,P], \tag{3.19}$$

$$f_1[P,P] \cdot x_1[P,P] + f_2[P,P] \cdot x_2[P,P] = 1. \tag{3.20}$$

Second, we add a new variable $x[1 - s_k + P^2]$ and a new equation (this equation is similar to (2.2)):

$$x[1 - s_k + P^2] + s_k - x[P^2] = 1. \tag{3.21}$$

Third, we introduce two auxiliary variables $x_1[1 - s_k + P^2]$ and $x_2[1 - s_k + P^2]$ and two factors $f_1[1 - s_k + P^2]$ and $f_2[1 - s_k + P^2]$ with $\mathbf{f}_1[1 - s_k + P^2] = \mathbf{f}_2[1 - s_k + P^2] = [-1,1]$, and add three new equations (these equations are similar to (3.8) – (3.10)):

$$f_1[1 - s_k + P^2] \cdot x[1 - s_k + P^2] = f_2[1 - s_k + P^2], \tag{3.22}$$

$$f_1[1 - s_k + P^2] \cdot x_1[1 - s_k + P^2] = 1, \tag{3.23}$$

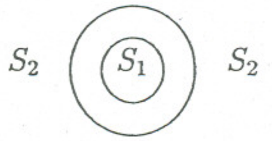
$$f_2[1 - s_k + P^2] \cdot x_2[1 - s_k + P^2] = 1. \tag{3.24}$$

We have shown how to describe all conditions in terms of systems of interval linear equations. By combining all these systems of interval linear equations, we get a large system that is solvable iff $x \in S$, i. e., for which S is a projection of the solution set. □

Remark 3.3. To make this complicated construction more understandable, in the next section, we will illustrate it on the example of a simple (two – component) semialgebraic set.

4. Example

Let us illustrate the above construction on the example of a semialgebraic set $S = S_1 \cup S_2$, where S_1 is the interior of the unit circle, and S_2 is the exterior of the open circle of radius 2 with a center in 0.



Here, $q = 2$, S_1 is described by an inequality $P_1 > 0$, where $P_1(x_1, x_2) = 1 - x_1^2 - x_2^2$, and S_2 is described by an inequality $P_2(x_1, x_2) \geq 0$, where $P_2(x_1, x_2) = x_1^2 + x_2^2 - 4$. We will follow the algorithm described in the proof step-by-step.

1 – 2. On Steps 1 – 2, we introduce the new variables $x[P_1]$, $x[P_2]$, $x[x_1^2]$, and $x[x_2^2]$, and the equations that follow the sample (3.1). These equations guarantee the proper relationship between the variables $x[P_i]$ that represent the polynomials and the variables that represent monomials.

3 – 4. On Steps 3 – 4, we add the following:

(a) For the monomial x_1^2, we introduce the new variables $x_1[x_1, x_1]$ and $x_2[x_1, x_1]$, the new factors $f_1[x_1, x_1]$ and $f_2[x_1, x_1]$ (with $\mathfrak{f}_1[x_1, x_1] = \mathfrak{f}_2[x_1, x_1] = [-1, 1]$) and add equations that follow the samples (3.2) – (3.4).

(b) For the monomial x_2^2, we introduce the new variables $x_1[x_2, x_2]$ and $x_2[x_2, x_2]$, the new factors $f_1[x_2, x_2]$ and $f_2[x_2, x_2]$ (with $\mathfrak{f}_1[x_2, x_2] = \mathfrak{f}_2[x_2, x_2] = [-1, 1]$) and add equations that follow the samples (3.2) – (3.4).

These equations guarantee that the values of the variables $x[m]$ that correspond to monomials is indeed equal to the value of the monomial $m(x_1, \ldots, x_q)$.

Since the given set S has two components ($u > 1$), we skip steps 5 – 7 and go straight to step 9.

9 – 10. We have two subsets S_k here, so, we add two new variables s_1 and s_2. For each of these variables, we do the following:

(a) We introduce two new factors: $f_1[s_1]$ and $f_2[s_1]$ with $\mathfrak{f}_1[s_1] = \mathfrak{f}_2[s_1] = [-1, 1]$, and add two new equations that follow the samples (3.11) – (3.12).

(b) We introduce two new factors: $f_1[s_2]$ and $f_2[s_2]$ with $\mathfrak{f}_1[s_2] = \mathfrak{f}_2[s_2] = [-1, 1]$, and add two new equations that follow the samples (3.11) – (3.12).

These equations guarantee that variables s_k are Boolean, i. e., take values only from the set $\{0, 1\}$.

Also, we add two new factors $f_3[s_1]$ and $f_3[s_2]$, $\mathfrak{f}_3[s_1] = \mathfrak{f}_2[s_2] = [0, 1]$, and a new equation that follows the samples (3.13). This equation guarantees that at least one of the variables s_k is equal to 1.

11. We must describe two conditional inequalities: "if $s_1 = 1$ then $P_1 > 0$" and "if $s_2 = 1$ then $P_2 \geq 0$."

11a. The second conditional inequality is easier to describe, so, we will start with it. We introduce three new variables: $x[s_2 \cdot P_2]$, $x_1[P_2, s_2]$, and $x_2[P_2, s_2]$, two auxiliary variables $f_1[P_2, s_2] \in [-1, 1]$ and $f_2[P_2, s_2] \in [-1, 1]$, and four new equations that follow the samples (3.14) – (3.17).

To describe the condition $s_2 \cdot P_2 \geq 0$, we follow Step 6: add an auxiliary factor $f[s_2 \cdot P_2]$ with $\mathbf{f}[s_2 \cdot P_2] = [0,1]$, and add an equation that follows the sample (3.7).

11b. The first condition is represented (as in 11.3) as a collection of two conditional inequalities "if $s_1 = 1$ then $P_1 \geq 0$" and a conditional inequality "if $s_1 = 1$ then $P_1 \neq 0$."

The first of these conditional inequalities can be represented in the exact same way as the inequality "if $s_2 = 1$ then $P_2 \geq 0$" that we analyzed in 11a: namely, first, we introduce three new variables: $x[s_1 \cdot P_1]$, $x_1[P_1, s_1]$, and $x_2[P_1, s_1]$, two auxiliary variables $f_1[P_1, s_1] \in [-1,1]$ and $f_2[P_1, s_1] \in [-1,1]$, and four new equations that follow the samples (3.14) – (3.17).

To describe the condition $s_1 \cdot P_1 \geq 0$, we follow Step 6: add an auxiliary factor $f[s_1 \cdot P_1]$ with $\mathbf{f}[s_1 \cdot P_1] = [0,1]$, and add an equation that follows the sample (3.7).

11.3. To describe the condition "if $s_1 = 1$ then $P_1 \neq 0$", we follow step 11.3:

(a) First, we add three new auxiliary variables $x[P_1^2]$, $x_1[P_1, P_1]$, and $x_2[P_1, P_1]$, and two new auxiliary factors $f_1[P_1, P_1]$ and $f_2[P_1, P_1]$ with $\mathbf{f}_1[P_1, P_1] = \mathbf{f}_2[P_1, P_1] = [-1,1]$. These auxiliary factors and variables will only be used in the equations that describe this relationship; these equations follow the samples (3.18) – (3.20).

(b) Second, we add a new variable $x[1 - s_1 + P_1^2]$ and a new equation that follows the sample (3.21).

(c) Third, we introduce two auxiliary variables $x_1[1-s_1+P_1^2]$ and $x_2[1-s_1+P_1^2]$ and two factors $f_1[1-s_1+P_1^2]$ and $f_2[1-s_1+P_1^2]$ with $\mathbf{f}_1[1-s_1+P_1^2] = \mathbf{f}_2[1-s_1+P_1^2] = [-1,1]$, and add three new equations that follow the samples (3.22) – (3.24).

Then (as we have shown in the proof of the theorem), the projection $\pi(X)$ of the solution set X of the resulting system of interval linear equations with dependent coefficients on $\mathbb{R}^2$ is the given set S: $\pi(X) = S$.

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