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# The basic properties of interval arithmetic, its software realizations and some applications

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**Abstract**

This article gives a short introduction to interval analysis and its possible applications. Furthermore an overview on existing programming languages for interval arithmetic is given. © 1998 Elsevier Science Ltd. All rights reserved.

## 1. Introduction

In this paper we give a basic introduction to the principles of interval arithmetic and its applications. The paper is written for readers who have no preliminary knowledge of interval arithmetic. Therefore after introducing some notation in Section 2 we introduce the arithmetic for intervals in Section 3 and discuss its most important properties. In Section 4 we discuss in a certain depth the problem of range inclusion of a real function defined on a set which contains an interval. The next Section 5 repeats one of the most important applications of interval arithmetic, namely the inclusion of solutions of real equations. Finally, in Section 6 an overview of existing programming languages in which interval arithmetic is realized is presented.

Meanwhile there exists a whole bunch of interesting and important applications. A reader who is interested in more information or details should consult the corresponding literature. Two references which contain the state of the art are the following.

Ref. [1].
Ref. [2].

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## 2. Notation

The set of reals is denoted by $\mathbb{R}$, its elements by $a, b, c, \ldots$. For closed bounded intervals contained in $\mathbb{R}$ we write the notation $[a] := [\underline{a};\overline{a}] = \{x \in \mathbb{R} | \underline{a} \le x \le \overline{a}$. The meaning of $f([x])$ for a real function $f$ is explained in the next section.

## 3. Real interval arithmetic and basic properties

In the set $\mathbb{R}$ of real numbers we consider closed and bounded intervals

$$[a] := [\underline{a};\overline{a}] = \{x \in \mathbb{R} \,|\, \underline{a} \le x \le \overline{a}\}.$$

The set of all such intervals is denoted by $I\mathbb{R}$. Real numbers $a$ can be considered as special elements of $I\mathbb{R}$ with $[a] = [a;a]$. We simply write $a$ in this case.

If "*" denotes one of the four operations $+, -, \times, /$ for real numbers then the corresponding operations for two elements $[a]$ and $[b]$ from $I\mathbb{R}$ are defined by

$$[a]*[b] = \{a*b | a \in [a],\ b \in [b]\}.$$

In the case of division $0 \notin [b]$ is assumed. Since the function $f(a,b) = a*b$, $a \in [a]$, $b \in [b]$, $* \in \{+, -, \times, /\}$ is continuous, $[a]*[b]$ is contained in $I\mathbb{R}$. A simple

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discussion gives the following rules for the four operations:

$$[a]+[b]=[\underline{a}+\underline{b};\overline{a}+\overline{b}],\qquad [a]-[b]=[\underline{a}-\overline{b};\overline{a}-\underline{b}],$$

$$[a]\times[b]=[\min\{\underline{a}\underline{b},\underline{a}\overline{b},\overline{a}\underline{b},\overline{a}\overline{b}\};\ \max\{\underline{a}\underline{b},\underline{a},\overline{a}\underline{b},\overline{a}\overline{b}\}],$$

$$[a]/[b]=[\underline{a};\overline{a}]\times\left[\frac{1}{\overline{b}};\frac{1}{\underline{b}}\right].$$

As for real numbers the multiplication sign " × " is usually replaced by "." or even omitted.

The multiplication of two intervals can be reduced to the multiplication of two real numbers in the case in which not simultaneously $0\in[a]$ and $0\in[b]$. Otherwise always four real multiplications have to be performed in the preceding formula for the multiplication. Recently Heindl [3] has shown, how to reduce this number to three multiplications.

Besides of these four basic operations we consider so-called unary operations in $I\mathbb{R}$: Let $r$ be a real continuous function defined on $\mathbb{R}$ (or a subset of $\mathbb{R}$). Then for $[a]$ contained in the domain of $r$ we define

$$r([a])=\{r(a)|a\in[a]\}\in I\mathbb{R}.$$

Examples are the elementary functions like sqr, sqrt, sine, cosine, exp, log, tan, ....

With the help of the four elementary operations for intervals and the preceding definition of $r([a])$ we are in the position to define for a real-valued function $f(a, b, \ldots, u, v)$ the so-called *interval arithmetic evaluation* of $f$ by $f([a], [b], \ldots, [u], [v])$.

For interval arithmetic evaluations the following for applications important rules hold:

(1) If $[a]\subseteq[\tilde{a}]$, $[b]\subseteq[\tilde{b}]$, ..., $[u]\subseteq[\tilde{u}]$, $[v]\subseteq[\tilde{v}]$ then

$$f([a],[b],\ldots,[u],[v])\subseteq f([\tilde{a}],[\tilde{b}],\ldots,[\tilde{u}],[\tilde{v}]).$$

This property is called *inclusion monotonicity*.

(2) If $a\in[a]$, $b\in[b]$, ..., $u\in[u]$, $v\in[v]$ then

$$f(a,b,\ldots,u,v)\in f([a],[b],\ldots,[u],[v]).$$

This property is a special case of the preceding one and is called *inclusion property*. It means that the interval arithmetic evaluation $f([a], [b], \ldots, [u], [v])$ always contains the range $R(f; [a], [b], \ldots, [u], [v])$ of the real function $f$ defined on the Cartesian product $[a]\times[b]\times\ldots\times[u]\times[v]$:

$$R(f;[a],[b],\ldots,[u],[v])=\{f(a,b,\ldots,u,v)|a\in[a],b\in[b],\ldots,u\in[u],v\in[v]\}\subseteq f([a],[b],\ldots,[u],[v]).$$

This is the property which makes interval arithmetic so important in applications.

Proofs of (1) and (2) follow immediately from the definition of the four basic operations and of $r([a])$.

**Example 1.** Let

$$f(x)=\frac{x}{1-x},\quad x\neq 1$$

and

$$[x]=[2;3].$$

Then

$$R(f;[x])=\left[-2;-\frac{3}{2}\right],$$

$$f([x])=\frac{[x]}{1-[x]}=\frac{[2;3]}{1-[2;3]}=[-3;-1]$$

and therefore

$$R(f;[x])\subset f([x])$$

as predicted by the preceding considerations.

For $x\neq 0$ we can rewrite $f(x)$ as

$$f(x)=\frac{x}{1-x}=\frac{1}{1/x-1},\quad x\neq 0.$$

For the interval arithmetic evaluation over [2; 3] we obtain

$$\tilde{f}([x])=\frac{1}{1/[2;3]-1}=\left[-2;-\frac{3}{2}\right]=R(f;[x]).\square$$

This example shows that the overestimation of the range of a given function by the interval arithmetic expression is strongly dependent on the arithmetic expression which is used for the interval arithmetic evaluation of the given function. The reason for this is based on the fact that interval arithmetic does not follow the same rules as the arithmetic for real numbers. We list a couple of exceptions:

(1) For $[x], [y], [z]\in I\mathbb{R}$ we have

$$[x]([y]+[z])\subseteq[x][y]+[x][z].$$

This property is called *subdistributivity*.

However, for $x\in\mathbb{R}$ it always holds

$$x([y]+[z])=x[y]+x[z].$$

(2) For $[x]\in I\mathbb{R}$, we have $[x]-[x]\neq 0$ if $[x]$ is a proper interval.

(3) For $[x]\in I\mathbb{R}$, $0\notin[x]$, we have $[x]/[x]\neq 1$ if $[x]$ is a proper interval.

The *distance* of two intervals $[x]=[\underline{x},\overline{x}]$ and $[y]=[\underline{y},\overline{y}]$ is defined as the real number

$$q([x],[y]):=\max\{|\underline{x}-\underline{y}|,|\overline{x}-\overline{y}|\}.$$

The *absolute value* of an interval $[x]=[\underline{x},\overline{x}]$ is defined as the distance of $[x]$ from 0:

$$|[x]|:=q([x],0)=\max\{|\underline{x}|,|\overline{x}|\}.$$

Among others the following relations hold:

$$|[x]| = \max\{|x| \mid x \in [x]\},$$
$$q([x]+[y],[x]+[z]) = q([y],[z]),$$
$$q(x[y],x[z]) = |x|q([y],[z]), \quad x \in \mathbb{R},$$
$$q([x][y],[x][z]) \le |[x]|q([y],[z]),$$
$$|[x]\pm[y]| \le |[x]| + |[y]|, \qquad |[x][y]| = |[x]||[y]|.$$

The *diameter* (or *width*) of an interval $[x] = [\underline{x},\overline{x}]$ is defined as

$$w([x]) = \overline{x} - \underline{x}.$$

The following rules hold:

$$w([x]\pm[y]) = w([x]) + w([y]),$$
$$w(x[y]) = |x|w([y]), \quad x \in \mathbb{R},$$
$$w([x][y]) \le w([x])|y| + |x|w([y]),$$
$$w([x][y]) \ge \max\{|[x]|w([y]), w([x])|[y]|\}.$$

## 4. Range inclusion

In Example 1 we have seen that the overestimation of the range of a real function by the interval arithmetic evaluation is dependent on the arithmetic expression which is used for the interval arithmetic evaluation. Moore [4] has shown that under reasonable assumptions the following inequality holds for the distance between $R(f;[x])$ and $f([x])$:

$$q(R(f;[x]), f([x])) \le \gamma w([x]), \quad \gamma \ge 0,$$

where $[x]$ is contained in some fixed interval $[x]^0$. This inequality means that the overestimation of $R(f;[x])$ by $f([x])$ goes linearly to zero with the diameter of $[x]$. (This estimation analogously holds for the interval arithmetic evaluation of functions of several variables.)

**Example 2**. Let

$$f(x) = x - x^2, \quad x \in [x]^0 = [0;1]$$

and

$$[x] = \left[\frac{1}{2} - r; \frac{1}{2} + r\right], \quad 0 \le r \le \frac{1}{2}.$$

A simple discussion gives

$$R(f;[x]) = \left[\frac{1}{4} - r^2; \frac{1}{4}\right].$$

For the interval arithmetic evaluation we obtain

$$f([x]) = \left[\frac{1}{2} - r; \frac{1}{2} + r\right] - \left[\frac{1}{2} - r; \frac{1}{2} + r\right]\left[\frac{1}{2} - r; \frac{1}{2} + r\right]$$
$$= \left[\frac{1}{4} - 2r - r^2; \frac{1}{4} + 2r - r^2\right].$$

Hence

$$q(R(f;[x]), f([x]))$$
$$= \max\left\{\left|\frac{1}{4} - 2r - r^2 - \frac{1}{4} + r^2\right|, \left|\frac{1}{4} + 2r - r^2 - \frac{1}{4}\right|\right\}$$
$$= \max\{2r, 2r - r^2\} = 2r = \gamma w([x]), \quad \gamma = 1,$$

as predicted by Moores result. □

The second part of Example 1 rises the question whether it is possible to rearrange the variables of the given function in such a manner that the interval arithmetic evaluation gives higher than linear convergence to the range of values. The answer is "yes". Before we state the general result we consider again an example.

**Example 3**. The function $f(x) = x - x^2$, $x \in [0;\ 1]$ from Example 2 can be written as

$$f(x) = x - x^2 = \frac{1}{4} - \left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right), \quad x \in [0;1].$$

Plugging in intervals we get for the interval arithmetic evaluation

$$\tilde{f}([x]) = \frac{1}{4} - \left(\left[\frac{1}{2} - r; \frac{1}{2} + r\right] - \frac{1}{2}\right)\left(\left[\frac{1}{2} - r; \frac{1}{2} + r\right] - \frac{1}{2}\right)$$
$$= \left[\frac{1}{4} - r^2; \frac{1}{4} + r^2\right].$$

Therefore we obtain

$$q(R(f;[x]), \tilde{f}([x]))$$
$$= \max\left\{\left|\frac{1}{4} - r^2 - \left(\frac{1}{4} - r^2\right)\right|, \left|\frac{1}{4} - \left(\frac{1}{4} + r^2\right)\right|\right\} = r^2$$
$$= \frac{1}{4}(w([x]))^2,$$

which means that the distance goes quadratically to zero with $w([x])$. □

The general result is as follows:

**Theorem 1**. (*The centered form*)

*Let the function* $f: \mathbb{R} \to \mathbb{R}$ *be represented in the so-called centered form*

$$f(x) = f(z) + (x - z) \cdot h(x)$$

*for some* $z \in [x]$. *If* $h(x)$ *has an interval arithmetic evaluation* $h([x])$ *then (under weak conditions on the arithmetic evaluation* $h([x])$) *for* $f([x])$ *defined by*

$$f([x]) := f(z) + ([x] - z) \cdot h([x])$$

*it holds that*

(*a*) $R(f;[x]) \subseteq f([x])$

*and*

(*b*) $q(R(f;[x]); f([x])) \le \gamma (w([x]))^2$. □

The property (b) is called *quadratic approximation property* of the centered form. The centered form was introduced by Moore in [4] where he conjectured that the quadratic approximation property holds. The conjecture was proved by Hansen in [5].

The question whether for a given (rational) function there exists a representation $\tilde{f}$ such that

$$q(R(f;[x]),\tilde{f}([x]))\leq\gamma(w([x]))^m,\quad \gamma\geq 0$$

with $m>2$ is open. Up to now such representations are only known under special assumptions.

## 5. Solution inclusion for real systems

We start with a single equation in one unknown. We assume that the real function

$$f:[x]\subset D\subset\mathbb{R}\longrightarrow\mathbb{R}$$

is differentiable in $D$ and that the derivative $f'(x)$ has an interval arithmetic evaluation $f'([x])$ which does not contain zero. Assume that $f$ has a zero $x^*$ in $[x]$. Then by the mean value theorem we have for an arbitrary $x\in[x]$ and for some $\xi$ between $x$ and $x^*$

$$f(x)-f(x^*)=f(x)=f'(\xi)(x-x^*)$$

and therefore

$$x^*=x-\frac{f(x)}{f'(\xi)}\in x-\frac{f(x)}{f'([x])}.$$

Hence

$$x^*\in\left\{x-\frac{f(x)}{f'([x])}\right\}\cap[x].$$

Defining $[x]^0:=[x]$ and denoting by $m([x])$ an arbitrary point contained in $[x]$ then by repeating the preceding steps we arrive at the following iteration method for repeated inclusion of $x^*$:

$$[x]^{k+1}=\left\{m([x]^k)-\frac{f(m([x]^k))}{f'([x]^k)}\right\}\cap[x]^k,\quad k=0,1,2,\ldots.$$

This method is called *Interval-Newton-Method*. If $0\notin f'([x]^0)$ this method is well defined, it holds that $x^*\in[x]^k$ and $\lim_{k\to\infty}[x]^k:=x^*$. Furthermore the sequence $\{w([x]^k)\}_{k=0}^{\infty}$ is under certain assumptions quadratically convergent to zero. Proofs and a whole bunch of other methods for enclosing zeroes of a real function can be found in [6].

The function

$$N([x]):=x-\frac{f(x)}{f'([x])}x\in[x]\in I\mathbb{R}$$

is called the *Interval-Newton-Operator*. It possesses a couple of interesting properties, which we obtain as special cases from the following discussion.

Consider now the function

$$f:[x]\subset D\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^n$$

where $[x]$ is a so-called *interval vector*. This is a vector whose components are compact real intervals. Assume that the partial derivatives of $f$ exist in $D$ and are continuous. Let $f'([x])$ denote the interval arithmetic evaluation of the Jacobi matrix $f'(x)$. The matrix $f'([x])$ contains intervals as elements and is called an *interval matrix*. Then analogously to the case $n=1$ we define the Interval-Newton-Operator by

$$N([x])=x-\mathrm{IGA}(f'([x]),f(x)),\quad x\in[x].$$

Here $\mathrm{IGA}(f'([x]),f(x))$ is an interval vector which is obtained by formally applying the formulas of the Gaussian algorithm to the interval matrix $f'([x])$ and to the right hand side $f(x)$. (interval arithmetic Gaussian algorithm). For more details see Ref. [7].

The operator $N([x])$ has the following interesting properties.

**Theorem 2**. *Let* $\mathrm{f}:\mathrm{D}\subseteq\mathbb{R}^n\to\mathbb{R}^n$ *be a continuously differentiable function.*

*Let the interval arithmetic evaluation* $\mathrm{f}'([\mathrm{x}])$ *exist for some* $[\mathrm{x}]\subseteq\mathrm{D}$. *Let* $\mathrm{x}\in[\mathrm{x}]$ *and assume that* $\mathrm{IGA}(\mathrm{f}'([\mathrm{x}]),\mathrm{f}(\mathrm{x}))$ *exists. Then the following hold:*

(1) *If* f *has a (necessarily unique) zero* $\mathrm{x}^*$ *in* $[\mathrm{x}]$ *then*

$$\mathrm{x}^*\in\mathrm{N}([\mathrm{x}]).$$

(2) *If*

$$\mathrm{N}([\mathrm{x}])\cap[\mathrm{x}]=\varnothing\quad\text{(empty set)}$$

*then* f *has no zero* $\mathrm{x}^*$ *in* $[\mathrm{x}]$.

(3) *If*

$$\mathrm{N}([\mathrm{x}])\subseteq[\mathrm{x}],$$

*then* f *has a unique zero in* $[\mathrm{x}]$.

A proof of Theorem 2 can be found in [7].

The property (2) allows to prove that a given interval vector $[x]$ contains no zero whereas (3) proves the existence of a zero in $[x]$.

If $[x]$ contains a zero $x^*$ in $[x]$ then analogously to the case $n=1$ we consider the Interval-Newton-Method for repeated inclusion of the zero:

$$[x]^{k+1}=\{m([x]^k)-IGA(f'([x]^k),f(m([x]^k)))\}\cap[x]^k,$$
$$k=0,1,2,\ldots.$$

In contrast to the case $n=1$ this method is in general not convergent to $x^*$ (provided it is well defined at all). However, we always have $x^* \in [x]^k$ under the assumptions of Theorem 2. Conditions for $\lim_{k\to\infty}[x]^k = x^*$ in the case $n > 1$ can be found in [7].

## 6. Languages for interval computation

In the preceding sections we have seen that inclusion monotonicity of interval arithmetic is of fundamental importance for interval arithmetic. If some algorithm — like the Interval-Newton-Method — is performed on a computer then inclusion monotonicity has to hold also on the computer. Otherwise it cannot be guaranteed that the final result on the computer really includes the unknown solution.

In this section we shortly report on a series of existing programming languages and implementations for performing interval arithmetic.

Interval arithmetic has been implemented in hardware, in firmware and in software on many different platforms and is supported by powerful programming languages.

The XSC (extended scientific computation) library provides powerful tools necessary for achieving high accuracy and reliability. It provides a large number of predefined numerical data types and operations to deal with *uncertain* data.

### *6.1. PASCAL-XSC [8, 9]*

It is a general purpose programming language. It provides special support for the implementation of numerical algorithms with mathematically verified results.

Compared with PASCAL, PASCAL-XSC provides an extended set of mathematical functions that are available for the types real, complex, interval and cinterval (complex interval) and delivers a result of maximum accuracy. Routines for solving numerical problems have been implemented in PASCAL-XSC. These routines compute an accurate enclosure of the solution and prove the existence and the uniqueness of the solution in the given interval.

PASCAL-XSC systems are available for personal computers, workstations, mainframes and supercomputers.

**Example 4**. (This can also be found in Ref. [10]).
*Interval-Newton-Method in PASCAL-XSC*.
Function $f(x) = \sqrt{x} + (x + 1)\cos x$.

```
programm inewt (input, output);
use
  i_ari; {i_ari: interval arithmetic}
var
  x, iy: interval;
function f(r: real): interval;
var
  x: interval;
begin
  x := r; {Converts r to type interval.}
  f := sqrt(x) + (x + 1)*cos(x)
end;
function der (x: interval): interval;
begin
  der := 1/(2*sqrt(x)) + cos(x) − (x + 1)*sin(x)
end;
  {The interval notation for I/O in PASCAL-XSC
     is [inf, sup]}
  {mid(x) is the midpoint of the interval x}
function criter (x: interval): boolean;
begin
  criter := (sup(f(inf(x))*f(sup(x))) < 0)  and  not
     (0 in der(x));
end;
begin
  read(y);
  while inf(y) < > sup(y) do
  begin
    if criter(y) then
      repeat
        x := y;
        writeln(x);
        y := (mid(x) − f(mid(x))/der(x))**x;
      until x = y
    else
      writeln("Criterion is not satisfied!");
    writeln;
    read(y);
  end;
end.
```

The results with the starting interval [6, 9] are:

```
[              2.0E + 000,              3.0E + 000]
[              2.0E + 000,              2.3E + 000]
[             2.05E + 000,             2.07E + 000]
[          2.05903E + 000,          2.05906E + 000]
[   2.059045253413E + 000,   2.059045253417E + 000]
[2.059045253415143E + 000, 2.059045253415145E + 000]
```

□

### 6.2. C-XSC [11]

It is a programming environment for verified scientific computing and numerical data processing and is a tool for the development of numerical algorithms delivering accurate and automatically verified results. C-XSC allows highlevel programming of numerical applications in C and C + + .

C-XSC provides the basic numerical data types real, interval, complex and cinterval with the corresponding arithmetic operators, relational operators and mathematical standard functions. Additionally the standard functions for the types interval and cinterval enclose the range of values in tight bounds, that means it supports the programming of algorithms which automatically enclose the solution of a given mathematical problem in verified bounds.

### 6.3. ACRITH-XSC [12]

It is an extension of FORTRAN 77. It was developed in a joint project between IBM/Germany and the Institute of Applied Mathematics of the University of Karlsruhe (Professor Kulisch). It can be used unfortunately only on machines with IBB/370-architecture that operates under the VMCMS operating system.

It is a FORTRAN-like programming library. Its features are dynamic arrays, subarrays, interval and vector arithmetic and problem solving routines for mathematical problems with verified results.

### 6.4. FORTRAN-XSC [13]

This language consists of a number of FORTRAN 90 modules providing accurate matrix arithmetic also with real and complex interval entries.

It is an easy and powerful programming tool for engineering applications. It provides problem-solving functions and programs that compute an accurate inclusion of the true solution and automatically proves the existence and uniqueness of a true result, that means that these programs provide solutions with error bounds and prove mathematical statements.

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