# The Cholesky Method for Interval Data

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*Dedicated to U. Kulisch on the occasion of his $60^{th}$ birthday*

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ABSTRACT

We apply the well-known Cholesky method to bound the solutions of linear systems with symmetric matrices and right-hand sides both of which are varying within given intervals. We derive criteria to guarantee the feasibility and the optimality of the method. Furthermore, we discuss some general properties.

## 1. INTRODUCTION

It is well known that the formulae of the Gaussian algorithm can be used to bound the solutions of the linear systems for which the coefficient matrices and the right-hand sides are varying within given intervals; see [11], or [3] and [13], where also criteria for the feasibility of this method can be found. A method which can be used systematically for linear systems with a *real symmetric and positive definite point matrix* is the Cholesky method. Compared with Gaussian elimination, it has among others certain advantages with respect to the amount of work which has to be performed.

The purpose of the present paper is to investigate the Cholesky method systematically when applied to systems with interval data. To our knowledge this has not been done before. After repeating some basic facts from interval analysis and matrix theory (Section 2), we introduce the interval Cholesky method in Section 3. A series of properties which may hold is illustrated by examples. In Section 4 we derive several sufficient criteria for the method to be feasible, and we prove some additional properties.




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## 2. PRELIMINARIES

By $\mathbf{R}^n, \mathbf{R}^{n\times n}, \mathbf{IR}, \mathbf{IR}^n, \mathbf{IR}^{n\times n}$ we denote the set of real vectors with $n$ components, the set of real $n \times n$ matrices, the set of intervals, the set of interval vectors with $n$ components, and the set of $n \times n$ interval matrices, respectively. By "interval" we always mean a real compact interval. We write interval quantities in brackets with the exception of point quantities (i.e., degenerate interval quantities), which we identify with the element which they contain. Examples are the null matrix 0 and the identity matrix $I$. We use the notation $[A] = [\underline{A}, \overline{A}] = ([a_{ij}]) = ([\underline{a}_{ij}, \overline{a}_{ij}]) \in \mathbf{IR}^{n\times n}$ simultaneously without further reference, and we proceed similarly for the elements of $\mathbf{R}^n$, $\mathbf{R}^{n\times n}$, $\mathbf{IR}$, and $\mathbf{IR}^n$. We write $\square\, S$ for the tightest interval enclosure of a given bounded subset $S \subseteq \mathbf{R}^n$ and call it the *interval hull of* $S$.

By $A \geqslant 0$ we denote a nonnegative $n \times n$ matrix, i.e., $a_{ij} \geqslant 0$ for $i, j = 1, \dots, n$. We call $x \in \mathbf{R}^n$ positive, writing $x > 0$, if $x_i > 0$, $i = 1, \dots, n$.

We also mention the standard notation from interval analysis [3, 13]:

$$|[a]| := \max\{|\tilde{a}| \,|\, \tilde{a} \in [a]\} = \max\{|\underline{a}|, |\overline{a}|\}$$

(*absolute value*), and

$$\langle [a] \rangle := \min\{|\tilde{a}| \,|\, \tilde{a} \in [a]\} = \begin{cases} \min\{|\underline{a}|, |\overline{a}|\} & \text{if} \quad 0 \notin [a], \\ 0 & \text{otherwise} \end{cases}$$

(*minimal absolute value*) for intervals $[a]$. For $[A] \in \mathbf{IR}^{n\times n}$ we obtain $|[A]| \in \mathbf{R}^{n\times n}$ by applying $|\cdot|$ entrywise, and we define the comparison matrix $\langle [A] \rangle = (c_{ij}) \in \mathbf{R}^{n\times n}$ by setting

$$c_{ij} := \begin{cases} -|[a_{ij}]| & \text{if} \quad i \neq j, \\ \langle [a_{ii}] \rangle & \text{if} \quad i = j. \end{cases}$$

Since real numbers can be viewed as degenerate intervals, $|\cdot|$ and $\langle \cdot \rangle$ can also be used for them.

By $\mathbf{Z}^{n\times n}$ we denote the set of real $n \times n$ matrices with nonpositive off-diagonal entries, by $\det A$ we mean the determinant of a matrix $A \in \mathbf{R}^{n\times n}$, and by $\rho(A)$ we denote its spectral radius.

In Section 4 we will consider several classes of matrices $A \in \mathbf{R}^{n\times n}$ for which we recall the definitions (cf. [5, 8, 17]):

$A$ is an *M-matrix* if $A$ is nonsingular, $A^{-1} \geqslant 0$, and $A \in \mathbf{Z}^{n\times n}$.

$A$ is a *Stieltjes matrix* if $A$ is a symmetric $M$-matrix.

$A$ is an *H-matrix* if $\langle A \rangle$ is an $M$-matrix.

$A$ is *diagonally dominant* if

$$|a_{ii}| \geqslant \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|, \qquad i = 1, \dots, n. \tag{2.1}$$

$A$ is *strictly diagonally dominant* if (2.1) holds with strict inequality.

$A$ is *irreducibly diagonally dominant* if $A$ is irreducible and if (2.1) holds with strict inequality for at least one index $i$.

$A$ is *totally positive* (totally nonnegative) if each minor of $A$ is positive (nonnegative).

$A$ is an *oscillatory matrix* if it is totally nonnegative and if at least one of its powers $A^k$ is totally positive.

An interval matrix $[A] \in \mathbf{IR}^{n\times n}$ is termed an *M-matrix* if each element $\tilde{A} \in [A]$ is an $M$-matrix. In the same way the term "$H$-matrix" can be extended to $\mathbf{IR}^{n\times n}$. It is easy to verify that

$[A]$ is an $M$-matrix if and only if $\underline{A}$ is an $M$-matrix and $\overline{a}_{ij} \leqslant 0$ for $i \neq j$, and that

$[A]$ is an $H$-matrix if and only if $\langle [A] \rangle$ is an $M$-matrix.

To prove these two statements one can refer to a very useful criterion for $M$-matrices due to Fan [6]:

LEMMA 2.1. *Let $A \in Z^{n\times n}$. Then $A$ is an $M$-matrix if and only if there exists a positive vector $x \in \mathbf{R}^n$ such that $Ax > 0$.*

We recall now some well-known results for symmetric positive definite matrices.

LEMMA 2.2. *If $A \in \mathbf{R}^{n\times n}$ is a symmetric matrix, then the following properties are equivalent:*

(i) *The matrix $A$ is positive definite.*

(ii) *Each principal submatrix of $A$ has a positive determinant.*

(iii) *Each eigenvalue of $A$ is positive.*

LEMMA 2.3 (Cf. [5, p. 141]). *$A \in \mathbf{R}^{n\times n}$ is a Stieltjes matrix if and only if $A$ is a symmetric and positive definite element of $\mathbf{Z}^{n\times n}$.*

LEMMA 2.4 (Cf. [7, p. 127]). *If $A \in \mathbf{R}^{n\times n}$ is a symmetric and positive definite tridiagonal matrix, then $A$ is an $H$-matrix.*

We equip $\mathbf{IR}, \mathbf{IR}^n, \mathbf{IR}^{n\times n}$ with the usual real interval arithmetic as described e.g. in [3, 11, 13]. We also assume that the reader is familiar with the properties of this arithmetic. We only mention the formulae (cf. [3], [13])

$$\left|\frac{1}{[a]}[b]\right| = \frac{1}{\langle [a]\rangle}|[b]| \quad \text{if} \quad 0 \notin [a], \qquad (2.2)$$
$$|[a] \pm [b]| \leqslant |[a]| + |[b]|$$

for intervals $[a]$, $[b]$, and we recall the definitions

$$\sqrt{[a]} := \{\sqrt{a}\,|\,a \in [a]\} \quad \text{for} \quad 0 \leqslant \underline{a} \qquad (2.3)$$

and

$$[a]^2 := \{a^2 | a \in [a]\}. \qquad (2.4)$$

Instead of $\sqrt{[a]}$ we also write $[a]^{1/2}$.

## 3. THE INTERVAL CHOLESKY METHOD

We start this section by specifying the problem which we want to attack. Let $[A] \in \mathbf{IR}^{n\times n}$ be an interval matrix satisfying $[A] = [A]^T$. Furthermore, let $[b] \in \mathbf{IR}^n$ be given. We want to bound the solution set

$$S_{\text{sym}} := \{x|\, Ax = b,\ A \in [A],\ A = A^T,\ b \in [b]\} \qquad (3.1)$$

by an interval vector $[x]$.

Such a vector can be obtained by using an iterative method as described in [3] or by applying the interval Gaussian algorithm, which also can be found in [3]. Since we are only interested in bounds for the solutions of linear systems with *symmetric* coefficient matrices $A$, we can hope to succeed also with an interval analogue of the Cholesky method which needs approximately half the operations of the interval Gaussian algorithm. By this analogue, we mean the construction of an interval vector $[x]^C = \text{ICh}([A], [b])$ by applying the following algorithm, which we divide into three steps. To formulate them we require $[A] = [A]^T$, and we assume that all the steps are feasible. This means that no division by an interval which contains zero appears, and that all square roots can be taken. Conditions which guarantee this feasibility will be derived in Section 4.

INTERVAL CHOLESKY METHOD.

*Step 1.* "$LL^T$ decomposition":

$$
\begin{array}{l}
\textbf{for } j := 1 \textbf{ to } n \textbf{ do} \\
\quad [l_{jj}] := ([a_{jj}] - \sum_{k=1}^{j-1} [l_{jk}]^2)^{1/2}; \\
\quad \textbf{for } i := j+1 \textbf{ to } n \textbf{ do} \\
\qquad [l_{ij}] := ([a_{ij}] - \sum_{k=1}^{j-1} [l_{ik}][l_{jk}])/[l_{jj}];
\end{array}
$$

*Step 2.* Forward substitution: (3.2)

$$
\begin{array}{l}
\textbf{for } i := 1 \textbf{ to } n \textbf{ do} \\
\quad [y_i] := ([b_i] - \sum_{j=1}^{i-1} [l_{ij}][y_j])/[l_{ii}];
\end{array}
$$

*Step 3.* Backward substitution:

$$
\begin{array}{l}
\textbf{for } i := n \textbf{ downto } 1 \textbf{ do} \\
\quad [x_i]^C := ([y_i] - \sum_{j=i+1}^{n} [l_{ji}][x_j]^C)/[l_{ii}]; \\
\mathrm{ICh}([A],[b]) := [x]^C.
\end{array}
$$

As usual, sums with an upper bound smaller than the lower one are defined to be zero. The squares in step 1 are evaluated by applying the interval square function (2.4), which yields for arbitrary $[a] \in \mathbf{IR}$ the inclusion

$$[a]^2 \subseteq [a] \cdot [a] \qquad \text{with equality if and only if } 0 \notin \mathrm{int}([a]). \tag{3.3}$$

We recall that we only intend to enclose the solutions for the *symmetric* matrices contained in $[A]$. This justifies the use of the squares $[l_{jk}]^2$ as defined in (2.4) in step 1 above.

As can be seen from the formulae in the interval Cholesky method, the feasibility of this method is independent of the right-hand side $[b]$. Therefore, the existence of $\mathrm{ICh}([A],[b])$ is also independent of $[b]$. Subsequently, we will simply write "$[x]^C$ exists" if we mean that $\mathrm{ICh}([A],[b])$ exists for any vector $[b] \in \mathbf{IR}^n$.

The three steps in the interval Cholesky method correspond to the three steps of the ordinary Cholesky method for a given symmetric matrix $A \in$

$\mathbf{R}^{n\times n}$ (provided that the feasibility is guaranteed):

(i) Decompose $A$ into $A = LL^T$ with a lower triangular matrix $L$ satisfying $l_{ii} > 0$, $i = 1, \dots, n$.
(ii) Solve $Ly = b$.
(iii) Solve $L^T x = y$.

As is well known, the decomposition in (i) is unique.

Define $[L]$ as the lower triangular matrix with $[l_{ij}]$ from step 1 of the interval Cholesky method. By the inclusion monotonicity of interval arithmetic it is clear that $L$ from (i) exists and is contained in $[L]$ for each matrix $A = A^T \in [A]$. This means that $A = LL^T \in [L][L]^T$ holds for symmetric matrices $A \in [A]$. In particular, $\underline{A}, \overline{A} \in [L][L]^T$, whence $[A] \subseteq [L][L]^T$, with strict inclusion being possible, as the example

$$[A] = \begin{pmatrix} [1,4] & 2 \\ 2 & 5 \end{pmatrix} \quad \text{with} \quad [L] = \begin{pmatrix} [1,2] & 0 \\ [1,2] & [1,2] \end{pmatrix}$$

and

$$[L][L]^T = \begin{pmatrix} [1,4] & [1,4] \\ [1,4] & [2,8] \end{pmatrix}$$

shows. Therefore, the name "$LL^T$ decomposition" in step 1 of the interval Cholesky method is in a certain sense misleading.

By the same reasoning as above, we obtain at once the following theorem.

THEOREM 3.1. *Let $[x]^C$ exist for $[A] = [A]^T \in \mathbf{IR}^{n\times n}$. Then*

$$S_{\text{sym}} \subseteq [x]^C. \tag{3.4}$$

The question arises quite naturally whether equality holds in (3.4) and whether the set

$$S := \{x \mid Ax = b,\ A \in [A],\ b \in [b]\},$$

in which now $A \neq A^T$ is allowed, is also contained in $[x]^C$. We answer these (and some more) questions by the following example.

EXAMPLE 3.2. Let

$$[A] := \begin{pmatrix} 4 & [-1,1] \\ [-1,1] & 4 \end{pmatrix}, \qquad [b] := \begin{pmatrix} 6 \\ 6 \end{pmatrix}.$$

Setting

$$A := \begin{pmatrix} 4 & \alpha \\ \beta & 4 \end{pmatrix} \quad \text{for} \quad A \in [A],$$

we get

$$A^{-1}b = \frac{6}{16 - \alpha\beta} \begin{pmatrix} 4 - \alpha \\ 4 - \beta \end{pmatrix} \qquad \text{with} \quad \alpha, \beta \in [-1, 1].$$

If $A = A^T \in [A]$, then $\beta = \alpha$ yields

$$A^{-1}b = \frac{6}{4 + \alpha} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus

$$\square S_{\mathrm{sym}} = \left(\left[\tfrac{18}{15}, 2\right], \left[\tfrac{18}{15}, 2\right]\right)^T, \qquad \square S = \left(\left[\tfrac{18}{17}, 2\right], \left[\tfrac{18}{17}, 2\right]\right)^T,$$

$$[x]^C = \left([1, 2], \left[\tfrac{18}{16}, 2\right]\right)^T, \qquad [x]^G = \left([1, 2], \left[\tfrac{18}{17}, 2\right]\right)^T,$$

where $[x]^G$ denotes the vector resulting from the interval Gaussian algorithm. The sets

$$S_{\mathrm{sym}} = \left\{ \frac{6}{4 + \alpha} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \middle| \, -1 \leqslant \alpha \leqslant 1 \right\} = \left\{ \gamma \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \middle| \, \frac{6}{5} \leqslant \gamma \leqslant 2 \right\}$$

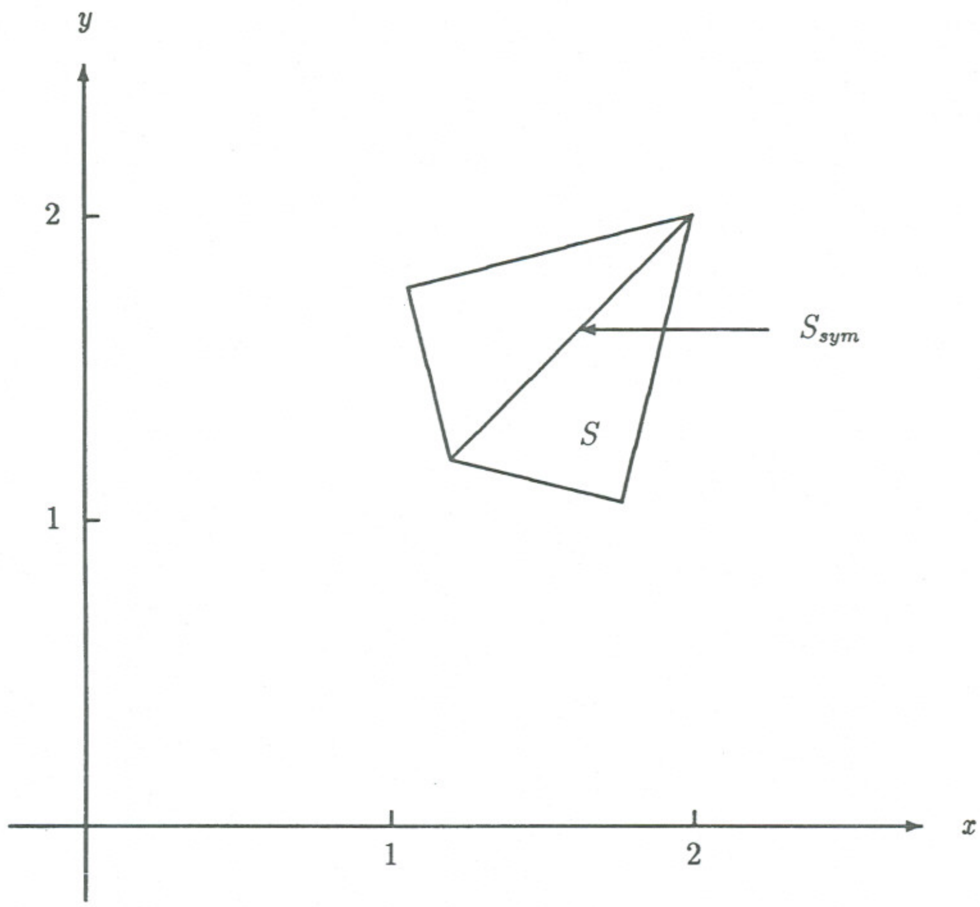


FIG. 1. The sets $S$ and $S_{\text{sym}}$.

and (see [9])

$$S = \text{convex hull}\left(\left\{\left(\tfrac{6}{5}, \tfrac{6}{5}\right)^T, (2,2)^T, \left(\tfrac{18}{17}, \tfrac{30}{17}\right)^T, \left(\tfrac{30}{17}, \tfrac{18}{17}\right)^T\right\}\right)$$

can be seen in Figure 1.

Example 3.2 illustrates that the following properties can occur:

(i) $\square S_{\text{sym}} \neq \square S$ (cf. [13]).
(ii) $\square S_{\text{sym}} \neq [x]^C$.
(iii) $\square S \neq [x]^G$.
(iv) $S \not\subseteq [x]^C$ (but $S_{\text{sym}} \subseteq [x]^C$; cf. Theorem 3.1).
(v) $[x]^C \subseteq [x]^G$ with $[x]^C \neq [x]^G$.

We enlarge this list by another property which is also possible:

(vi) $[x]^G \subseteq [x]^G$ with $[x]^G \neq [x]^C$.

Our next example illustrates this property.

EXAMPLE 3.3. Let

$$[A] := \begin{pmatrix} [1,4] & [0,1] \\ [0,1] & 3 \end{pmatrix}, \qquad [b] := \begin{pmatrix} 2 \\ [0,2] \end{pmatrix}.$$

Then $[x]^G = ([0.25, 3], [-1, 1])^T \subset [x]^C = ([0, 3], [-1, 1])^T$.

The reason why these two examples above work is best seen by expressing $[x]^C$ and $[x]^G$ in terms of the input data. One obtains

$$[x_2]^C = \frac{1}{[a_{22}] - [a_{12}]^2/[a_{11}]} \left\{ [b_2] - \frac{[a_{12}]}{[a_{11}]}[b_1] \right\},$$

$$[x_1]^C = \frac{1}{\sqrt{[a_{11}]}} \left\{ \frac{[b_1]}{\sqrt{[a_{11}]}} - \frac{[a_{12}]}{\sqrt{[a_{11}]}}[x_2]^C \right\},$$

$$[x_2]^G = \frac{1}{[a_{22}] - [a_{12}][a_{12}]/[a_{11}]} \left\{ [b_2] - \frac{[a_{12}]}{[a_{11}]}[b_1] \right\},$$

$$[x_1]^G = \frac{1}{[a_{11}]} \left\{ [b_1] - [a_{12}][x_2]^G \right\}.$$

Hence, by (3.3), we always get $[x_2]^C \subseteq [x_2]^G$. If, however, $0 \notin \mathrm{int}([a_{12}])$, then $[x_2]^C = [x_2]^G$, and the subdistributivity of the interval arithmetic causes $[x_1]^G \subseteq [x_1]^C$. Similar phenomena can appear in higher dimensions, too.

We now turn to an alternative representation of $[x]^C$. As with the result of the interval Gaussian algorithm (cf. e.g. [2] or [15]), the vector $\mathrm{ICh}([A], [b])$ can be expressed as a product of certain diagonal matrices $[D^s]$, $s = 1, \ldots, n$, and lower triangular matrices $[L^s]$, $s = 1, \ldots, n-1$, which are defined by

$$
\begin{aligned}
\left[d_{ij}^s\right] &:= \begin{cases} 1 & \text{if} \quad i = j \neq s, \\ 1/[l_{ss}] & \text{if} \quad i = j = s, \\ 0 & \text{otherwise}, \end{cases} \\
\left[l_{ij}^s\right] &:= \begin{cases} 1 & \text{if} \quad i = j, \\ -[l_{is}] & \text{if} \quad i > j = s, \\ 0 & \text{otherwise}. \end{cases}
\end{aligned}
\tag{3.5}
$$

By executing the steps 2 and 3 of the interval Cholesky algorithm, one gets the proof of the following theorem.

THEOREM 3.4. *Let the elements of* $[D^s], [L^s] \in \mathbf{IR}^{n \times n}$ *be defined as in* (3.5). *Then for the vectors* $[y]$ *and* $[x]^C$ *of* (3.2) *we get*

$$
[y] = [D^n]\Big([L^{n-1}]\Big([D^{n-1}]\Big(\cdots\Big([L^2]\big([D]^2([L^1]([D^1][b])))\big)\cdots\Big)\Big)\Big), \tag{3.6}
$$

$$
[x]^C = [D^1]\Big([L^1]^T\Big([D^2]\Big(\cdots\Big([L^{n-2}]^T\Big([D^{n-1}]\big([L^{n-1}]^T([D^n][y])\big)\Big)\Big)\cdots\Big)\Big)\Big). \tag{3.7}
$$

Note that the parentheses cannot be omitted in general, since the multiplication of interval matrices is not associative. For point matrices $[A] \equiv A$, the matrices $[D^s] \equiv D^s$ and $[L^s] \equiv L^s$ are point matrices, too. Hence, for a point vector $[b] \equiv b$ we get

$$
\begin{aligned}
y &= D^n D^{n-1}\{D^{-(n-1)} L^{n-1} D^{n-1}\} D^{n-2} \\
&\quad \times \{D^{-(n-2)} L^{n-2} D^{n-2}\} \cdots D^1\{D^{-1} L^1 D^1\} b \\
&= D^n D^{n-1} \cdots D^1 \tilde{L}^{n-1} \tilde{L}^{n-2} \cdots \tilde{L}^1 b \\
&= \tilde{D} \tilde{L} b
\end{aligned}
\tag{3.8}
$$

with $D^{-s} := (D^s)^{-1}$, $\tilde{D} := D^n D^{n-1} \cdots D^2 D^1$, $\tilde{L}^s := D^{-s} L^s D^s$, and $\tilde{L} := \tilde{L}^{n-1} \cdots \tilde{L}^2 \tilde{L}^1$. In (3.8) we used the fact that $D^s$ commutes with $L^r$ for $r > s$ because of the particular shape of $D^s$ and $L^r$. By the same reasoning, we get

$$\begin{aligned} x &= D^1(L^1)^T D^2(L^2)^T \cdots D^{n-1}(L^{n-1})^T D^n y \\ &= \left\{D^1(L^1)^T D^1\right\}\left\{D^2(L^2)^T D^2\right\} \cdots \left\{D^{n-1}(L^{n-1})^T D^{n-1}\right\}(D^n)^2 \tilde{L} b \\ &= \tilde{U}\tilde{L}b, \end{aligned} \tag{3.9}$$

where $\tilde{U} := \{D^1(L^1)^T D^1\}\{D^2(L^2)^T D^2\} \cdots \{D^{n-1}(L^{n-1})^T D^{n-1}\}(D^n)^2$.

Since $\tilde{L}$ is a lower triangular matrix with ones in its diagonal, the same holds for its inverse $\tilde{L}^{-1}$. Thus $\tilde{L}^{-1}\tilde{U}^{-1} = A$ is the $LU$ decomposition of $A$ resulting from the Gaussian algorithm without permuting rows or columns. This well-known relation between the Cholesky decomposition and the $LU$ decomposition of $A$ cannot be generalized to nondegenerate interval matrices $[A]$, again because the multiplication of interval matrices is not associative. In addition, inverses of such matrices do not exist in the usual algebraic sense.

We end this section with a different description of step 1 in the interval Cholesky method.

DEFINITION 3.5. Let either $[A] = ([a_{11}]) \in \mathbf{IR}^{1\times 1}$ or

$$[A] = \begin{pmatrix} [a_{11}] & [c]^T \\ [c] & [A'] \end{pmatrix} = [A]^T \in \mathbf{IR}^{n\times n}, \qquad n > 1, \quad [c] \in \mathbf{IR}^{n-1}, \quad [A'] \in \mathbf{IR}^{(n-1)\times(n-1)}.$$

(a) $\Sigma_{[A]} := [A'] - (1/[a_{11}])[c][c]^T \in \mathbf{IR}^{(n-1)\times(n-1)}$ is termed the *Schur complement* (of the (1, 1) entry $[a_{11}]$) provided $n > 1$ and $0 \notin [a_{11}]$. In the product $[c][c]^T$ we assume that $[c_i][c_i]$ is evaluated as $[c_i]^2$ [see (2.4)]. $\Sigma_{[A]}$ is not defined if $n = 1$ or if $0 \in [a_{11}]$.

(b) We call the pair $([L], [L]^T)$ the *Cholesky decomposition* of $[A]$ if $0 < \underline{a}_{11}$ and if either $n = 1$ and $[L] = (\sqrt{[a_{11}]})$ or

$$[L] = \begin{pmatrix} \sqrt{[a_{11}]} & 0 \\ \dfrac{[c]}{\sqrt{[a_{11}]}} & [L'] \end{pmatrix}, \tag{3.10}$$

where $([L'], [L']^T)$ is the Cholesky decomposition of $\Sigma_{[A]}$.

Definition 3.5(a) is a modification of the Schur complement defined in [13, p. 155], where the square of an interval $[a]$ is computed as $[a]\cdot[a]$.

THEOREM 3.6. *The matrix* $[L]$ *in* (3.2) *exists if and only if* $[L]$ *from* (3.10) *exists. In this case, the two matrices are identical.*

*Proof.* We prove the assertion by induction with respect to the number $n$ of rows or columns of $[A]$.

If $n = 1$, the assertion follows from $\sqrt{[a_{11}]}\sqrt{[a_{11}]} = [a_{11}]$ for $0 \leqslant \underline{a}_{11}$.

Let the assertion be true for some $n$, and choose $[A]$ from $\mathbf{IR}^{(n+1)\times(n+1)}$. For ease of argumentation we replace $[L]$, $[L']$ in Definition 3.5 by $[M]$, $[M']$.

Assume first that $[L]$ exists, where $[L]$ is computed by the interval Cholesky method (3.2). We show that $[A]$ has the Cholesky decomposition $([M],[M]^T)$ satisfying $[M]=[L]$. Since $[L]$ exists, we obtain $\underline{a}_{11} > 0$. Hence $[l_{i1}] = [m_{i1}]$ for $i = 1,\dots,n+1$.

For $j \geqslant 2$, the formulae in the interval Cholesky method can be reformulated as

$$
\begin{aligned}
[l_{jj}] &= \left(\left([a_{jj}] - [l_{j1}]^2\right) - \sum_{k=2}^{j-1}[l_{jk}]^2\right)^{1/2} \\
&= \left(\left([a_{jj}] - \frac{[a_{j1}]^2}{\sqrt{[a_{11}]}\sqrt{[a_{11}]}}\right) - \sum_{k=2}^{j-1}[l_{jk}]^2\right)^{1/2} \\
&= \left(\left([a_{jj}] - \frac{[a_{j1}]^2}{[a_{11}]}\right) - \sum_{k=2}^{j-1}[l_{jk}]^2\right)^{1/2}, \qquad (3.11) \\
[l_{ij}] &= \frac{1}{[l_{jj}]}\left(\left([a_{ij}] - [l_{i1}][l_{j1}]\right) - \sum_{k=2}^{j-1}[l_{ik}][l_{jk}]\right) \\
&= \frac{1}{[l_{jj}]}\left(\left([a_{ij}] - \frac{[a_{i1}][a_{j1}]}{[a_{11}]}\right) - \sum_{k=2}^{j-1}[l_{ik}][l_{jk}]\right).
\end{aligned}
$$

These formulae can be interpreted as the interval Cholesky method applied to $\Sigma_{[A]} \in \mathbf{IR}^{n\times n}$, which results in a lower triangular matrix $[L']$. By the hypotheses made for this induction, the matrix $[M']$ of Definition 3.5(b) exists and equals $[L']$. Thus $[M]$ exists and satisfies $[M]=[L]$.

Assume now conversely that $[M]$ exists. Then, again, $\underline{a}_{11} > 0$, $[l_{i1}] = [m_{i1}]$ for $i = 1, \ldots, n+1$, and $[L'] = [M']$ by the hypotheses and by (3.11). This finishes the proof. ■

We remark that Definition 3.5(b) is a formulation which is an analogue of the triangular decomposition of $[A]$ made in [13, p. 155].

## 4. FEASIBILITY

In this section, we first consider the feasibility of the interval Cholesky method. We start with an example which shows that the method need not be feasible for interval matrices $[A] = [A]^T$ even if it is for any symmetric matrix $A \in [A]$.

EXAMPLE 4.1. Let

$$[A] := \begin{pmatrix} 1 & [a] & [a] \\ [a] & 1 & [a] \\ [a] & [a] & 1 \end{pmatrix} \qquad \text{with} \quad [a] := \left[0, \tfrac{2}{3}\right].$$

This matrix and a slightly modified one have already been used to illustrate that the interval Gaussian algorithm is not feasible although it is for any matrix $A \in [A]$ (cf. [10, 12, 14]).

Let $A \in [A]$ be symmetric. Then

$$A = \begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix} \qquad \text{with} \quad a, b, c \in \left[0, \tfrac{2}{3}\right].$$

The determinants $D_1, D_2, D_3$ of the leading principal matrices have the values $D_1 = 1 > 0$, $D_2 = 1 - a^2 > 0$, and $D_3 = 1 - c^2 - a(a - bc) + b(ac - b) = 1 - a^2 - b^2 - c^2 + 2abc$. The continuous function $D_3 = D_3(a, b, c)$ has a minimum at some point $(a_0, b_0, c_0)$ of the Cartesian product $[a]^3 := [a] \times [a] \times [a]$, since $[a]^3$ is compact. If at least one of the three coordinates $a_0, b_0, c_0$ is zero, we get $D_3(a_0, b_0, c_0) \geqslant 1 - (\frac{2}{3})^2 - (\frac{2}{3})^2 > 0$. If $a_0 = b_0 = c_0 = \frac{2}{3}$, then $D_3(a_0, b_0, c_0) = \frac{7}{27} > 0$. If at least one of the three coordinates $a_0, b_0, c_0$ is contained in the interior of $[0, \frac{2}{3}]$, we can w.l.o.g. assume that $c_0 \in (0, \frac{2}{3})$. Then

$$\frac{\partial D_3(a_0, b_0, c_0)}{\partial c} = -2c_0 + 2a_0 b_0 = 0,$$

which implies

$$\begin{aligned} D_3(a_0, b_0, c_0) &= 1 - a_0^2 - b_0^2 + c_0(-c_0 + a_0 b_0) + a_0 b_0 c_0 \\ &\geqslant 1 - \tfrac{4}{9} - \tfrac{4}{9} > 0. \end{aligned}$$

Thus, for any choices $a, b, c \in [a]$, the matrix $A$ is symmetric and positive definite by Lemma 2.2, and the ordinary Cholesky method is feasible (cf. [16, pp. 174–175]). But the interval Cholesky method fails, since $[l_{11}] = 1$, $[l_{21}] = [l_{31}] = [0, \frac{2}{3}]$, $[l_{22}] = [\sqrt{5}/3, 1]$, $[l_{32}] = [-4/(3\sqrt{5}), 2/\sqrt{5}]$, and $[a_{33}] - [l_{31}]^2 - [l_{32}]^2 = [-\frac{11}{45}, 1]$ contains zero, i.e., $[l_{33}]$ does not exist.

We now present a class of matrices for which (3.2) is feasible.

THEOREM 4.2. *Let $[A] \in \mathbf{IR}^{n\times n}$ be an H-matrix satisfying $[A] = [A]^T$ and $0 < \underline{a}_{ii}$, $i = 1, \ldots, n$. Then $[x]^C$ exists, and $[L]$ is again an H-matrix.*

*Proof.* By the assumptions, $\hat{A} := \langle [A] \rangle$ is a Stieltjes matrix; in particular, it is symmetric and positive definite by Lemma 2.3. Hence $\hat{A}$ can be represented as $\hat{A} = \hat{L}\hat{L}^T$ by using the Cholesky method (cf. [16, pp. 174–175]). From the formulae of this method it follows immediately that the triangular matrix $\hat{L}$ is contained in $\mathbf{Z}^{n\times n}$ and has positive diagonal entries. Therefore, it is an $M$-matrix. We show by induction with respect to the column index $j$ that $[L]$ exists and that

$$\hat{L} \leqslant \langle [L] \rangle \tag{4.1}$$

holds.

For $j = 1$, $[l_{11}] = \sqrt{[a_{11}]}$ exists, since we assumed $\underline{a}_{11} > 0$. We get $\langle [l_{11}] \rangle = \sqrt{\langle [a_{11}] \rangle} = \hat{l}_{11}$, $[l_{i1}] = [a_{i1}]/[l_{11}]$ exists, and it follows from (2.2) that

$$|[l_{i1}]| = \left| \frac{[a_{i1}]}{[l_{11}]} \right| = \frac{|[a_{i1}]|}{\langle [l_{11}] \rangle} = -\hat{l}_{i1}, \qquad i = 2, \ldots, n.$$

Let all columns of $[L]$ exist which have an index less than $j > 1$. Assume that (4.1) holds for all these columns, and define

$$[s] = [\underline{s}, \bar{s}] := \sum_{k=1}^{j-1} [l_{jk}]^2$$

and

$$[t] := [a_{jj}] - [s].$$

Then using (2.2), $\underline{a}_{jj} > 0$, and the induction hypothesis we obtain

$$\begin{aligned} 0 < \hat{l}_{jj}^2 &= \langle [a_{jj}] \rangle - \sum_{k=1}^{j-1} \hat{l}_{jk}^2 \leqslant \langle [a_{jj}] \rangle - \sum_{k=1}^{j-1} |[l_{jk}]|^2 \\ &= \underline{a}_{jj} - \bar{s} = \underline{t} = \langle [t] \rangle = \left\langle [a_{jj}] - \sum_{k=1}^{j-1} [l_{jk}]^2 \right\rangle. \end{aligned}$$

Hence $0 \notin [a_{jj}] - \sum_{k=1}^{j-1} [l_{jk}]^2$. Therefore, $[l_{jj}]$ exists and satisfies $\langle [l_{jj}] \rangle \geqslant \hat{l}_{jj}$.

For $i > j$ we get

$$\begin{aligned} |[l_{ij}]| &\leqslant \frac{1}{\langle [l_{jj}] \rangle} \left( |[a_{ij}]| + \sum_{k=1}^{j-1} |[l_{ik}]| |[l_{jk}]| \right) \\ &\leqslant \frac{1}{\hat{l}_{jj}} \left( |[a_{ij}]| + \sum_{k=1}^{j-1} \hat{l}_{ik} \hat{l}_{jk} \right) = -\hat{l}_{ij}. \end{aligned}$$

This implies $\hat{l}_{ij} \leqslant -|[l_{ij}]|$.

Thus, $[x]^C$ exists, and the $H$-matrix property of $[L]$ follows from Corollary 3.7.4 in [13]. ■

Note that an analogue of Theorem 4.2 holds also for the interval Gaussian algorithm, as was shown in [1].

COROLLARY 4.3. *Let* $[A] = [A]^T \in \mathbf{IR}^{n \times n}$ *with* $0 < \underline{a}_{ii}$, $i = 1, \ldots, n$. *Then in each of the following cases,* $[A]$ *is an H-matrix and* $[x]^C$ *exists.*

(i) $\langle [A] \rangle$ *is strictly diagonally dominant.*
(ii) $\langle [A] \rangle$ *is irreducibly diagonally dominant.*
(iii) $\langle [A] \rangle$ *is regular and diagonally dominant.*
(iv) $\langle [A] \rangle$ *is positive definite.*

*Proof*. By Theorem 2 in [10], $[A]$ is an H-matrix in each of the four cases; hence $[x]^C$ exists by Theorem 4.2. ■

EXAMPLE 4.4. Let $[a]$, $[b]$, and $[c] \in \mathbf{IR}$ and

$$[A] = \begin{pmatrix} [5,6] & [a] & [a] & [b] & [c] \\ [a] & [8,9] & [b] & [b] & [a] \\ [a] & [b] & 6 & [c] & [a] \\ [b] & [b] & [c] & [7,8] & [b] \\ [c] & [a] & [a] & [b] & [7,8] \end{pmatrix} \in \mathbf{IR}^{5\times 5}.$$

Then $0 < \underline{a}_{ii}$, $i = 1,\ldots,5$, $[A] = [A]^T$, and

$$\langle [A] \rangle = \begin{pmatrix} 5 & -1 & -1 & -2 & -\frac{1}{2} \\ -1 & 8 & -2 & -2 & -1 \\ -1 & -2 & 6 & -\frac{1}{2} & -1 \\ -2 & -2 & -\frac{1}{2} & 7 & -2 \\ -\frac{1}{2} & -1 & -1 & -2 & 7 \end{pmatrix}$$

in all of the following cases, for example:

(i) $[a] = [\frac{1}{2}, 1]$, $[b] = [1,2]$, $[c] = [\frac{1}{4}, \frac{1}{2}]$,
(ii) $[a] = [-\frac{1}{2}, 1]$, $[b] = [1,2]$, $[c] = [\frac{1}{4}, \frac{1}{2}]$,
(iii) $[a] = [-\frac{1}{2}, 1]$, $[b] = [-1,2]$, $[c] = [\frac{1}{4}, \frac{1}{2}]$,
(iv) $[a] = [-\frac{1}{2}, 1]$, $[b] = [-1,2]$, $[c] = [-\frac{1}{4}, \frac{1}{2}]$,
(v) $[a] = [-1, -\frac{1}{2}]$, $[b] = [-2,1]$, $[c] = [-\frac{1}{2}, \frac{1}{4}]$.

Since $\langle [A] \rangle$ is strictly diagonally dominant, $[x]^C$ exists in all cases.

In our next corollary we consider the particular case of $2 \times 2$ matrices.

COROLLARY 4.5. *If* $[A] = [A]^T \in \mathbf{IR}^{2\times 2}$, *then the following properties are equivalent*:

(i) $[x]^C$ *exists*.
(ii) *Any symmetric matrix* $A \in [A]$ *is positive definite*.
(iii) *The Cholesky method is feasible for any symmetric matrix* $A \in [A]$.

*Proof.* (i) $\Rightarrow$ (ii) follows from Lemma 2.2, since $a_{11} = l_{11}^2 > 0$ and

$$\det A = a_{22}a_{11} - a_{12}^2 = \left(a_{22} - \frac{a_{12}^2}{a_{11}}\right)\cdot a_{11} = l_{22}^2 l_{11}^2 > 0.$$

(ii) $\Rightarrow$ (iii) follows from Theorem 4.3.3 in [16].

(iii) $\Rightarrow$ (i): Choose $A \in [A]$ such that $A = A^T$ and $\langle A \rangle = \langle [A] \rangle$ holds. By the hypothesis, $l_{11}, l_{22} > 0$; hence

$$a_{11} = l_{11}^2 > 0,$$

$$a_{22} = l_{22}^2 + \frac{a_{12}^2}{a_{11}} > 0.$$

This implies

$$\langle [a_{11}] \rangle = \langle a_{11} \rangle = a_{11} > 0,$$

$$\det\langle [A] \rangle = |a_{11}|\,|a_{22}| - |[a_{12}]|^2 = a_{11}a_{22} - a_{12}^2 = l_{22}^2 l_{11}^2 > 0.$$

Therefore, $\langle [A] \rangle$ is symmetric and positive definite, and $\underline{a}_{11} = a_{11} > 0$, $\underline{a}_{22} = a_{22} > 0$. The assertion follows from Corollary 4.3. ∎

COROLLARY 4.6. *If $[A] \in \mathbf{IR}^{n\times n}$ is an M-matrix satisfying $[A] = [A]^T$, then $[x]^C$ exists.*

As the example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

illustrates, not every symmetric $H$-matrix is an $M$-matrix. But symmetric $H$-matrices are closely related to positive definite matrices, as the following theorem shows.

THEOREM 4.7. *Let $[A] \in \mathbf{IR}^{n\times n}$ be an H-matrix satisfying $[A] = [A]^T$ and $0 < \underline{a}_{ii}$, $i = 1, \dots, n$. Then each symmetric matrix $A \in [A]$ is positive definite.*

*Proof.* Since $\langle [A] \rangle$ is an $M$-matrix, $\langle A \rangle \geqslant \langle [A] \rangle$ is an $M$-matrix, too. Because $\underline{a}_{ii} > 0$, the matrix $A$ has a nonnegative diagonal part $D$. Split $A$

into $A = D - B$. Then $\langle A \rangle = D - |B| = sI - (sI - D + |B|)$, $s \in \mathbf{R}$. By a property which is equivalent to the definition of an $M$-matrix (cf. e.g. [5, (1.2) and $(N_{38})$], $s$ can be chosen such that

$$s > \rho(sI - D + |B|) \quad \text{and} \quad sI - D + |B| \geqslant 0; \tag{4.2}$$

hence $sI \geqslant D$, and

$$|sI - A| = |sI - D + B| \leqslant |sI - D| + |B| = sI - D + |B|$$

implies

$$\rho(sI - A) \leqslant \rho(sI - D + |B|) < s$$

by results in [17, §2.1], following from the Perron-Frobenius theorem. Therefore, all eigenvalues $\lambda$ of $A$ satisfy $|s - \lambda| < s$, whence $\lambda > 0$. This proves the assertion by Lemma 2.2. ∎

Note that the converse of Theorem 4.7 is not true. This is shown by the matrix of Example 4.1. Every symmetric matrix $A \in [A] = [A]^T$ is positive definite, and $[A]$ satisfies $\underline{a}_{ii} > 0$, $i = 1, \dots, n$. But $[A]$ is not an $H$-matrix, since otherwise, $[x]^C$ would exist by Theorem 4.2.

The fact that the interval Cholesky factorization need not exist for an interval matrix whose symmetric element matrices all are positive definite can make preconditioning necessary. An algorithm will be investigated in a future paper.

For tridiagonal matrices we have the following result.

THEOREM 4.8. *Let $[A] = [A]^T \in \mathbf{IR}^{n \times n}$ be a tridiagonal matrix, and let $\tilde{A} \in [A]$ be any symmetric matrix which satisfies $\langle \tilde{A} \rangle = \langle [A] \rangle$ and which is positive definite. Then $[A]$ is an $H$-matrix; in particular, all symmetric matrices $A \in [A]$ are positive definite, and $[x]^C$ exists.*

*Proof.* Since $\tilde{A}$ is assumed to be positive definite, all diagonal entries $\tilde{a}_{ii}$ are positive. Therefore, $\langle \tilde{A} \rangle = \langle [A] \rangle$ and $\tilde{A} \in [A]$ imply $\underline{a}_{ii} > 0$, $i = 1, \dots, n$. By Lemma 2.4, $\tilde{A}$ is an $H$-matrix; hence $[A]$ is an $H$-matrix, too. Here we have used the equality $\langle \tilde{A} \rangle = \langle [A] \rangle$ once more. The assertion follows now from Theorems 4.2 and 4.7. ∎

COROLLARY 4.9. *Let $[A] = [A]^T \in \mathbf{IR}^{n \times n}$ be a tridiagonal matrix, and let $\tilde{A} \in [A]$ be any symmetric matrix which satisfies $\langle \tilde{A} \rangle = \langle [A] \rangle$. If $\tilde{A}$ can* *be chosen such that it fulfills one of the three properties*

(i) $\tilde{A}$ *is totally positive,*
(ii) $\tilde{A}$ *is regular and totally nonnegative,*
(iii) $\tilde{A}$ *is oscillatory,*

*then* $[A]$ *is an H-matrix; in particular, all symmetric matrices* $A \in [A]$ *are positive definite, and* $[x]^C$ *exists.*

*Proof.* In the case of (i), the leading principal minors are positive; hence $\tilde{A}$ is symmetric positive definite, and Theorem 4.8 proves the assertion.

In the case of (ii), the assumptions yield det $\tilde{A} > 0$. Thus Lemma 2.2 combined with the inequality (116) in [8, p. 443] shows that the assumptions of Theorem 4.8 hold. Therefore, the corollary is proved in case (ii).

Since det $\tilde{A}^k > 0$ for some integer $k$ implies det $\tilde{A} \neq 0$, (iii) is a particular case of (ii). ∎

Example 4.1 and Theorems 4.2 and 4.7 show that $[x]^C$ does not necessarily exist for interval matrices $[A] = [A]^T$ of which all symmetric element matrices $A$ are positive definite, but that for an important subclass of such matrices the existence of $[x]^C$ is guaranteed.

We will now show that for an $M$-matrix $[A] = [A]^T$ the bounds of the matrix $[L]$ in step 1 of (3.2) can be obtained independently of each other from the Cholesky decomposition of the bounds $\underline{A}$, $\overline{A}$.

THEOREM 4.10. *Let* $[A] = [A]^T \in \mathbf{IR}^{n\times n}$ *be an M-matrix, and let* $\underline{A} = L^{(l)}(L^{(l)})^T$, $\overline{A} = L^{(u)}(L^{(u)})^T$ *be the Cholesky decompositions of* $\underline{A}$ *and* $\overline{A}$, *respectively. Then* $L^{(l)}$, $L^{(u)}$ *are M-matrices. The matrix* $[L]$ *from the Cholesky decomposition of* $[A]$ *can be represented as*

$$[L] = [L^{(l)}, L^{(u)}]; \tag{4.3}$$

*in particular,* $[L]$ *is an M-matrix.*

*Proof.* Since $\underline{A}$, $\overline{A}$ are Stieltjes matrices, the formulae in (3.2) show at once that $L^{(l)}, L^{(u)} \in \mathbf{Z}^{n\times n}$. Theorem 4.2, applied to $\underline{A}$ and to $\overline{A}$, respectively, implies that they are $M$-matrices.

We now prove (4.3) by induction with respect to the column index $j$.

For $j = 1$ we get at once

$$[l_{11}] = \sqrt{[a_{11}]} = \left[\sqrt{\underline{a}_{11}}, \sqrt{\overline{a}_{11}}\right] = \left[l_{11}^{(l)}, l_{11}^{(u)}\right]$$

and

$$[l_{i1}] = \frac{[a_{i1}]}{[l_{11}]} = \left[\frac{\underline{a}_{i1}}{l_{11}^{(l)}}, \frac{\overline{a}_{i1}}{l_{11}^{(u)}}\right] = \left[l_{i1}^{(l)}, l_{i1}^{(u)}\right], \quad i > 1, \qquad \text{with} \quad l_{i1}^{(u)} \leqslant 0,$$

where we have taken into account $\overline{a}_{i1} \leqslant 0$ for $i > 1$.

Assume now that (4.3) holds for all columns with an index less than $j > 1$. Then

$$[l_{jj}] = \left([a_{jj}] - \sum_{k=1}^{j-1}\left[\left(l_{jk}^{(u)}\right)^2, \left(l_{jk}^{(l)}\right)^2\right]\right)^{1/2}$$

$$= \left[l_{jj}^{(l)}, l_{jj}^{(u)}\right]$$

and

$$[l_{ij}] = \left([a_{ij}] - \sum_{k=1}^{j-1}\left[l_{ik}^{(u)}l_{jk}^{(u)}, l_{ik}^{(l)}l_{jk}^{(l)}\right]\right) \cdot \left[\frac{1}{l_{jj}^{(u)}}, \frac{1}{l_{jj}^{(l)}}\right]$$

$$= \left[l_{ij}^{(l)}, l_{ij}^{(u)}\right] \qquad \text{for} \quad i > j,$$

since $\overline{a}_{ij} \leqslant 0$ and $l_{ij}^{(u)} \leqslant 0$ for $i > j$. This proves the assertion. ■

We now consider the quality of the enclosure of $[x]^C$ with respect to $S_{\text{sym}}$ from (3.1).

THEOREM 4.11. *Let* $[A] = [A]^T \in \mathbf{IR}^{n \times n}$ *be an M-matrix, and let* $[b] \in \mathbf{IR}^n$ *satisfy* $\underline{b} \geqslant 0$ *or* $0 \in [b]$ *or* $\overline{b} \leqslant 0$. *Then* $[x]^C = \square S_{\text{sym}}$.

*Proof.* Denote by $(D^{(l)})^s, (L^{(l)})^s$ and $(D^{(u)})^s, (L^{(u)})^s$ the matrices in the representation (3.7) when the interval Cholesky method is applied to $\underline{A}$ and $\overline{A}$, respectively. By Theorem 4.10 and by (3.5), these matrices are nonnegative, and

$$[D]^s = \left[\left(D^{(u)}\right)^s, \left(D^{(l)}\right)^s\right], \qquad [L]^s = \left[\left(L^{(u)}\right)^s, \left(L^{(l)}\right)^s\right].$$

Hence Theorem 3.4 proves

$$[x]^C = \begin{cases} \left[\underline{A}^{-1}\underline{b},\, \overline{A}^{-1}\overline{b}\right] & \text{if } \overline{b} \leqslant 0, \\ \left[\underline{A}^{-1}\underline{b},\, \underline{A}^{-1}\overline{b}\right] & \text{if } 0 \in [b], \\ \left[\overline{A}^{-1}\underline{b},\, \underline{A}^{-1}\overline{b}\right] & \text{if } \underline{b} \geqslant 0. \end{cases} \qquad \blacksquare$$

COROLLARY 4.12. *Let* $[A] = [A]^T \in \mathbf{IR}^{n\times n}$ *be an M-matrix, and let* $[b] \in \mathbf{IR}^n$ *satisfy* $\underline{b} \geqslant 0$ *or* $0 \in [b]$ *or* $\overline{b} \leqslant 0$. *Then* $[x]^C = \square S_{\text{sym}} = \square S = [x]^G$, *where* $[x]^G$ *denotes the vector resulting from the interval Gaussian algorithm applied to* $[A]$ *and* $[b]$.

*Proof.* The proof follows at once from Theorem 4.11 and from results in [4]. ■

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