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On the Approximation of the Range of Values by Interval Expressions

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Abstract — Zusammenfassung

On the Approximation of the Range of Values by Interval Arithmetic Expressions. If the real-valued mapping f has a representation of the form $f(x) = \varphi_0 + \ell(x) \cdot h(x)$, $x \in X$, where for the diameter of $h(X)$ the inequality $d(h(X)) \leq \sigma d(X)$ holds and for the absolute value of $\ell(X)$ we have $|\ell(X)| \leq \tau d(X)^n$, then we introduce an interval expression for f which approximates the range of values of f over the compact interval X with order $n + 1$. Our result contains as a special case the theorem on higher order centered forms from [2] and a series of representations of f not discussed before.

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Über die Approximation des Wertebereichs durch Intervallausdrücke. Die reele Funktion f besitze über dem Intervall X eine Darstellung der Form $f(x) = \varphi_0 + \ell(x) \cdot h(x)$, $x \in X$. Für den Durchmesser von $h(X)$ gelte $d(h(X)) \leq \sigma d(X)$ und für den Absolutbetrag von $\ell(X)$ bestehe die Ungleichung $|\ell(X)| \leq \tau d(X)^n$. Dann geben wir einen Intervallausdruck an, der den Wertebereich von f über dem Intervall X mit der Ordnung $n + 1$ approximiert. Unser Resultat enthält als Spezialfall das Theorem aus [2] über zentrierte Formen höherer Ordnung und außerdem eine Reihe von Darstellungen von f, für die diese Resultate bisher nicht bekannt waren.

1. Introduction

A fundamental property of interval arithmetic is the fact that it allows to include the range of values of real functions over an interval. It is well known that the distance of such a result to the exact range is strongly dependent on the representation of the function. For some details see, for example, the discussion in Chapter 3 of [1]. In this paper we discuss a generalization of the following result (see [2]):

Let the real valued function f be defined on the real compact interval $X \subseteq \mathbb{R}$. Assume that for some $c \in X$ and for some integer $n \geq 1$ the function f can be represented as

$$f(x) = f(c) + (x - c)^n \cdot h(x) \tag{1}$$

where h is a continuous function defined on X. Let $d(X) = x_2 - x_1$ be the diameter of the interval $X = [x_1, x_2]$ and denote by $q(A, B)$ the Hausdorff distance of two compact intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$:

(2) $$q(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\}.$$

Assume that $h(X)$ is a real compact interval for which

(3) $$h(x) \in h(X), \qquad x \in X$$

and

(4) $$d(h(X)) \le \sigma \cdot d(X)$$

hold. Defining the compact interval $f(X)$ by

(5) $$f(X) := f(c) + W((x - c)^n, X) \cdot h(X)$$

where $W((x - c)^n, X)$ denotes the range of $(x - c)^n$ over X then it follows that

(6) $$f(x) \in f(X), \qquad x \in X$$

and for $W(f, X)$, the range of values of f over X, it holds

(7) $$q(W(f, X), f(X)) \le \kappa d(X)^{n+1}$$

where κ is a nonnegative constant.

Because of (7) the representation (5) is called a centered form of f of order $n + 1$.

It was already mentioned in [2] that it is not an easy task to find a representation (1) of f (if it exists at all). In this note we show that the representation (1) of f is only a special case of a more general form which has the same approximation property. We first consider an example.

2. Example

Let there be given the real polynomial

$$f(x) = x^3 - 6x^2 + (12 - \varepsilon^2)x - (8 - 2\varepsilon^2), \qquad \varepsilon \ge 0$$

which has the zeroes

$$x_1 = 2 - \varepsilon, \qquad x_2 = 2, \qquad x_3 = 2 + \varepsilon.$$

Consider the interval $X = [2 - \delta, 2 + \delta]$, $0 \le \delta \le 2$, which contains the zeroes if $\varepsilon \le \delta$.

An elementary discussion gives the following result.

a) If $\delta^3 - \varepsilon^2\delta \ge \frac{2}{9}\sqrt{3}\varepsilon^3$ then f assumes on its maximum and its minimum at the boundary points of X and we get for the range of values of f over X the interval

$$W(f, X) = [-\delta^3 + \varepsilon^2\delta, \delta^3 - \varepsilon^2\delta].$$

b) Otherwise, that is if $\delta^3 - \varepsilon^2\delta < \frac{2}{9}\sqrt{3}\varepsilon^3$, we get

$$W(f, X) = \tfrac{2}{9}\sqrt{3}\varepsilon^3[-1, 1].$$

A) Assume now that we write $f(x)$ in the form

$$f(x) = \varphi_0 + \ell(x) \cdot h(x)$$

where $\varphi_0 = 0$, $\ell(x) = x - 2$, $h(x) = x^2 - 4x + 4 - \varepsilon^2$.

Then we get

$$\begin{aligned} f(X) &:= \varphi_0 + W(\ell, X)\cdot h(X) \\ &= [-\delta, \delta]([2-\delta, 2+\delta][2-\delta, 2+\delta] - 4[2-\delta, 2+\delta] + 4 - \varepsilon^2) \\ &= [-\delta, \delta][\delta^2 - \varepsilon^2 - 8\delta, \delta^2 - \varepsilon^2 + 8\delta] \\ &= [-\delta^3 + \delta\varepsilon^2 - 8\delta^2, \delta^3 - \delta\varepsilon^2 + 8\delta^2]. \end{aligned}$$

Hence in case a) we get

$$q(W(f, X), f(X)) = 8\delta^2 = c_1 d(X)^2$$

and in case b) because of $\varepsilon \leq \delta = \frac{1}{2}d(X)$

$$\begin{aligned} q(W(f, X), f(X)) &= \max\{|-\tfrac{2}{9}\sqrt{3}\,\varepsilon^3 + \delta^3 - \delta\varepsilon^2 + 8\delta^2|, |\tfrac{2}{9}\sqrt{3}\,\varepsilon^3 - \delta^3 + \delta\varepsilon^2 - 8\delta^2|\} \\ &= |\tfrac{2}{9}\sqrt{3}\,\varepsilon^3 - \delta^3 + \delta\varepsilon^2 - 8\delta^2| \leq c_2 d(X)^2. \end{aligned}$$

These results are not surprising since they are covered by the quadratic approximation property of the centered form. See (1) for $n = 1$ and (7).

B) Assume now that we write $f(x)$ in the form

$$f(x) = \varphi_0 + \ell(x)\cdot h(x)$$

where $\varphi_0 = 0$, $\ell(x) = (x-2)(x-(2+\varepsilon))$, $h(x) = x - (2-\varepsilon)$.

Then we get

$$\begin{aligned} f(X) &:= \varphi_0 + W(\ell, X)h(X) \\ &= \left[\ell\left(2+\frac{\varepsilon}{2}\right), \ell(2-\delta)\right]([2-\delta, 2+\delta] - (2-\varepsilon)) \\ &= \left[-\frac{\varepsilon^2}{4}, \delta^2 + \varepsilon\delta\right][-(\delta-\varepsilon), \delta+\varepsilon] \\ &= \left[\min\left\{-\frac{\varepsilon^2}{4}(\delta+\varepsilon), -(\delta-\varepsilon)(\delta^2+\varepsilon\delta)\right\}, \max\left\{\frac{\varepsilon^2}{4}(\delta-\varepsilon), (\delta^2+\varepsilon\delta)(\delta+\varepsilon)\right\}\right]. \end{aligned}$$

Using again the fact that $\varepsilon \leq \delta = \frac{1}{2}d(X)$ it is easy to see that both in case a) and in case b)

$$q(W(f, X), f(X)) \leq c_3 d(X)^3$$

holds.

Note that this result is not covered by the Theorem mentioned in the preceding Chapter. It is a special case of a more general result which we are now going to state and to prove.

3. Results

The result of the example from the preceding chapter is covered by the following main result of this paper.

Theorem. *Let the real valued function f be defined on a compact interval $X \subseteq \mathbb{R}$. Assume that the function f can be represented as*

(8) $$f(x) = \varphi_0 + \ell(x) \cdot h(x), \qquad x \in X,$$

where $\varphi_0 \in \mathbb{R}$ and where ℓ and h are continuous functions on X. Assume that $h(X)$ is a real compact interval for which

(9) $$h(x) \in h(X), \qquad x \in X$$

and

(10) $$d(h(X)) \leq \sigma d(X).$$

Furthermore assume that $\ell(X)$ is a real compact interval for which

(11) $$\ell(x) \in \ell(X), \qquad x \in X$$

and

(12) $$|\ell(X)| \leq \tau d(X)^n$$

where $|\cdot| := q(\cdot, 0)$ denotes the absolute value of an interval. Defining the compact interval $f(X)$ by

(13) $$f(X) := \varphi_0 + W(\ell, X) \cdot h(X)$$

it follows that

(14) $$q(f(X), W(f, X)) \leq \kappa d(X)^{n+1}$$

where κ is a nonnegative constant. □

Of course, we also have $W(f, X) \subseteq f(X)$ since by (9)

$$\begin{aligned} W(f, X) &= \{\varphi_0 + \ell(x)h(x) \mid x \in X\} \\ &\subseteq \varphi_0 + W(\ell, X) \cdot W(h, X) \\ &\subseteq \varphi_0 + W(\ell, X) \cdot h(X). \end{aligned}$$

Before we go into the details of a proof of this Theorem we include some remarks:

1. Assume that $\ell(x) = (x - c)^n$ for some $c \in X$. It follows from (8) that $\varphi_0 = f(c)$. If we define $\ell(X) := (X - c)^n$ (product of n terms each equal to the interval $X - c$), we have

$$\begin{aligned} |\ell(X)| &= |(X - c)^n| \leq |(X - X)^n| \\ &= |X - X|^n = d(X)^n \end{aligned}$$

and therefore (12) holds.

Hence our Theorem contains the results from [2] as a special case.

2. Assume that $\ell(x) = (x - x_1)(x - x_2) \cdot \ldots \cdot (x - x_n)$ where $x_i \in X$, $i = 1(1)n$. It follows from (8) that $\varphi_0 = f(x_i)$, $i = 1(1)n$.

If we define $\ell(X) := (X - x_1) \cdot (X - x_2) \cdot \ldots \cdot (X - x_n)$ (product of the n intervals $X - x_i$, $i = 1(1)n$), we have

$$|\ell(X)| = |(X - x_1)(X - x_2) \cdot \ldots \cdot (X - x_n)|$$

$$\leq |(X - X)(X - X)\cdot\ldots\cdot(X - X)|$$
$$= d(X)^n$$

and therefore (12) holds.

For $x_i = c$, $i = 1(1)n$, we have the special case which was discussed in Remark 1. Hence the Theorem of this paper is more general than the results from [2].

3. Assume that $\ell(x) = (s_1(x) - c_1)\cdot\ldots\cdot(s_n(x) - c_n)$ where the real functions s_i, $i = 1(1)n$, are continuous over the compact interval X. Denote by $s_i(X)$ compact intervals for which $s_i(x) \in s_i(X)$, $x \in X$, and $d(s_i(X)) \leq \tau_i d(X)$, $i = 1(1)n$, hold. Choose $c_i \in s_i(X)$, $i = 1(1)n$. Defining

$$\ell(X) := (s_1(X) - c_1)\cdot\ldots\cdot(s_n(X) - c_n)$$

it follows that

$$\begin{aligned}|\ell(X)| &= |(s_1(X) - c_1)\cdot\ldots\cdot(s_n(X) - c_n)|\\ &\leq |(s_1(X) - s_1(X))\cdot\ldots\cdot(s_n(X) - s_n(X))|\\ &\leq d(s_1(X))\cdot\ldots\cdot d(s_n(X))\\ &\leq \tau_1\cdot\ldots\cdot\tau_n d(X)^n = \tau d(X)^n,\end{aligned}$$

where $\tau = \tau_1\cdot\ldots\cdot\tau_n$ and therefore (12) holds.

4. As the example from the preceding chapter shows there may exist a representation (8) of $f(x)$ with $\ell(x) = (x - x_1)\cdot\ldots\cdot(x - x_n)$ where the x_i are pairwise distinct. However a representation with all x_i equal does not exist or may not be readily available. On the other hand it must be stressed that $W(\ell, X)$ can be computed easily only in very special cases. This is for example the case if $\ell(x) = (x - c)^n$ for some integer n. For $\ell(x) = (x - x_1)\cdot\ldots\cdot(x - x_n)$ with pairwise different numbers x_i, $i = 1(1)n$, only small values of n are of practical value.

Proof of the Theorem.
Let $h_0 \in h(X)$. Defining $g(x) = \varphi_0 + \ell(x)h_0$, $r(x) = \ell(x)\cdot(h(x) - h_0)$ we have

(15) $$f(x) = g(x) + r(x), \qquad x \in D.$$

Setting

(16) $$r(X) := \ell(X)\cdot(h(X) - h_0)$$

we have $r(x) \in r(X)$ because of (9) and (11).

For $f(X)$ defined by (13) we have

$$\begin{aligned}f(X) &= \varphi_0 + W(\ell, X)\cdot h(X)\\ &= \varphi_0 + W(\ell, X)\cdot\{h_0 + h(X) - h_0\}\\ &\subseteq \varphi_0 + W(\ell, X)\cdot h_0 + W(\ell, X)\cdot(h(X) - h_0)\\ &\subseteq W(g, X) + r(X) =: \tilde{f}(X).\end{aligned}$$

$\tilde{f}(X)$ is a remainder form of f over X (see the Appendix).

Because of

$$W(f, X) \subseteq f(X) \subseteq \tilde{f}(X)$$

it follows from property b) of remainder forms (see the Appendix) that

$$q(W(f,X), f(X)) \le q(W(f,X), \tilde{f}(X)) \le 2|r(X)|.$$

Using the definition (16) of $r(X)$, we get by (10) and (12)

$$\begin{aligned} |r(X)| &= |\ell(X)\cdot(h(X) - h_0)| \\ &= |\ell(X)|\cdot|h(X) - h_0| \\ &\le |\ell(X)|\cdot|h(X) - h(X)| \\ &\le |\ell(X)|\cdot d(h(X)) \\ &\le \tau d(X)^n \cdot \sigma d(X) \\ &= \kappa d(X)^{n+1}, \qquad \kappa = \sigma\tau. \end{aligned}$$

□

4. *Appendix. Remainder Form* (Cornelius-Lohner [2]).
Let $f: D \to \mathbb{R}$ be a real function and suppose that f can be represented as

(17) $$f(x) = g(x) + r(x), \qquad x \in D,$$

where $g: D \to \mathbb{R}$ is continuous and $r: D \to \mathbb{R}$ is bounded. Assume that for some compact interval $X \subset D$ we have a compact interval $r(X)$ which contains $W(r, X)$.

Define

(18) $$\tilde{f}(X) := W(g, X) + r(X).$$

Then

a) $$W(f, X) \subseteq \tilde{f}(X)$$

b) $$q(W(f,X), \tilde{f}(X)) \le d(r(X)) \le 2|r(X)|.$$

These results can be found in [2], [4].

(18) is called a remainder form of f over X.

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