Existence of Solutions and Iterations for Nonlinear Equations

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1. Introduction. In this paper we discuss the use of interval analysis in order to prove the existence of solutions of an equation. In Chapter 2 we repeat the generalization of the bisection process using interval arithmetic tools. The use of the Brouwer fixed-point theorem is demonstrated in Chapter 3. We show in Example 1 that by using interval arithmetic it is sometimes possible to improve known existence statements. Since the proof of the Brouwer fixed-point theorem is nontrivial it seems worthwhile to investigate if one can prove the existence of fixed points by using interval arithmetic tools alone. Some ideas in this direction are described in Chapter 4. In the final Chapter 5 the Interval-Newton-Method is reconsidered again and a new statement concerning the order of convergence is given. The terminology used in this paper is the same as in [5].

2. Bisection. It is well-known that if for a real continuous function $f : \mathbb{R} \to \mathbb{R}$ there exist reals a and b, $a < b$, such that $f(a)f(b) \leq 0$, then there exists an $x^* \in [a,b]$ such that $f(x^*) = 0$. Furthermore x^* can be computed by

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the well known bisection process. If, however, the condition $f(a)f(b) \leq 0$ does not hold, then there is no statement possible whether there is a zero in $[a,b]$ or not.
Let f have an interval arithmetic evaluation $f([x])$ where $[x] = [a,b]$. If $0 \notin f([x])$ then f has no zero in $[x]$. This follows from the fact that the range of values of f over $[x]$ which we denote by $R(f;[x])$ is contained in $f([x])$. If, however, the interval arithmetic evaluation $f([x])$ contains zero, then no statement is possible whether there is a zero in $[a,b]$ or not. The reason for this conclusion is the fact that in general $f([x])$ is a proper superset of $R(f;[x])$. In this case we bisect the given interval into two equal parts and then proceed by computing the interval arithmetic evaluations for both subintervals. Now the same conclusions can be drawn as for the original interval. By repeating this process we can find subsets of the original interval which do not contain zeroes of f and subsets which possibly contain zeroes of f, respectively. Since with decreasing diameters the interval arithmetic evaluation is converging to the range we can locate the zeroes of f in arbitrarily close subintervals of $[a,b]$. For more details and references see the introductory part of Chapter 7 in [5].
The procedure described can in principle also be applied to mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ in order to locate zeroes of the system $f(x) = 0$ in n-dimensional intervals, so-called interval vectors. It is obvious that for larger n the

computing time increases exponentially with n and that the data organization becomes a nontrivial problem. On the other hand the advantage of this procedure is that besides the existence of the interval arithmetic evaluation no further assumptions - like differentiability - have to be imposed on f.

3. Brouwer Fixed-Point Theorem. This theorem is a very old result from 1912. A systematic application in Numerical Analysis in connection with interval arithmetic tools started after the publication of R.E. Moore's paper [12] in which he used the so-called Krawczyk-operator in order to prove the existence of fixed points of a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$. Nearly ten years earlier one can already find an application of the Brouwer fixed-point theorem in a note by Hansen [7], p. 23. See also Stetter [18], page 43, Satz 6.5.

Theorem 1. (Brouwer) *Let* $f : S \subset \mathbb{R}^n \to \mathbb{R}^n$ *be continuous on the compact, convex, nonempty set* S, *and suppose that* $f(x) \in S$ *for all* $x \in S$. *Then* f *has a fixed point in* S. □

A proof of this fundamental result can be found, for example, in [15]. The details of the proof are far from being obvious.

If one tries to apply the Brouwer fixed-point theorem it is usually not an easy task to verify that $f(x) \in S$ for all $x \in S$. The reason for this is the fact that (besides being continuous) the mapping f and the (compact, convex, nonempty) set S can be very general. If, however, f has an interval arithmetic evaluation then it is easy to give <u>sufficient conditions</u> for $f(x) \in S$ for all $x \in S$ if S is an n-dimensional interval vector $[x]$. This is the content of the following well-known result.

<u>Theorem 2</u>. <u>Let</u> $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ <u>be continuous and suppose that for some interval vector</u> $[x] \subseteq D$ <u>the interval arithmetic evaluation</u> $f([x])$ <u>of</u> f <u>exists</u>. <u>If</u> $f([x]) \subseteq [x]$ <u>then</u> f <u>has a fixed point in</u> $[x]$.

<u>Proof</u>. By the inclusion property of interval arithmetic we have $f(x) \in f([x])$ for $x \in [x]$. Therefore the assumption $f([x]) \subseteq [x]$ implies that $f(x) \in [x]$ for all $x \in [x]$. Since $[x]$ is compact, convex and nonempty the assertion follows by applying the Brouwer theorem. □

We note that under practical aspects the requirement that the interval arithmetic evaluation exists is not very restrictive. Most mappings which appear in numerical computation are composed of the four algebraic operations and of the elementary functions (trigonometric functions, exponential functions etc.) for which interval arithmetic evaluations can be defined in a natural manner. The great advantage of Theorem 2 is the fact that $f([x])$ can be computed systematically by following the rules of interval arithmetic. No sophisticated considerations or estimations are necessary in order to prove that $f(x) \in [x]$ for all $x \in [x]$. We will now explain this in some detail on a special problem. The discussion will even show that using interval arithmetic directly (which means to try to apply Theorem 2) gives in a precisely defined sense better results compared with verifying the hypothesis of the Brouwer theorem without interval analysis.

Example 1. (Error bounds for polynomial root approximations)
Let there be given the complex polynomial

$$p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_o$$

where $a_i \in \mathbb{C}$, $i = 0(1)n-1$, and furthermore pairwise different complex numbers w_i, $i = 1(1)n$, as approximations to the zeroes of p:

$$w_i \neq w_j \; ; \quad i \neq j \; ,$$
$$p(w_i) \neq 0 \; , \quad i = 1(1)n \; .$$

Consider now the n-th degree polynomial

$$f(z) = \prod_{i=1}^{n} (z-w_i) + \sum_{j=1}^{n} \left[\prod_{k=1,k\neq j}^{n} \frac{z-w_k}{w_j-w_k} \right] \cdot p(w_j) .$$

This polynomial has leading coefficient one and furthermore $f(w_j) = p(w_j)$, $j = 1(1)n$. From this it follows that $f(z) = p(z)$. Using this representation of $p(z)$ we get

$$p(z) = \prod_{k=1}^{n} (z-w_k) \cdot \left\{ 1 + \sum_{j=1}^{n} \frac{s_j}{z-w_j} \right\} , \quad z \neq w_j$$

where

$$s_j = \frac{p(w_j)}{\prod_{k=1,k\neq j}^{n} (w_j-w_k)} , \quad j = 1(1)n .$$

Since the w_k are by assumption not zeroes of p it follows that $p(z) = 0$ iff

$$1 + \sum_{j=1}^{n} \frac{s_j}{z-w_j} = 0 ,$$

or iff

$$z = w_i - s_i - (z - w_i) \cdot \sum_{j=1,j\neq i}^{n} \frac{s_j}{z-w_j} = : T_i(z) .$$

Hence the existence of a fixed point of T in a certain set guarantees the existence of a zero of p in this same set.

Theorem 3 (J.W. Schmidt [17]) Assume that n circular discs $K_j = \langle w_j, r_j \rangle$, $j = 1(1)n$, are given and that for some $i \in \{1,2,\ldots,n\}$

$$\min_{j=1, j\neq i}^{n} |w_i - w_j| > r_i$$

and

$$h_i(r_i) := |s_i| + r_i \cdot \sum_{j=1, j\neq i}^{n} \frac{|s_j|}{|w_i - w_j| - r_i} \leq r_i$$

hold. Then p has a zero z_i for which

$$|z_i - w_i| \leq h_i(r_i) . \qquad \square$$

The proof is performed by verifying the hypothesis of the Brouwer fixed-point theorem.

We now apply interval arithmetic in order to verify $T_i(z) \in K_i = \langle w_i, r_i \rangle$ for all $z \in K_i$. We have $T_i(z) \in K_i = \langle w_i, r_i \rangle \iff |T_i(z) - w_i| \leq r_i$.
The last inequality holds if $|T_i(K_i) - w_i| \leq r_i$ where $T_i(K_i)$ is obtained by replacing z by the complex circular disc K_i and computing $T_i(K_i)$ following the laws of circular disc arithmetic. See [5], Chapter 4, for example.
We obtain

$$|T_i(K_i) - w_i| = | \, s_i + (K_i - w_i) \cdot \sum_{j=1, j\neq i}^{n} \frac{s_j}{K_i - w_j} \, | =: g_i(r_i) .$$

The following result holds.

Theorem 4 (Frommer and Straub [6], Straub [19]).

If

$$\min_{j=1, j\neq i}^{n} |w_i - w_j| > r_i$$

then

$$g_i(r_i) \leq h_i(r_i) .$$

Equality holds iff the centers of the circular discs $s_j/(K_i - w_j)$, $j = 1(1)n$, $j \neq i$, are all located in the same quadrant of the complex plane and are all lying on a line which passes through the origin.

□

From this theorem it follows that for given circular discs $K_j = < w_j, r_j >$ it is in general easier to bound a zero of p by using $T_i(K_i)$ compared with trying to apply Theorem 3. Furthermore note that if $T_i(K_i) \subseteq K_i$ then there exists a zero z_i in $T_i(K_i)$ which means that $|z_i - w_i| \leq g_i(r_i)$. Hence if both approaches work then the interval arithmetic approach gives the better inclusion. □

Without going into details we mention that there are known a series of further classical results which can be improved or which lead at least to the same results if one uses interval arithmetic tools directly.

4. Avoiding the Brouwer Fixed-Point Theorem. We have already

mentioned that the details of the proof of the Brouwer fixed-point theorem are not very obvious. Since furthermore we are considering here only mappings which have an interval arithmetic evaluation the question naturally arises if the content of Theorem 2 could be proved without referring to the Brouwer fixed-point theorem. We will see that this is possible for certain simple mappings. For the general case of a nonlinear mapping we have to modify the interval arithmetic laws in order to perform the same proof. Nevertheless it turns out that some well known existence statements can be proved in this manner. Finally we note that in [14] the inverse function theorem was used in order to prove the existence of a fixed point for a special mapping.

We start by repeating some well known facts. The <u>midpoint operator</u> or simply <u>midpoint</u> of a real interval $[a] = [a_1, a_2]$ is defined to be the center of $[a]$:

$$m[a] = \frac{1}{2}(a_1 + a_2) .$$

If $[a]$ and $[b]$ are real intervals then

$$m([a] \pm [b]) = m[a] \pm m[b] ,$$
$$m(a \cdot [b]) = a \cdot m[b] , \quad a \in \mathbb{R} ,$$
$$m([a] : b) = m[a] : b , \quad b \in \mathbb{R} ,$$

but, in general,

$$m([a][b]) \neq m[a]\, m[b]$$

$$m([a]:[b]) \neq m[a]:m[b] \; .$$

For interval vectors and interval matrices the midpoint is defined via the components and elements, respectively. Similar rules as for intervals hold. For example, for a real matrix A and an interval vector $[x]$ it holds that $m(A[x]) = A\, m[x]$. For two interval vectors $[x]$ and $[y]$ we have $m([x]+[y]) = m[x] + m[y]$.

Consider now the real system

$$x = Ax + b$$

where the matrix A and the vector b are given. Assume that for some interval vector $[x]^{o}$ we have $f([x]^{o}) \subseteq [x]^{o}$ for $f(x) = Ax + b$. We consider then the iteration method $[x]^{k+1} = f([x]^{k})$, $k = 0,1,2,\ldots$. Using inclusion monotonicity of interval arithmetic it follows by complete induction that $[x]^{k+1} \subseteq [x]^{k}$ and therefore $\lim_{k\to\infty} [x]^{k} = [x]^{*}$. Since f is a continuous mapping from the set of interval vectors into itself it follows that $[x]^{*} = f([x]^{*})$. Applying the midpoint operator to this equation we get $m[x]^{*} = m(f([x]^{*})) = m(A[x]^{*}+b) = Am[x]^{*}+b$ which means that the center of $[x]^{*}$ is a solution of the equation $x = Ax + b$.

The preceding result holds under more general conditions.

Theorem 5. (See [16]). *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an inclusion monotone mapping which has an interval arithmetic evaluation $f([x]^o)$ for some interval vector $[x]^o$ with $f([x]^o) \subseteq [x]^o$. Assume that the mapping with domain consisting of the interval vectors $[x]$ contained in $[x]^o$ and with range consisting of $\{f([x]) \mid [x] \subseteq [x]^o\}$ is continuous. Assume furthermore that for the midpoint operator $m(f([x])) = f(m[x])$ for all $[x] \subseteq [x]^o$. Then $f : \mathbb{R}^n \to \mathbb{R}^n$ has a fixed point x^* in $[x]^o$.* □

The details of a proof proceed exactly as in the preceding special case.

Under practical aspects Theorem 5 is not very far reaching since the mapping f is not allowed to contain any multiplication and/or division. If this would be the case the equation $m(f([x])) = f(m[x])$ would not hold in general, as we have seen above. Therefore the question arises if it is possible to modify the multiplication and division of intervals in such a manner that also for these operations the midpoint of the operation can be obtained by performing the operation with the midpoints. Furthermore in order that the preceding Theorem 5 can be applied the new operations have to be inclusion monotone. We introduce these new operations simultaneously for real intervals and for circular disc intervals in the complex plane. By $< m[a], r[a] >$ we either denote a real interval $[a] = [r[a]-m[a] \,,\, r[a]+m[a]]$ or a circular disc in the complex plane where $m[a] \in \mathbb{C}$ is the center and $r[a]$ is the radius. We now <u>define</u>:

Let $[a] = < m[a] \,,\, r[a] >$, $[b] = < m[b] \,,\, r[b] >$. Then

$$[a] \underline{\times} [b] = < m[a]\, m[b], |m[a]|r[b]+|m[b]|r[a]+r[a]r[b] >$$

and (for $0 \notin [b]$)

$$[a] \underline{:} [b] = < m[a] \,,\, r[a] > \underline{\times} < \frac{1}{m[b]} \,,$$

$$\left| \frac{1}{m[b]} - \frac{\overline{m[b]}}{|m[b]|^2-(r[b])^2} \right| + \frac{r[b]}{|m[b]|^2-(r[b])^2} >.$$

Note that in the case of circular disc intervals the multiplication "$\underline{\times}$" is identical to the usual one. See [5], Chapter 5.

The following result holds (see [16]).

<u>Theorem 6</u>. <u>Assume that</u> $[a],[b],[c]$ <u>and</u> $[d]$ <u>are real intervals or circular disc intervals</u>. <u>Then</u>

(1) a) $[a]\cdot[b] \subseteq [a] \underline{\times} [b]$

b) $[a] : [b] \subseteq [a] \underline{:} [b] \qquad (0 \neq [b])$

(2) $[a] \subseteq [c]$ and $[b] \subseteq [d] \Longrightarrow$

a) $[a] \underline{\times} [b] \subseteq [c] \underline{\times} [d]$

b) $[a] \underline{:} [b] \subseteq [c] \underline{:} [d] \qquad (0 \neq [d])$.

(Inclusion monotonicity)

(3) a) $m([a] \underline{\times} [b]) = m[a] \cdot m[b]$

b) $m([a] \underline{:} [b]) = m[a] : m[b]$

(4) $m[a] = m[b] = 0 \Longrightarrow [a] \cdot [b] = [a] \underline{\times} [b]$. □

Proofs can be found in [16]. Multiplication of an interval matrix by an interval vector can be defined by using the introduced operation "$\underline{\times}$" for intervals. In this case we write $[A] \underline{\times} [x]$ for the product.

We consider now two examples which show that using the new operations more or less well known existence statements can be proven by applying Theorem 5.

<u>Example 2</u> Suppose that

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_o$$

is a real polynomial. Let

$$p'(x) = n a_n x^{n-1} + (n-1)a_{n-1} x^{n-2} + \dots + a_1$$

and define for a given interval $[x]$

$$p'([x]) = n a_n [x]^{n-1} + (n-1)a_{n-1} [x]^{n-2} + \dots + a_1$$

where

$$[x]^k = [x] \underline{\times} [x] \underline{\times} \dots \underline{\times} [x] \ , \quad \text{(k factors)} \ .$$

Furthermore let the real numbers c_i , $i = 1(1)n$, be defined by

$$c_{i-1} = \sum_{j=i}^{n} a_j \tilde{x}^{j-i} \ , \ i = 1(1)n \ ,$$

where $\tilde{x} \in \mathbb{R}$, and let

$$\delta p([x],\tilde{x}) = \Big(\sum_{i=1}^{n} c_{i-1}[x]^{i-1}\Big)_{H,\underline{\times}}$$

be the interval arithmetic evaluation of the slope of p . The index "H" means that the Horner-scheme has to be used to compute the sum and "$\underline{\times}$" indicates that all multiplications have to be performed by the new definition. Obviously

$$m(\delta p([x],\tilde{x})) = \sum_{i=1}^{n} c_{i-1}(m[x])^{i-1} \ .$$

Furthermore it is easy to show that for $\tilde{x} \in [x]$ it holds that $\delta p([x],\tilde{x}) \subseteq p'([x])$ (see [1]).

<u>Theorem 7</u>. <u>Let the real polynomial</u> $p(x)$ <u>be given and assume that for some</u> $\tilde{x} \in [x]^o$ <u>and a real number</u> $r \neq 0$ <u>we have</u>

$$\tilde{x} - r \cdot p(\tilde{x}) + (1 - r \cdot p'([x]^O)) \underline{\times} ([x]^O - \tilde{x}) \subseteq [x]^O .$$

<u>Then</u> p <u>has a zero in</u> $[x]^O$.

<u>Proof</u>. Since $\delta p([x]^O,\tilde{x}) \subseteq p'([x]^O)$ we also have

$$\tilde{x} - r \cdot p(\tilde{x}) + (1-r\cdot\delta p([x]^O,\tilde{x}) \underline{\times} ([x]^O-\tilde{x}) \subseteq [x]^O .$$

Define the real function $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \tilde{x} - r \cdot p(\tilde{x}) + (1 - r \cdot \delta p(x,\tilde{x}))(x-\tilde{x})$$

and its interval arithmetic evaluation as

$$f([x]) = \tilde{x} - r \cdot p(\tilde{x}) + (1 - r \cdot \delta p([x],\tilde{x})) \underline{\times} ([x]-\tilde{x}) .$$

Then all assumptions of Theorem 5 hold which means that f has a fixed point x^* in $[x]^O$:

$$\begin{aligned} x^* &= \tilde{x} - r \cdot p(\tilde{x}) + (1- r\cdot\delta p(x^*,\tilde{x})(x^*-\tilde{x}) \\ &= x^* - r \cdot p(x^*) \end{aligned}$$

where we have used the fact that for the slope $\delta p(x^*,\tilde{x})$ the equation $p(x^*)-p(\tilde{x}) = \delta p(x^*,\tilde{x})(x^*-\tilde{x})$ holds. Since $r \neq 0$ we have $p(x^*) = 0$. □

The proof of Theorem 7 can be generalized to functions different from polynomials without any complications. In order to do this one has to explain how to compute $\delta f([x],\tilde{x})$ if f is not a polynomial. This was done by R. Krawczyk and A. Neumaier in [9]. Furthermore Theorem 7 can be generalized to mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$. □

<u>Example 3</u> (<u>Alefeld</u> [3]). Consider the eigenvalue problem for the real matrix A . Assume that (λ,x) is an approximation to an eigenpair of A . In order to find bounds for λ and x it is sufficient to find bounds for μ and $\tilde{y}$ for which

$$A(x+\tilde{y}) = (\lambda+\mu)(x+\tilde{y}) \ . \tag{*}$$

Since $x+\tilde{y}$ is not unique we set $\tilde{y}_s = 0$ where s is defined by the equation $\|x\|_\infty = |x_s|$. Let the vector $y = (y_i)$ be defined by

$$y_i = \begin{cases} \tilde{y}_i \ , & i \neq s \\ \mu \ , & i = s \end{cases} \ .$$

Furthermore set

$$r = \lambda x - Ax$$

and

$$B = ((A-\lambda I)_1,\ldots,(A-\lambda I)_{s-1},-x,(A-\lambda I)_{s+1},\ldots,(A-\lambda I)_n) \ .$$

Then equation (*) can be rewritten as

$$By = r + y_s\tilde{y}$$

or as

$$y = Lr + (I-LB)y + L(y_s\tilde{y})$$

where L is some approximation of the inverse of B . It has been shown in [3] that in dependence of r , L and B one can find an interval vector $[y]^o = -[y]^o$ such that

$$Lr + (I-LB)[y]^o + L([y]^o_s[\tilde{y}]^o) \subseteq [y]^o \ .$$

By Theorem 6, (4) we have $[y]^o_s[\tilde{y}]^o = [y]^o_s \underline{\times} [\tilde{y}]^o$ and therefore

$Lr + (I-LB)[y]^o + L([y]^o_s \underline{\times} [\tilde{y}]^o) \subseteq [y]^o$.

Now define

$f(y) = Lr + (I-LB)y + L(y_s\tilde{y})$

and

$f([y]) = Lr + (I-LB)[y] + L([y]_s \underline{\times} [\tilde{y}])$.

Then all assumptions of Theorem 5 hold. Hence f has a fixed point y^* in $[y]^o$ which is a solution of the equation $By = r + y_s\tilde{y}$.

5. Iteration methods. In the preceding chapters we have already repeatedly used iteration methods $[x]^{k+1} = f([x]^k)$, $k = 0,1,\ldots$. (See the proof of Theorem 5, for example.) We cannot give a survey of all iteration methods which are based on interval arithmetic tools. Instead we refer to [5] and to the other contributions of this volume. We concentrate our discussion on the Interval-Newton-Method for a single equation (see [5], Chapter 7): Let $f : D \subseteq \mathbb{R}^1 \to \mathbb{R}^1$ have an interval arithmetic evaluation of the derivative for all $[x] \subseteq [x]^o$ where $[x]^o$ contains a zero x^* of f . Then

$$[x]^{k+1} = N[x]^k \cap [x]^k , \quad k = 0,1,2,\ldots,$$

where

$$N[x] = m[x] - \frac{f(m[x])}{f'([x])}$$

is called the Interval-Newton-Method. If $0 \notin f'([x]^o)$ then the sequence $\{[x]^k\}$ is well defined and $\lim_{k\to\infty} [x]^k = x^*$.

Furthermore if $d(f'([x])) \leq \gamma\, d[x]$, $\gamma \geq 0$, $[x] \subseteq [x]^o$, then $d([x]^{k+1}) \leq c(d[x]^k)^2$ which means that the diameters are converging quadratically to zero.

It is well-known that the classical Newton-Method is cubically convergent if besides $f(x^*) = 0$, $f'(x^*) \neq 0$, the equation $f''(x^*) = 0$ holds. In [4] we have demonstrated by a simple example that this is not true for the Interval-Newton-Method. However the following theorem shows that we can get cubic convergence if we replace $f'([x])$ by the centered form or by the mean value form of the derivative, respectively.

Theorem 8. Let $f : D \subseteq \mathbb{R}^1 \to \mathbb{R}^1$ have a continuous second derivative in D and suppose $0 \notin f'([x]^o)$, $[x]^o \subseteq D$. Define

$$N_M[x] = m[x] - \frac{f(m[x])}{f'_M([x]) \cap f'([x]^o)}$$

where $f'_M([x])$ denotes the mean value form of the derivative (see [5]) and

$$[x]^{k+1} = N_M[x]^k \cap [x]^k .$$

If $f(x^*) = f''(x^*) = 0$ for some $x^* \in [x]^o$ then

$$d[x]^{k+1} \leq \gamma (d[x]^k)^3$$

provided

$$d(f''([x])) \le \alpha\, d[x] \ , \ \alpha > 0 \ , \ [x] \subseteq [x]^{\circ} \ .$$

The same result holds if the mean value form

$f'_M([x])$ of f' is replaced by the centered

form of f'. □

Details of the proof can be found in [4]. In concluding we remark that the content of this theorem can be generalized to systems of equations.

References

[1] G. Alefeld, Bounding the slope of polynomial operators and some applications. Computing 26, 227-237 (1981)

[2] G. Alefeld, Componentwise Inclusion and Exclusion Sets for Solutions of Quadratic Equations in Finite Dimensional Spaces. Numer. Math. 48, 391-416 (1985)

[3] G. Alefeld, Berechenbare Fehlerschranken für ein Eigenpaar unter Einschluß von Rundungsfehlern bei Verwendung des genauen Skalarprodukts. Z. angew. Math. Mech. 67, 145-152 (1987)

[4] G. Alefeld, Über die Konvergenzordnung des Intervall-Newton-Verfahrens. Submitted for publication.

[5] G. Alefeld, J. Herzberger, Introduction to Interval Computations, Academic Press, New York and London (1983)

[6] A. Frommer, W. Straub, Fehlerschranken für Polynomwurzelapproximationen. Under preparation.

[7] E. Hansen (ed.), Topics in Interval Analysis. Proceedings of a symposium held in Culham, Great Britain in January 1968, Oxford University Press, London (1969)

[8] E. Hansen, On linear algebraic equations with interval coefficients. In [7], p. 35.

[9] R. Krawczyk, A. Neumaier, Interval slopes for rational functions and associated centered forms. SIAM J. Numer. Anal. 22, 604-616 (1985)

[10] R.E. Moore, Interval Analysis. Prentice Hall, Englewood Cliffs, N.J. (1966)

[11] R.E. Moore, Introduction to algebraic problems. In [7], p.3.

[12] R.E. Moore, A test for existence of solutions to nonlinear systems. SIAM J. Numer. Anal. 14, 611-615 (1977)

[13] R.E. Moore, Methods and Applications of Interval Analysis. SIAM Studies, Philadelphia (1979)

[14] A. Neumaier, Interval iteration for zeroes of systems of equations. BIT 25, 256-273 (1985)

[15] J.M. Ortega, W.C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several variables. Academic Press, New York and London (1970)

[16] M. Rostock, Über die Anwendung des Brouwerschen Fixpunktsatzes in der Intervallrechnung. Diplomarbeit, Universität Karlsruhe, 1987 (Not available)

[17] J.W. Schmidt, Eine Anwendung des Brouwerschen Fixpunktsatzes zur Gewinnung von Fehlerschranken für Näherungen von Polynomnullstellen. Beiträge zur Num. Math. 6, 158-163 (1977)

[18] H. Stetter, Numerische Mathematik I, Manuscript at the Technical University Vienna, 1968

[19] W. Straub, Verbesserte Einschließung von Polynomnullstellen mit Hilfe der Intervallrechnung. Diplomarbeit, Universität Karlsruhe, 1987 (Not available)

[20] H.J. Symm, J.H. Wilkinson, Realistic error bounds for a simple eigenvalue and its associated eigenvector. Numer. Math. 35, 113-126 (1980)