Computing 33, 165–169 (1984)

Computing

Short Communications / Kurze Mitteilungen

The Centered Form and the Mean Value Form — A Necessary Condition That They Yield the Range

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Received March 15, 1982

Abstract — Zusammenfassung

The Centered Form and the Mean Value Form — A Necessary Condition That They Yield the Range. It is shown that if the interval arithmetic evaluation of either the centered form or the mean value form is equal to the range of values then the function attains its absolute extreme values only at the endpoints of the interval.

Die zentrische Form und die Mittelwertform — eine notwendige Bedingung für die Bestimmtheit des Wertebereichs. Gezeigt wird: Ist die intervallarithmetische Auswertung der zentrischen oder der Mittelwertform gleich dem Wertebereich, dann nimmt die Funktion ihre absoluten Extremwerte nur an den Endpunkten des Intervalls an.

1. Introduction

As is well known the range of values $W(f, X)$ of a real mapping f defined over the compact interval $X=[x_1, x_2]$ can be enclosed using interval arithmetic. The goodness of the computed interval is strongly dependent on the representation of the function f. See, for example, the discussion in [1, section 3], [2], [3, chapter 6], [4, chapter 3], and [5].

Assume, for example, that f is represented by the so-called centered form

$$f(x)=f(c)+(x-c)\,h(x-c)$$

where $c\in X$ is fixed (see [5]). If $h(x-c)$ is bounded for $x\in X$, that is

$$h(x-c)\in h(X-c) \tag{1}$$

where $h(X-c)$ is a real compact interval then for the compact interval

$$f(X):=f(c)+(X-c)\,h(X-c) \tag{2}$$

it holds that

$$f(x)\in f(X) \quad \text{for } x\in X. \tag{3}$$

The interval $f(X)$ defined by (2) is called the interval arithmetic evaluation of the centered form.

If

$$q(A,B)=\max\{|a_1-b_1|,|a_2-b_2|\}$$

denotes the Hausdorff distance of the intervals $A=[a_1,a_2]$ and $B=[b_1,b_2]$ then the interval $f(X)$ defined by (2) has the so-called quadratic approximation property

$$q\big(f(X),W(f,X)\big)\leq \alpha\, d^2(X)$$

where $d(X)=x_2-x_1$ is the width of the interval $X=[x_1,x_2]$. (See [1], p. 26, Theorem 4, for the exact formulation of the conditions under which this relation holds.)

The same is true for the so-called mean value form which is defined in the following manner (see, for example [3, p. 47 ff.] and [1, p. 28, Theorem 6]):

Let the function f have a bounded derivative over the interval $X=[x_1,x_2]$, that is it holds that

$$f'(x)\in f'(X),\quad x\in X, \tag{1'}$$

where $f'(X)$ is a compact interval. Let $c\in X$ be fixed and define the compact interval $f(X)$ by

$$f(X):=f(c)+(X-c)f'(X). \tag{2'}$$

Then it holds that

$$f(x)\in f(X)\qquad \text{for } x\in X. \tag{3'}$$

Usually one chooses $c=m(X)$ in (2) and (2′) where $m(X)=(x_1+x_2)/2$. Subsequently we always assume that c is equal to this value.

Occasionally one ovserves that besides of the quadratic approximation property the equation $f(X)=W(f,X)$ holds for (2) or (2′). We demonstrate this using a simple

Example (see [7], p. 29 ff.):

Let

$$f(x)=x^3-3x^2+3x,\quad X=[0.9,\ 1.1].$$

Then

$$W(f,X)=[0.999,\ 1.001].$$

Using $c=1$ we get for the centered form (1)

$$f(x)=1+(x-1)(x-1)^2.$$

Its interval arithmetic evaluation by (2) yields $f(X)=W(f,X)$.

(This result is independent of using either

$$h(X-c)=(X-1)^2:=[-0.1,0.1]\cdot[-0.1,0.1]=[-0.01,0.01]$$

or

$$h(X-c):=\{(x-1)^2\,|\,x\in[0.9,1.1]\}=[0,0.01].)\quad \blacksquare$$

2. The Necessary Condition

The objective of this note is to prove that for either (2) or (2′) the equation $f(X)=W(f,X)$ can only hold if the function f attains its absolute minimum and absolute maximum only at the boundary points of the interval $X=[x_1,x_2]$. In other words: If in the interval $X=[x_1,x_2]$ the function f attains its absolute minimum and/or absolute maximum (also) at an interior point then (2) and (2′) cannot yield the range of f over X. Notice that this condition is only a necessary one: From the assumption that f takes on its absolute extreme values only at boundary points it does not follow that either (2) or (2′) yields the range. This can be demonstrated by simple counterexamples.

In order to prove the above mentioned assertion we only need the following relations for the width $d(A)$, distance $q(A,B)$ and absolute value $|A|=q(A,0)$ of intervals A and B:

$$A\subseteq B \Rightarrow d(B)-d(A)\geq q(A,B). \tag{4}$$

(see [1, p. 17, (21)])

$$d(a+B)=d(B),\ a\in\mathbb{R}. \tag{5}$$

$$d(AB)\geq d(A)\,|B|. \tag{6}$$

(see [1, p. 15, (13)])

$$a\in A\Rightarrow |a|\leq |A|. \tag{7}$$

Theorem: *Let the nonconstant function $f(x)$ be defined on the interval $X=[x_1,x_2]$ and assume that it has a bounded derivative on X (which implies that (1′) holds). If for the range of f over X, $W(f,X)$, the equation*

$$W(f,X)=f(X) \tag{8}$$

holds, where $f(X)$ is defined by (2′), then f attains its absolute extreme values only at the boundary points of the interval X. The same is true for the centered form (2).

Proof: Using (5) and (6) we get from the definition of $f(X)$ by (2′) the inequality

$$d\big(f(X)\big)=d\big((X-c)f'(X)\big)\geq d(X)\,|f'(X)|. \tag{9}$$

Let $W(f,X)=[f(u),f(v)]$, $u,v\in X$.

Then applying the mean value theorem and using (1′) and (7) it follows that

$$d\big(W(f,X)\big)=f(v)-f(u)=f'(\xi)(v-u)\leq |f'(X)|\,|v-u|. \tag{10}$$

If we define

$$U=[\min(u,v),\ \max(u,v)]$$

then we have $U\subseteq X$. Using (8), (9) and (10) and applying (4) it follows that

$$\begin{aligned} 0&=d\big(f(X)\big)-d\big(W(f,X)\big)\\ &\geq |f'(X)|\,\{d(X)-|v-u|\}\\ &=|f'(X)|\,\{d(X)-d(U)\}\\ &\geq |f'(X)|\,q(X,U)\geq 0, \end{aligned}$$

and therefore that

$$|f'(X)|\,q(X,U)=0.$$

Since $f(x)$ is nonconstant we have $|f'(X)|>0$ and therefore $q(X,U)=0$, that is $X=U$. From this it follows that the absolute extreme values are only attained at the endpoints of the interval. This concludes the proof for the mean value form. Analogously to (9) one gets

$$d(f(X)) \geq d(X)\,|h(X-c)| \tag{9'}$$

for (2).

The corresponding relation to (10) is

$$\begin{aligned} d(W(f,X)) &= f(v)-f(u) \\ &= (v-c)\,h(v-c)-(u-c)\,h(u-c) \\ &\leq (|v-c|+|u-c|)\,|h(X-c)| \end{aligned} \tag{10'}$$

and therefore it follows that

$$\begin{aligned} 0 &= d(f(X))-d(W(f,X)) \\ &\geq \{d(X)-(|v-c|+|u-c|)\}\cdot|h(X-c)|. \end{aligned}$$

Because of $c=m(X)$ and since $u,v\in X$ it follows that the right-hand side is nonnegative. Since $|h(X-c)|>0$ we have

$$d(X)=|v-c|+|u-c|$$

from which it follows that u and v are boundary points of the interval X. This concludes the proof for the centered form and therefore of the theorem. ■

In concluding we remark that in the example discussed above f actually attains its absolute extreme values only at the endpoints of the interval.

Added in proof:

As H. Ratschek [6] has pointed out the different proofs of the preceding Theorem for the mean value form and for the centered form, respectively, can be combined into one single proof. The reason for this is that *theoretically* the mean value form (2') may be considered to be a special case of the interval arithmetic evaluation (2) of the centered form. In order to see this one defines

$$h(x-c)=\begin{cases} \dfrac{f(x)-f(c)}{x-c}, & x\neq c \\ f'(x), & x=c. \end{cases} \tag{*}$$

By the mean value theorem one then has $h(x-c)\in h(X-c)$ by setting $h(X-c)=f'(X)$. This means that (1) holds where $h(X-c)$ is a real compact interval. Hence using (*) we have represented $f(x)$ by the centered form

$$f(x)=f(c)+(x-c)\cdot h(x-c)$$

and it holds that

$$f(x)\in f(X):=f(c)+(X-c)\,f'(X).$$

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Printed in Austria
Druck: Paul Gerin, A-1021 Wien