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ON THE CONVERGENCE OF SOME INTERVAL-ARITHMETIC MODIFICATIONS OF NEWTON'S METHOD*

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Abstract. In this paper we consider the meanwhile well-known interval-arithmetic modifications of the Newton's method and of the so-called simplified Newton's method for solving systems of nonlinear equations. Starting with an interval-vector which contains a zero, we give for the first time sufficient conditions for the convergence of these methods to this solution. If the starting interval-vector contains no solution, then under the very same conditions the methods under consideration will break down after a finite number of steps. The interval-arithmetic evaluation of the first derivative is only involved in these conditions.

1. Introduction. In order to motivate the results of the later sections, we first consider a single equation with one unknown. If f has a zero x^* which is contained in the interval $X^0 \subseteq X$, where $f: X \subset \mathbb{R} \to \mathbb{R}$, and if the interval-arithmetic evaluation $f'(X^k)$ of the derivative exists, then, if $0 \notin f'(X^k)$, the method

$$
\begin{aligned}
&m(X^k) \in X^k, \qquad (m(X^k) \in \mathbb{R}),\\
&X^{k+1} = \left\{ m(X^k) - \frac{f(m(X^k))}{f'(X^k)} \right\} \cap X^k
\end{aligned}
\tag{1}
$$

is well defined. Furthermore $x^* \in X^k$ and $\lim_{k\to\infty} X^k = x^*$ hold. The order of convergence is (under some additional conditions) at least two. (There are, meanwhile, several people who are claiming authorship for this nice result. Therefore we omit to include a list of papers where the method (1) was discussed. Instead we refer to [2, § 7], where this method was discussed, surely not for the first time.)

The paper under consideration discusses the question, how far the results about (1) can be proven for the corresponding generalization of (1) to systems of nonlinear equations. This question has so far not been answered in a satisfactory manner.

At first we discuss some tools from interval-analysis. Among these is an explicit representation of the result of the Gaussian algorithm if it is applied to systems with intervals as coefficients and in the right-hand side. This representation was first given by H. Schwandt in [4]. A simple conclusion (Lemma 2), which we prove by applying this representation is the important new tool of this paper.

Because of its fundamental importance, we then introduce an example which shows that the results on (1) obviously do not all hold for nonlinear systems. In part a) of the following theorem we prove that under certain conditions on the starting interval-vector the modifications of Newton's method are convergent. These conditions are dependent only on the interval-arithmetic evaluation of the first derivative. In part b) of this theorem we prove that under certain conditions the methods are breaking down if the starting interval does not contain a zero. The idea of the proof for this result was first used by H. Cornelius in [3].

2. Preliminaries. Concerning the notation and basic facts of interval-analysis, we refer to [2]. Therefore we only list the most important concepts. The set of compact intervals becomes a complete metric space if we introduce the *metric* (distance)

$$q(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\}.$$

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The *width* or *diameter* of an interval $A=[a_1, a_2]$ is defined by

$$d(A)=a_2-a_1.$$

The *absolute value* of the interval A is defined to be the distance

$$|A|:=q(A,0).$$

For interval-matrices we define these concepts via the elements. If $\mathscr{A}=(A_{ij})$ is an interval-matrix, then $d(\mathscr{A})=(d(A_{ij}))$, for example.

Among others we use the following equations:

α) For interval-matrices $\mathscr{A}$ and $\mathscr{B}$ and a point-matrix $\underset{\cdot}{\mathscr{C}}$, it holds that

$$(\mathscr{A}+\mathscr{B})\underset{\cdot}{\mathscr{C}}=\mathscr{A}\underset{\cdot}{\mathscr{C}}+\mathscr{B}\underset{\cdot}{\mathscr{C}}.$$

(See [2, § 10, formula (7)].)

β) For interval-matrices $\mathscr{A}$ and $\mathscr{B}$ it holds that

$$d(\mathscr{A}\pm\mathscr{B})=d(\mathscr{A})+d(\mathscr{B}).$$

(See [2, § 10, formula (12)].)

γ) For an interval-matrix $\mathscr{A}$ and a point-vector $\underset{\cdot}{\mathscr{b}}$ it holds that

$$d(\mathscr{A}\underset{\cdot}{\mathscr{b}})=d(\mathscr{A})|\underset{\cdot}{\mathscr{b}}|.$$

(See [2, § 10, formula (19)].)

δ) For an interval-matrix $\mathscr{A}$ and a point-vector $\underset{\cdot}{\mathscr{b}}$ it holds that

$$\mathscr{A}\underset{\cdot}{\mathscr{b}}=\{\underset{\cdot}{\mathscr{A}}\underset{\cdot}{\mathscr{b}}\,|\,\underset{\cdot}{\mathscr{A}}\in\mathscr{A}\}.$$

(See [2, § 10, formula (1)].)

In general we have

$$\mathscr{A}(\mathscr{B}\underset{\cdot}{c})\subseteq(\mathscr{A}\mathscr{B})\underset{\cdot}{c} \tag{2}$$

for interval-matrices $\mathscr{A}$, $\mathscr{B}$ and a point-vector $\underset{\cdot}{c}$.

This was proven in [4, p. 15]. If $\underset{\cdot}{c}$ is equal to one of the unit-vectors then the equality-sign holds in (2).

This is the content of the next lemma.

LEMMA 1. *If $\underset{\cdot}{c}=\underset{\cdot}{e}_i$ (=ith unit-vector) then the equality-sign holds in* (2).

Proof. Denote the columns of the matrix $\mathscr{B}$ by $\mathscr{b}_i$, $1\leqq i\leqq n$. Then it holds that

$$\mathscr{B}\underset{\cdot}{e}_i=(\mathscr{b}_1,\cdots,\mathscr{b}_n)\underset{\cdot}{e}_i=\mathscr{b}_i$$

and therefore

$$\mathscr{A}(\mathscr{B}\underset{\cdot}{e}_i)=\mathscr{A}\mathscr{b}_i.$$

On the other hand we have

$$(\mathscr{A}\mathscr{B})\underset{\cdot}{e}_i=(\mathscr{A}\mathscr{b}_1,\cdots\mathscr{A}\mathscr{b}_n)\underset{\cdot}{e}_i=\mathscr{A}\mathscr{b}_i$$

and therefore the assertion follows. □

Assume now that we have given an n by n interval-matrix $\mathscr{A}=(A_{ij})$ and an interval-vector $\mathscr{b}=(B_i)$ with n components. By applying the formulas of the Gaussian algorithm we compute an interval-vector $x=(X_i)$ for which the relation

$$\{\underset{\cdot}{x}=\underset{\cdot}{\mathscr{A}}^{-1}\underset{\cdot}{\mathscr{b}}\,|\,\underset{\cdot}{\mathscr{A}}\in\mathscr{A},\underset{\cdot}{\mathscr{b}}\in\mathscr{b}\}\subseteq x$$

holds. See [2, § 15] or [4, pp. 20ff], for example. If we set $A_{ij}^{(1)}:=A_{ij}$, $1\leqq i,j\leqq n$, and $B_i^{(1)}:=B_i$, $1\leqq i\leqq n$, then the formulas are as follows (see [4, p. 23], for example). We explicitly list these formulas since we have to make explicit use of them:

for $k = 1(1)n-1$ **do**
begin
  for $i = k+1(1)n$ **do**
  begin
    for $j = k+1(1)n$ **do**

$$A_{ij}^{(k+1)} := A_{ij}^{(k)} - A_{kj}^{(k)} \frac{A_{ik}^{(k)}}{A_{kk}^{(k)}}$$

$$B_i^{(k+1)} := B_i^{(k)} - B_k^{(k)} \frac{A_{ik}^{(k)}}{A_{kk}^{(k)}}$$

  end;
  for $l = 1(1)k$ **do**
  begin
    for $j = l(1)n$ **do**
      $A_{lj}^{(k+1)} := A_{lj}^{(k)}$
      $B_l^{(k+1)} := B_l^{(k)}$
  end
end;
$X_n = B_n^{(n)} / A_{nn}^{(n)}$
for $i = n-1(-1)1$ **do**

$$X_i = \left(B_i^{(n)} - \sum_{j=i+1}^{n} A_{ij}^{(n)} X_i \right) \Big/ A_{ii}^{(n)}.$$

We have assumed that no division by an interval which contains zero occurs. In this case we say that the feasibility of the Gaussian algorithm is guaranteed. The feasibility is not dependent on the right-hand side vector ℓ.

In the formulas above we have not taken into account exchanges of rows or column which are eventually necessary in order to prevent division by an interval which contains zero. If one programs the above formulas, then the upper index can be suppressed.

If we define the interval-matrices

$$\mathscr{G}_k := \begin{bmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & 0 & & -\dfrac{A_{k+1,k}^{(k)}}{A_{kk}^{(k)}} & & \\ & & & \vdots & \ddots & \\ & & & -\dfrac{A_{nk}^{(k)}}{A_{kk}^{(k)}} & 0 & 1 \end{bmatrix}, \qquad 1 \leqq k \leqq n-1,$$

$$\mathscr{D}_k := \begin{bmatrix} 1 & & & & \\ & \ddots & & 0 & \\ & & \dfrac{1}{A_{kk}^{(n)}} & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix}, \qquad 1 \leqq k \leqq n,$$

$$\mathscr{T}_k := \begin{bmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & -A_{k,k+1}^{(n)} & \cdots & -A_{kn}^{(n)} \\ & & & 1 & & \\ & & & & \ddots & 0 \\ & 0 & & & & \\ & & & & & 1 \end{bmatrix}, \qquad 1 \leqq k \leqq n-1,$$

then we have for the interval-vector calculated above, using

$$\tilde{\mathscr{b}} := \mathscr{G}_{n-1}(\mathscr{G}_{n-2}(\cdots(\mathscr{G}_2(\mathscr{G}_1\mathscr{b})\cdots),$$

the representation

$$\mathscr{x} = \mathscr{D}_1(\mathscr{T}_1(\mathscr{D}_2(\mathscr{T}_2(\cdots(\mathscr{D}_{n-1}(\mathscr{T}_{n-1}(\mathscr{D}_n\tilde{\mathscr{b}})\cdots),$$

which was first given by H. Schwandt in [4, pp. 24ff]. As in [4] we denote this interval-vector by IGA $(\mathscr{A}, \mathscr{b})$, that is we have the representation

$$\text{IGA}\,(\mathscr{A}, \mathscr{b}) = \mathscr{D}_1(\mathscr{T}_1(\cdots \mathscr{T}_{n-1}(\mathscr{D}_n(\mathscr{G}_{n-1}(\cdots(\mathscr{G}_2(\mathscr{G}_1\mathscr{b})\cdots). \tag{3}$$

Notice that it is not possible to omit the parentheses in general.

In order to formulate the next result, we *define* an interval-matrix IGA $(\mathscr{A})$ by using the interval-matrices occurring on the right-hand side of (3):

$$\text{IGA}\,(\mathscr{A}) := \mathscr{D}_1(\mathscr{T}_1(\cdots \mathscr{D}_n(\mathscr{G}_{n-1}(\cdots(\mathscr{G}_2\mathscr{G}_1)\cdots). \tag{4}$$

Then the following holds.

LEMMA 2. *For $1 \leqq i \leqq n$ it holds that*

$$\text{IGA}\,(\mathscr{A}, e_i) = \text{IGA}\,(\mathscr{A}) \cdot e_i$$

where e_i denotes the ith unit-vector.

Proof. Starting with the representation (3) of IGA $(\mathscr{A}, e_i)$ we get the assertion by applying repeatedly Lemma 1. □

The last lemma states that the ith column of the matrix IGA $(\mathscr{A})$ is equal to the interval-vector which one obtains if one applies the Gaussian algorithm to the interval-matrix $\mathscr{A}$ and the right-hand side e_i. In order to compute IGA $(\mathscr{A})$ it is therefore not necessary to know the matrices appearing on the right-hand side of (4) explicitly. IGA $(\mathscr{A})$ can be computed by "formally inverting" the interval-matrix $\mathscr{A}$ by applying the Gaussian algorithm. Finally we need the following result.

LEMMA 3. *For an interval-matrix $\mathscr{A}$ and a point-vector b it always holds that*

$$\text{IGA}\,(\mathscr{A}, b) \subseteq \text{IGA}\,(\mathscr{A}) \cdot b.$$

The proof can be found in [4, p. 33]. It is performed by applying (2) in the representation (3) of IGA $(\mathcal{A}, \mathit{b})$.

Assume now that there is given a mapping $f: x^0 \subseteq \vartheta \subseteq V_n(\mathbb{R}) \to V_n(\mathbb{R})$. We consider the following methods for computing a zero x^* of f in x^0:

(Simplified Newton's method)

$$
\begin{aligned}
& m(x^k) \in x^k, \qquad (m(x^k) \in V_n(\mathbb{R})), \\
\text{(SN)}\quad & \mathrm{SN}\,(x^k) = m(x^k) - \mathrm{IGA}\,(f'(x^0), f(m(x^k))), \\
& x^{k+1} = \mathrm{SN}\,(x^k) \cap x^k;
\end{aligned}
$$

(Newton's method)

$$
\begin{aligned}
& m(x^k) \in x^k, \qquad (m(x^k) \in V_n(\mathbb{R})), \\
\text{(N)}\quad & \mathrm{N}\,(x^k) = m(x^k) - \mathrm{IGA}\,(f'(x^k), f(m(x^k))), \\
& x^{k+1} = \mathrm{N}\,(x^k) \cap x^k.
\end{aligned}
$$

Usually one chooses $m(x^k) \in x^k$ to be the center of the interval-vector x^k if there is no specific information about the location of x^* in x^k. We do not assume this choice, however.

Both the methods (SN) and (N) compute sequences of interval-vectors enclosing the zero x^*. (SN) uses the fixed matrix $f'(x^0)$ whereas $f'(x^k)$ has to be computed in each step of (N). As a consequence of this (N) is at least quadratically convergent whereas (SN) normally converges only linearly. (N) may be considered to be the immediate generalization of (1) to systems and is identical for $n = 1$ to (1).

Without going into the details of a proof, we mention the following existence statement (see [1, p. 70, Satz 3.4]).

LEMMA 4. *If*

$$\mathrm{SN}\,(x^0) = \mathrm{N}\,(x^0) \subseteq x^0$$

then f has a solution in x^0 and no solution in $x^0 \setminus \mathrm{SN}\,(x^0)$. If $x^0 \cap \mathrm{SN}\,(x^0) = \emptyset$ then f has no solution in x^0. □

Both the methods (SN) and (N) have been considered repeatedly. See [5], [6]. Nevertheless results do not exist which ensure the convergence of these methods. The following simple example shows that the convergence can not be assured under the analogous weak conditions as it was the case for (1). This example was discussed by H. Schwandt in [4, pp. 85ff]. We repeat the discussion because of its fundamental importance.

Example. Let $x = \binom{x}{y}$ and

$$f(x) = \begin{pmatrix} -x^2 + y^2 - 1 \\ x^2 - y \end{pmatrix}.$$

The vector

$$x^* = \begin{pmatrix} \sqrt{\dfrac{1+\sqrt{5}}{2}} \\ \dfrac{1+\sqrt{5}}{2} \end{pmatrix}$$

is the unique solution of the system $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{o}$ in the interval-vector

$$\underset{\sim}{x}^0 = \frac{1}{10}\begin{pmatrix}[11, 19]\\ [11, 19]\end{pmatrix}.$$

Choosing $m(\underset{\sim}{x}^0)$ as the center of $\underset{\sim}{x}^0$ then one gets

$$\mathrm{SN}\,(\underset{\sim}{x}^0) = \mathrm{N}\,(\underset{\sim}{x}^0) = \begin{pmatrix} \left[-\frac{3}{88}, \frac{90771}{12584}\right] \\ \left[\frac{7}{8}, \frac{5801}{1144}\right] \end{pmatrix} \supset \underset{\sim}{x}^0$$

and therefore $\underset{\sim}{x}^1 = \underset{\sim}{x}^0$. Therefore, if we choose $m(\underset{\sim}{x}^k)$ to be the center of $\underset{\sim}{x}^k$, we have

$$\underset{\sim}{x}^0 = \underset{\sim}{x}^1 = \cdots = \underset{\sim}{x}^k = \cdots \quad \text{and} \quad \lim_{k\to\infty} \underset{\sim}{x}^k = \underset{\sim}{x}^0 \neq \underset{\sim}{x}^*. \qquad \square$$

3. Convergence statements. In the theorem following, we state and prove some convergence results concerning the methods (SN) and (N).

THEOREM. *Let there be given an interval-vector $\underset{\sim}{x}^0 \subseteq \vartheta \subseteq V_n(\mathbb{R})$ and a mapping $\underset{\sim}{f}: \underset{\sim}{x}^0 \subseteq \vartheta \subseteq V_n(\mathbb{R}) \to V_n(\mathbb{R})$, whose Fréchet-derivative has an interval-arithmetic evaluation. (See* [2, § 3], *for example.)*

We assume that IGA $(\underset{\sim}{f}'(\underset{\sim}{x}^0))$ *exists.*

(a) *Suppose that $\rho(\underset{\sim}{\mathcal{A}}) < 1$ (ρ = spectral radius) where*

$$\underset{\sim}{\mathcal{A}} = |\underset{\sim}{\mathcal{E}} - \mathrm{IGA}\,(\underset{\sim}{f}'(\underset{\sim}{x}^0)) \cdot \underset{\sim}{f}'(\underset{\sim}{x}^0)| \tag{5}$$

or that $\rho(\underset{\sim}{\mathcal{B}}) < 1$ where

$$\underset{\sim}{\mathcal{B}} = d(\mathrm{IGA}\,(\underset{\sim}{f}'(\underset{\sim}{x}^0))) \cdot |\underset{\sim}{f}'(\underset{\sim}{x}^0)|. \tag{6}$$

If $\underset{\sim}{f}$ has a zero $\underset{\sim}{x}^*$ in $\underset{\sim}{x}^0$ then the sequences computed by (SN) *or* (N) *are well defined and it holds that $\lim_{k\to\infty} \underset{\sim}{x}^k = \underset{\sim}{x}^*$. If one chooses $m(\underset{\sim}{x}^k)$ to be the center of $\underset{\sim}{x}^k$ then the condition $\rho(\underset{\sim}{\mathcal{B}}) < 1$ can be replaced by $\rho(\underset{\sim}{\mathcal{B}}) < 2$.*

(b) *If $\rho(\underset{\sim}{\mathcal{A}}) < 1$ where $\underset{\sim}{\mathcal{A}}$ is defined by* (5) *and if $\underset{\sim}{f}$ has no zero in $\underset{\sim}{x}^0$, then there is a $k_0 \geqq 0$ depending on the method such that both* (SN) *and* (N) *are well-defined for $0 \leqq k \leqq k_0$. It holds, however, that*

$$\mathrm{SN}\,(\underset{\sim}{x}^{k_0}) \cap \underset{\sim}{x}^{k_0} = \varnothing \quad \text{and} \quad N(\underset{\sim}{x}^{k_0}) \cap \underset{\sim}{x}^{k_0} = \varnothing,$$

that is both methods break down after a finite number of steps because of empty intersections.

Proof. At first we recall the following fact (see [2, § 19], for example): If $\underset{\sim}{f}$ has a zero $\underset{\sim}{x}^*$ in $\underset{\sim}{x}^0$ and if IGA $(\underset{\sim}{f}'(\underset{\sim}{x}^0), \underset{\sim}{f}(m(\underset{\sim}{x}^0)))$ exists, then $\underset{\sim}{x}^* \in N(\underset{\sim}{x}^0)$ and therefore $\underset{\sim}{x}^* \in N(\underset{\sim}{x}^0) \cap \underset{\sim}{x}^0 = \underset{\sim}{x}^1$, that is the intersection is not empty. If now $\underset{\sim}{x}^* \in \underset{\sim}{x}^k$ for some $k \geqq 0$ and *if* the vector IGA $(\underset{\sim}{f}'(\underset{\sim}{x}^k), \underset{\sim}{f}(m(\underset{\sim}{x}^k)))$ *exists*, then using induction one shows $\underset{\sim}{x}^* \in N(\underset{\sim}{x}^k) \cap \underset{\sim}{x}^k$, from which it follows that the sequence $\{\underset{\sim}{x}^k\}$ computed by using (N) is well-defined. The existence of IGA $(\underset{\sim}{f}'(\underset{\sim}{x}^k), \underset{\sim}{f}(m(\underset{\sim}{x}^k)))$ can be seen in the following manner: Assume that $\underset{\sim}{x}^k$ exists for some $k > 0$. Then because of forming intersections in (N) we have $\underset{\sim}{x}^k \subseteq \underset{\sim}{x}^0$ and using inclusion monotonicity it follows that $\underset{\sim}{f}'(\underset{\sim}{x}^k) \subseteq \underset{\sim}{f}'(\underset{\sim}{x}^0)$. Using again inclusion monotonicity it follows the existence of IGA $(\underset{\sim}{f}'(\underset{\sim}{x}^k))$ since IGA $(\underset{\sim}{f}'(\underset{\sim}{x}^0))$ exists. From the existence of IGA $(\underset{\sim}{f}'(\underset{\sim}{x}^k))$ it follows that IGA $(f'(\underset{\sim}{x}^k), \underset{\sim}{b})$ exists for an arbitrary interval-vector $\underset{\sim}{b}$. Thus we have shown that (N) is well defined if IGA $(\underset{\sim}{f}'(\underset{\sim}{x}^0))$ exists and if $\underset{\sim}{f}$ has a zero $\underset{\sim}{x}^*$ in $\underset{\sim}{x}^0$.

The proof that (SN) is well defined can be performed similarly.

We now assume $\rho(\mathcal{A})<1$ and prove (a). In doing this, we use the fact that $f(m(x^k))$ can be represented as

$$f(m(x^k)) = f(m(x^k)) - f(x^*) = \mathcal{J}(m(x^k), x^*)\cdot(m(x^k)-x^*) \tag{7}$$

where $\mathcal{J}(mx^k), (x^*)$ is a real point-matrix for which

$$\mathcal{J}(m(x^k)), x^*) \in f'(x^k) \tag{8}$$

holds. See the beginning of [2, § 19], for example. Using, besides these facts, Lemma 3, the relation (2) and the relation α) from the preliminaries, we obtain the following:

$$\begin{aligned}
\mathrm{N}\,(x^k)-x^* &= m(x^k)-x^*-\mathrm{IGA}\,(f'(x^k), f(m(x^k)))\\
&\subseteq m(x^k)-x^*-\mathrm{IGA}\,(f'(x^k))\cdot f(m(x^k))\\
&= m(x^k)-x^*-\mathrm{IGA}\,(f'(x^k))\cdot\{\mathcal{J}(m(x^k), x^*)(m(x^k)-x^*)\}\\
&\subseteq m(x^k)-x^*-\mathrm{IGA}\,(f(x^k))\cdot\{f'(x^k)(m(x^k)-x^*)\}\\
&\subseteq m(x^k)-x^*-\{\mathrm{IGA}\,(f'(x^k))\cdot f'(x^k)\}(m(x^k)-x^*)\\
&= (\mathcal{E}-\mathrm{IGA}\,(f'(x^k))\cdot f'(x^k))(m(x^k)-x^*).
\end{aligned}$$

Because $m(x^k)\in x^k$ and $|\mathcal{E}-\mathrm{IGA}\,(f'(x^k))\cdot f'(x^k)|\leqq \mathcal{A}$, it follows that

$$|\mathrm{N}\,(x^k)-x^*|\leqq \mathcal{A}|x^k-x^*|.$$

Using

$$\begin{aligned}
q(x^k, x^*) &= |x^k-x^*|,\\
q(\mathrm{N}\,(x^k), x^*) &= |\mathrm{N}\,(x^k)-x^*|,
\end{aligned}$$

this can be written as

$$q(\mathrm{N}\,(x^k), x^*)\leqq \mathcal{A}q(x^k, x^*).$$

Because $x^*\in \mathrm{N}\,(x^k)\cap x^k = x^{k+1}\subseteq \mathrm{N}\,(x^k)$ it also holds that

$$q(x^{k+1}, x^*)\leqq q(\mathrm{N}\,(x^k), x^*)\leqq \mathcal{A}q(x^k, x^*)$$

and therefore

$$q(x^{k+1}, x^*)\leqq \mathcal{A}^{k+1}q(x^0, x^*).$$

From this inequality follows the assertion $\lim_{k\to\infty} x^k = x^*$.

The proof of (a) for the method (SN) under the assumption $\rho(\mathcal{A})<1$ needs only some minor modifications compared with the preceding proof for (N). We omit the details.

We now prove (a) under the assumption $\rho(\mathcal{B})<1$. In this case we perform the proof for (SN). Because of the remarks at the beginning of the proof of this theorem it is clear that the sequence $\{x^k\}$ is well defined. Applying Lemma 3, we get for $k\geqq 0$

$$\begin{aligned}
\mathrm{SN}\,(x^k) &= m(x^k)-\mathrm{IGA}\,(f'\,(x^0), f(m(x^k)))\\
&\subseteq m(x^k)-\mathrm{IGA}\,(f'\,(x^0))\cdot f(m(x^k)).
\end{aligned}$$

Using both β) and γ) from the preliminaries as well as (7), (8), of this paper and taking into account that $|m(x^k)-x| \leqq d(x^k)$, it follows that

$$\begin{aligned} d(\mathrm{SN}\,(x^k)) &\leqq d(\mathrm{IGA}\,(f'(x^0)))\cdot|f(m(x^k))| \\ &= d(\mathrm{IGA}\,(f'(x^0)))\cdot|\mathcal{J}(m(x^k), x^*)(m(x^k)-x^*)| \\ &\leqq d(\mathrm{IGA}\,(f'(x^0)))\cdot|f'(x^k)|d(x^k) \\ &\leqq d(\mathrm{IGA}\,(f'(x^0)))\cdot|f'(x^0)|d(x^k). \end{aligned}$$

Because of $x^{k+1} = \mathrm{SN}\,(x^k)\cap x^k$ it follows that

$$d(x^{k+1}) \leqq \mathcal{B} d(x^k)$$

and therefore

$$d(x^{k+1}) \leqq \mathcal{B}^{k+1} d(x^0),$$

from which the assertion $\lim_{k\to\infty} x^k = x^*$ follows.

If one chooses $m(x^k)$ to be the center of x^k, then it follows that

$$|m(x^k)-x^*| \leqq \tfrac{1}{2} d(x^k).$$

Using this relation the proof can be completed also in the case of $\rho(\mathcal{B}) < 2$.

The proof of (a) for (N) under the assumption $\rho(\mathcal{A}) < 1$, $\rho(\mathcal{B}) < 1$ $(\rho(\mathcal{B}) < 2)$ can be performed similarly. We omit the details.

We now prove (b). We assume that for all $k \geqq 0$ the intersections $\mathrm{SN}\,(x^k)\cap x^k$ and $\mathrm{N}\,(x^k)\cap x^k$ are not empty. Then both methods are well defined for all $k \geqq 0$ and it holds that

$$x^0 \supseteq x^1 \supseteq \cdots x^k \supseteq x^{k+1} \cdots,$$

from which it follows that the sequence is converging to an interval-vector z^*. We now consider the sequence $\{m(x^k)\}$. This sequence is contained in the compact set x^0. By applying the Bolzano–Weierstrass theorem, we conclude that there exists a convergent subsequence $\{m(x^{k_i})\}_{i=0}^{\infty}$. Suppose that $\lim_{i\to\infty} m(x^{k_i}) = z^*$. For the elements of the sequence $\{m(x^{k_i})\}$ it holds that $m(x^{k_i}) \in x^{k_i}$. From this remark it follows that, besides $\lim_{i\to\infty} x^{k_i} = z^*$, we also have $z^* \in z^*$.

Using the continuity of the operations involved in the method (SN), we get from the equations

$$\begin{aligned} \mathrm{SN}\,(x^{k_i}) &= m(x^{k_i}) - \mathrm{IGA}\,(f'(x^0), f(m(x^{k_i}))), \\ x^{k_i+1} &= \mathrm{SN}\,(x^{k_i})\cap x^{k_i}, \end{aligned}$$

the relations

$$\begin{aligned} w^* &= z^* - \mathrm{IGA}\,(f'(x^0), f(z^*)), \\ z^* &= w^* \cap z^*, \end{aligned}$$

where $w^* := \lim_{i\to\infty} \mathrm{SN}\,(x^{k_i}) = \mathrm{SN}\,(z^*)$.

From the second equation it follows that $z^* \subseteq w^*$ and therefore that $z^* \in w^*$. Hence

$$z^* \in z^* - \mathrm{IGA}(f'(x^0), f(z^*)),$$

or, by applying Lemma 3,

$$o \in \mathrm{IGA}\,(f'(x^0), f(z^*)) \subseteq \mathrm{IGA}\,(f'(x^0))\cdot f(z^*).$$

Applying δ) from the preliminaries, it follows that there exists a point-matrix $\mathscr{H} \in$ IGA $f'(x^0)$) such that $o = \mathscr{X} \cdot f(z^*)$. If $\mathscr{X}$ is nonsingular, we have the contradiction $f(z^*) = o$. The nonsingularity of $\mathscr{X}$ follows from the condition $\rho(\mathscr{A}) < 1$ in the following manner: If $\mathscr{Y} \in f'(x^0)$, then

$$(9) \qquad |\mathscr{E} - \mathscr{X}\mathscr{Y}| \leqq |\mathscr{E} - \mathrm{IGA}\,(f'(x^0)) \cdot f'(x^0)| = \mathscr{A}.$$

Applying the Perron–Frobenius theorem for nonnegative matrices, it follows that

$$\rho(\mathscr{E} - \mathscr{X}\mathscr{Y}) \leqq \rho(|\mathscr{E} - \mathscr{X}\mathscr{Y}|) \leqq \rho(\mathscr{A}) < 1.$$

Hence

$$(\mathscr{X}\mathscr{Y})^{-1} = (\mathscr{E} - (\mathscr{E} - \mathscr{X}\mathscr{Y}))^{-1}$$

exists, from which the existence of $\mathscr{X}^{-1}$ follows. This is the proof of (b) for (SN). The proof of (b) for (N) needs only some small supplements which we omit. □

We close this section with two remarks:

1. From the inequality (9) it follows that the assertion (b) of the theorem already holds if $\rho(\tilde{\mathscr{A}}) < 1$ where

$$\tilde{\mathscr{A}} = |\mathscr{E} - \mathrm{IGA}\,(f'(x^0)) \cdot \mathscr{Y}|$$

for *some* $\mathscr{Y} \in f'(x^0)$. Of course this is a weaker condition than $\rho(\mathscr{A}) < 1$. This fact was pointed out by an (anonymous to me) referee.

2. It is an open question whether (b) of the theorem holds if $\rho(\mathscr{B}) < 1$ (or if $\rho(\mathscr{B}) < 2$).

4. Some further remarks. As already mentioned above, (SN) exhibits only linear convergence whereas (N) shows under some additional assumptions at least quadratic convergence behaviour. From this point of view the following fact, which holds both for (SN) and (N), is of interest: If one chooses $m(x^k)$ to be the center of x^k then at least one of the components of x^{k+1} has its width smaller than the half of the width of the corresponding component of x^k. This follows from the following Lemma 5 by choosing $m(x^k)$ to be the center of x^k.

LEMMA 5. *Let the assumptions of the preceding theorem hold. Suppose that for the matrix* $\mathscr{A}$ *defined by* (5) *the condition* $\rho(\mathscr{A}) < 1$ *holds. If* f *has a zero* x^* *in the interval-vector* x^0 *then both for* (SN) *and* (N) *it holds that* $m(x^k) \notin x^{k+1}$ *if* $m(x^k) \neq x^*(m(x^k) \in x^k)$.

Proof. We perform the proof for (N). Suppose that $m(x^k) \in x^{k+1}$. Then it follows that $m(x^k) \in \mathrm{N}\,(x^k)$. Using the relation

$$\mathrm{N}\,(x^k) - x^* \subseteq (\mathscr{E} - \mathrm{IGA}\,(f'(x^k)) \cdot f'(x^k))(m(x^k) - x^*),$$

which was derived in the proof of the preceding Theorem, we therefore get

$$m(x^k) - x^* \subseteq N(x^k) - x^* \subseteq (\mathscr{E} - \mathrm{IGA}\,(f'(x^k)) \cdot f'(x^k))(m(x^k) - x^*).$$

From this relation it follows that

$$\begin{aligned} |m(x^k) - x^*| &\leqq |\mathscr{E} - \mathrm{IGA}\,(f(x^k)) \cdot f(x^k)|\,|m(x^k) - x^*| \\ &\leqq \mathscr{A}|m(x^k) - x^*|. \end{aligned}$$

Since $\rho(\mathscr{A}) < 1$, we obtain the contradiction that $m(x^k) = x^*$. The proof for (SN) can be performed similarly. We omit the details. □

The next lemma shows that the statements of the preceding lemma are also true if there exists no zero if f in x^0.

LEMMA 6. *Let the assumptions of the theorem hold. Suppose that for the matrix* $\mathcal{A}$ *defined by* (5) *the condition* $\rho(\mathcal{A})<1$ *holds. If* f *has no zero in* x^0 *then both for* (N) *and* (SN) *it holds that* $m(x^k)\notin x^{k+1}$ *if* $m(x^k)\in x^k$. (Of course we assume that $k\leqq k_0$ where k_0 is defined by the statement (b) of the theorem.)

Proof. Consider again the method (N). Assume that $m(x^k)\in x^{k+1}$. Then it follows that $m(x^k)\in \mathrm{N}(x^k)$ and hence that

$$o\in \mathrm{IGA}\,(f'(x^k))\cdot f(m(x^k)).$$

Using the corresponding reasoning as in the proof of part (b) of the theorem we arrive at the contradiction that $m(x^k)$ is a zero of f. □

Because of Lemma 6 we can hope that it will not take a large number of steps until the intersection becomes empty, that is, until we have shown that there is no zero in x^0.

We finally comment on the spectral radius conditions $\rho(\mathcal{A})<1$ and $\rho(\mathcal{B})<1$ where the matrices $\mathcal{A}$ and $\mathcal{B}$ are defined by (5) and (6): For the interval arithmetic evaluation $f'(x^0)$ of the derivative of $f(x)$ the condition $\lim_{d(x^0)\to\mathcal{O}} d(f'(x^0))=\mathcal{O}$ holds. See [2], § 2. Hence, by continuity arguments, it also follows that $\lim_{d(x^0)\to\mathcal{O}} (d(\mathrm{IGA}\,(f'(x^0)))=\mathcal{O}$. Therefore $\rho(\mathcal{A})<1$ and $\rho(\mathcal{B})<1$ hold if $d(x^0)$ is not too large.

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