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On the Convergence of the Symmetric SOR Method for Matrices with Red-Black Ordering

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Summary. We prove that if the matrix A has the structure which results from the so-called "red-black" ordering and if A is an H-matrix then the symmetric SOR method (called the SSOR method) is convergent for $0<\omega<2$. In the special case that A is even an M-matrix we show that the symmetric single-step method cannot be accelerated by the SSOR method. Symmetry of the matrix A is not assumed.

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1. Introduction

Let there be given the linear system

$$Ax=b, \tag{1}$$

where A is a nonsingular n by n matrix. If the diagonal part D of the matrix A is nonsingular then we define the strictly lower triangular matrix L and the strictly upper triangular matrix U by the corresponding parts of the matrix

$$\mathscr{J}_1=I-D^{-1}A=L+U.$$

$\mathscr{J}_1$ is usually called the total-step or Jacobi matrix belonging to A. If we define

$$\mathscr{L}_\omega=(I-\omega L)^{-1}\{(1-\omega)I+\omega U\}$$
$$\mathscr{U}_\omega=(I-\omega U)^{-1}\{(1-\omega)I+\omega L\}$$

and finally

$$\mathscr{S}_\omega=\mathscr{U}_\omega\mathscr{L}_\omega \tag{2}$$

then the iteration method

$$u^{k+1}=\mathscr{S}_\omega u^k+\omega(2-\omega)(I-\omega U)^{-1}(I-\omega L)^{-1}b$$

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is called symmetric successive overrelaxation iteration method (SSOR). In the special case $\omega=1$ it is called symmetric single-step method (SSM). If we want to express the dependency of $\mathcal{S}_\omega$ on A then we write $\mathcal{S}_\omega^A$. Similarly we write $\mathcal{J}_1^A$ instead of $\mathcal{J}_1$ and $\mathcal{L}_\omega^A$ instead of $\mathcal{L}_\omega$.

W. Niethammer [2] has pointed out that with the exception of the first step, (SSOR) can be performed with the same amount of work one needs for the successive overrelaxation method. (SSOR) is convergent for all starting values if the spectral radius of the matrix $\mathcal{S}_\omega$ is smaller than one, $\rho(\mathcal{S}_\omega)<1$.

2. Results

In Theorem 2.1 in [6, p. 463] the following result was proven:

If A is symmetric then $\rho(\mathcal{S}_\omega)<1$ if and only if A is positive definite and $0<\omega<2$.

In [1] we were able to prove the following Theorem: *If $A\in\mathbb{C}_{\Pi}^{n,n}$ then the following statements are equivalent:*

a) A is a nonsingular H-Matrix.

b) For all B from the set of equimodular matrices $\Omega(A)$ and for all ω from the interval $0<\omega<2/(1+\rho(|\mathcal{J}_1^A|))$ it holds that $\rho(\mathcal{S}_\omega^B)<1$.

In this Theorem $\mathbb{C}_{\Pi}^{n,n}$ denotes the set of all n by n complex matrices with all diagonal entries nonzero. Given any complex matrix $A=(a_{ij})\in\mathbb{C}_{\Pi}^{n,n}$, we define its comparison matrix $\mathcal{M}(A)=(\alpha_{ij})$ by

$$\begin{aligned}\alpha_{ii}&=|a_{ii}|, \qquad 1\leqq i\leqq n,\\ \alpha_{ij}&=-|a_{ij}|, \qquad i\neq j.\end{aligned}$$

Furthermore we define for an arbitrary matrix $A=(a_{ij})\in\mathbb{C}_{\Pi}^{n,n}$ the set of equimodular matrices by

$$\Omega(A)=\{B=(b_{ij})\,|\,|b_{ij}|=|a_{ij}|,\ 1\leqq i,j\leqq n\}.$$

It is obvious that A and $\mathcal{M}(A)$ are elements of the set $\Omega(A)$. Any real matrix $A=(a_{ij})$ with $a_{ij}\leqq 0$, $i\neq j$, can be written as

$$A=\tau I-C$$

satisfying $\tau>0$ and $C\geqq 0$.

Following Ostrowski [3] such a matrix is called a *nonsingular M-Matrix* if $\tau>\rho(C)$ holds. A complex matrix A is called *nonsingular H-matrix* if $\mathcal{M}(A)$ is a nonsingular M-matrix.

In this paper we will make special assumptions about the structure of A. In other respects we assume that the same assumptions hold as in [1]. The structure which we assume that A has is

$$A=\begin{pmatrix}D_1 & H\\ K & D_2\end{pmatrix} \tag{3}$$

where D_1 and D_2 are diagonal matrices. Matrices having this structure are the result of the discretization of elliptic boundary value problems if the unknowns are collected using the so-called red-black ordering. See [6, p. 159], for example.

We don't assume symmetry of the matrix A. In general, symmetry does indeed not hold for the aforementioned class of problems.

Before we state our result we mention that one can prove that $0<\omega<2$ is a necessary condition for the convergence of (SSOR). We omit the details and simply refer to the proof of Kahan's theorem in [4, p. 75].

After these preliminaries we prove the following generalization of the aforementioned Theorem from [1]:

Theorem 1. *Let* $A=(a_{ij})\in\mathbb{C}_{II}^{n,n}$ *and let* A *have the form* (3). *Then the following statements are equivalent:*

(*a*) *A is a nonsingular H-matrix.*

(*b*) *For all $B\in\Omega(A)$ and for all $0<\omega<2$ it holds that $\rho(\mathcal{S}_\omega)<1$.*

Proof. We consider the (SSOR)-matrix $\mathcal{S}_\omega=\mathcal{U}_\omega\mathcal{L}_\omega$ belonging to A. Because of the special form (3), $\mathcal{L}_\omega$ can be written as

$$\mathcal{L}_\omega=\begin{pmatrix}(1-\omega)I_1 & \omega F\\ (1-\omega)\omega G & \omega^2 GF+(1-\omega)I_2\end{pmatrix}.$$

I_1 and I_2 are unit matrices and we have set

$$F:=-D_1^{-1}H,\qquad G:=-D_2^{-1}K.$$

See [6, p. 237], for example. Completely analogously one shows

$$\mathcal{U}_\omega=\begin{pmatrix}(1-\omega)I_1+\omega^2 FG & (1-\omega)F\\ \omega G & (1-\omega)I_2\end{pmatrix}.$$

If we define (see [6, p. 256])

$$\mathcal{L}_{\omega,0}=\begin{pmatrix}(1-\omega)I_1 & \omega F\\ 0 & I_2\end{pmatrix}$$

and

$$\mathcal{L}_{0,\omega}=\begin{pmatrix}I_1 & 0\\ \omega G & (1-\omega)I_2\end{pmatrix}$$

then

$$\mathcal{L}_{0,\omega}\mathcal{L}_{\omega,0}=\mathcal{L}_\omega. \tag{4}$$

Correspondingly one has

$$\mathcal{L}_{\omega,0}\mathcal{L}_{0,\omega}=\mathcal{U}_\omega$$

and therefore

$$\mathcal{S}_\omega=\mathcal{U}_\omega\mathcal{L}_\omega=\mathcal{L}_{\omega,0}\mathcal{L}_{0,\omega}\mathcal{L}_{0,\omega}\mathcal{L}_{\omega,0}.$$

Since the spectral radius $\rho(\mathscr{S}_\omega)$ remains unchanged if the order of the matrices appearing on the right-hand side is changed (see [6, p. 15, Theorem 1.11]) we have the result that $\mathscr{S}_\omega$ has the same spectral radius as the matrix

$$\hat{\mathscr{S}}_\omega := \mathscr{L}_{0,\omega}^2 \mathscr{L}_{\omega,0}^2$$

has. If we define $\hat{\omega}=\omega(2-\omega)$ then it holds that

$$\mathscr{L}_{0,\omega}^2 = \mathscr{L}_{0,\hat{\omega}}, \qquad \mathscr{L}_{\omega,0}^2 = \mathscr{L}_{\hat{\omega},0},$$

(see [6, p. 256ff]). Therefore, using (4), we have

$$\hat{\mathscr{S}}_\omega = \mathscr{L}_{0,\hat{\omega}} \mathscr{L}_{\hat{\omega},0} = \mathscr{L}_{\hat{\omega}} \tag{5}$$

where

$$\mathscr{L}_{\hat{\omega}} = (I-\omega(2-\omega)L)^{-1}\{(1-\omega)^2 I + \omega(2-\omega)U\}. \tag{6}$$

We have $0<\omega<2$ iff $0<\hat{\omega}\leqq 1$. Therefore, assuming (3), the following holds:

(i) $\rho(\mathscr{L}_{\hat{\omega}}^A)<1$ for some $\hat{\omega}$, $0<\hat{\omega}\leqq 1$, iff $\rho(\mathscr{S}_\omega^A)<1$ for the corresponding ω, $0<\omega<2$.

In order to perform the proof of this theorem we use the following equivalent statements:

(ii) A is a nonsingular H-matrix.

(iii) For all $B\in\Omega(A)$ it holds that $\rho(\mathscr{J}_1^B)\leqq\rho(\mathscr{J}_1^{\mathscr{M}(A)})<1$.

(iv) For all $B\in\Omega(A)$ and for all

$$0<\hat{\omega}<2/(1+\rho(|\mathscr{J}_1^B|)) \text{ it holds that } \rho(\mathscr{L}_{\hat{\omega}}^B)<1.$$

(See [5, Theorem 1].)

(a)⇒(b): If A is a nonsingular H-matrix, that is if (ii) holds, then it follows from (iv) that $\rho(\mathscr{L}_{\hat{\omega}}^B)<1$ for all $0<\hat{\omega}\leqq 1$ and for all $B\in\Omega(A)$. Using (i) it follows that $\rho(\mathscr{S}_\omega^B)<1$ for all $0<\omega<2$ and for arbitrary $B\in\Omega(A)$.

(b)⇒(a): If on the other hand $\rho(\mathscr{S}_\omega^B)<1$ for all $0<\omega<2$ and for arbitrary $B\in\Omega(A)$ then we can choose $B=\mathscr{M}(A)$. Because of $\mathscr{J}_1^{\mathscr{M}(A)}\geqq 0$ it follows $\rho(\mathscr{L}_1^{\mathscr{M}(A)})<1 \Leftrightarrow \rho(\mathscr{J}_1^{\mathscr{M}(A)})<1$ by the Stein-Rosenberg-Theorem. For $\omega=1$ it follows from (i) that $\rho(\mathscr{L}_1^{\mathscr{M}(A)})<1$ and therefore that $\rho(\mathscr{J}_1^{\mathscr{M}(A)})<1$. Since (ii) follows from (iii) we have that A is a nonsingular H-matrix. □

Without going into details we mention that both the Corollaries from [1] hold for matrices of the form (3), where ω is now allowed to lie in the interval $0<\omega<2$.

The next Theorem discusses the case that A is an M-matrix.

Theorem 2. *If $A=(a_{ij})$ is an M-matrix of the form* (3) *then*

$$\min_{0<\omega<2} \rho(\mathscr{S}_\omega)=\rho(\mathscr{S}_1).$$

Proof. We consider the splitting $A=M-N$ where

$$M=\frac{D}{\omega(2-\omega)}(I-\omega(2-\omega)L),$$

$$N = \frac{D}{\omega(2-\omega)}((1-\omega)^2 + \omega(2-\omega)U) = \frac{(1-\omega)^2}{\omega(2-\omega)}D + DU.$$

We have $M^{-1}N = \mathscr{L}_{\hat{\omega}}$. Because of $\rho(\mathscr{S}_{\omega}) = \rho(\mathscr{L}_{\hat{\omega}})$ and since $(1-\omega)^2/[\omega(2-\omega)]$ takes on its minimum for $\omega = 1$ in the interval $0 < \omega < 2$ the result follows from a simple generalization of Theorem 3.15 in [4]. $\square$

We therefore have the negative result, that also in the case that A is an M-matrix of the form (3), (SSM) can not be accelerated by using (SSOR). If A is a symmetric and positive definite matrix of the form (3) the same result can already be found in [6, p. 464, Theorem 2.2].

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