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Some Convergence Results for the PEACEMAN-RACHFORD Method in the Noncommutative Case

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1. Introduction

We consider the system of simultaneous linear equations

$$Au = b \; .$$

Let the matrix A be expressed as the sum

$$A = H + V$$

of two matrices H and V. Then we consider the following iteration method for solving the system given above:

$$\begin{cases} (r_k I + H)x_{k+\frac{1}{2}} = (r_k I - V)x_k + b \\[2ex] (r_k I + V)x_{k+1} = (r_k I - H)x_{k+\frac{1}{2}} + b \; , \end{cases}$$

$$k = 0,1,2,\ldots \; ,$$

$$(r_k > 0 \; , \quad I = \text{unit matrix}) \; .$$

This method is called <u>PEACEMAN-RACHFORD iterative method</u> (PRM). If $r_k = r$, $k = 0,1,2,\ldots$, then the method is called <u>stationary</u> otherwise <u>nonstationary</u>.

Most known results concerning the convergence of the stationary PRM consider the case in which both H and V are Hermitian and nonnegative definite and where at least one of the matrices H and V is positive definite ([13,14,16]).

In the nonstationary case very satisfactory practical experience has been made. But proofs of convergence and optimizing the parameter sequence (r_k) have been performed only under even more restrictive conditions ([13,14,16]). Particularly the matrices H and V have to commute, this means that $HV = VH$ holds. Although the nonstationary method shows very good convergence behavior also in most noncommutative cases there are scarcely criteria known which assure at least convergence in these cases. See however [2,6,7,11,15]. On the other side there are linear systems arising from boundary value problems for which it is possible to choose a parameter sequence (r_k) such that PRM does not converge ([12]). Because of these reasons it seems quite desirable to look for new convergence criteria for the non-stationary method.

In this paper we first report on some results from ALEFELD [1] concerning the convergence of PRM (Section 2). These results can immediately be applied to discrete versions of elliptic boundary value problems (Section 3). Finally we prove a new convergence result for an iterative method for $m \geqq 2$ space variables which was introduced in [5] (Section 4).

2. A Convergence Theorem for PRM

Let $\mathbb{C}^{n,n}$ be the set of all $n \times n$ matrices $A = [a_{ij}]$ with elements taken from $\mathbb{C}$ (= the set of complex numbers). Analogously $R^{n,n}$ is defined. For $A = [a_{ij}] \in \mathbb{C}^{n,n}$ we set $|A| := [|a_{ij}|] \in R^{n,n}$. Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be decomposed into the sum $A = H + V$ and let

$$H = D_H - B_H \qquad \text{and} \qquad V = D_V - B_V .$$

Here D_H and D_V represent the diagonal parts of H and V whereas B_H and B_V stand for the off-diagonal parts of H and V respectively. The set of matrices $\tilde{\Omega}(A)$ is now defined by

$$\tilde{\Omega}(A) = \{C = [c_{ij}] \in \mathbb{C}^{n,n} \mid C = \tilde{H} + \tilde{V},\ \tilde{H} = D_{\tilde{H}} - B_{\tilde{H}},\ \tilde{V} = D_{\tilde{V}} - B_{\tilde{V}},$$
$$D_{\tilde{H}} = D_H, |B_{\tilde{H}}| = |B_H|,\ D_{\tilde{V}} = D_V,\ |B_{\tilde{V}}| = |B_V|\}.$$

We have $A \in \tilde{\Omega}(A)$.

The spectral radius of a matrix A is denoted by $\rho(A)$. Consider now any $B = [b_{ij}] \in R^{n,n}$ with $b_{ij} \leqq 0,\ i \neq j$. Then B can be expressed as the difference

$$B = \kappa I - C$$

where $\kappa = \max_{1 \leqq i \leqq n}\{b_{ii}\}$ and where $C = [c_{ij}] \in R^{n,n}$, satisfying $C \geqq 0$, has its entries given by

$$c_{ii} = \kappa - b_{ii} \geqq 0 , \quad 1 \leqq i \leqq n$$

$$c_{ij} = -b_{ij} \geqq 0 , \quad i \neq j , \quad 1 \leqq i,j \leqq n .$$

Following OSTROWSKI [9] such a matrix B is called a nonsingular M-matrix iff $\kappa > \rho(C)$. By Theorem 3.8 and by Theorem 3.10 in [13] a nonsingular M-matrix has positive diagonal elements. The proof of the following theorem is given in detail in [1].

Theorem 1. Let the matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be expressed as the sum $A = H + V$ of two matrices $H = [h_{ij}]$ and $V = [v_{ij}]$. Let H and V both have only real diagonal elements. Let τ be defined by

$$\tau = \max_{1 \leqq i \leqq n} \{h_{ii}, v_{ii}\}$$

and assume that the matrices

$$\tau I + D_H - |B_H| \quad \text{and} \quad \tau I + D_V - |B_V|$$

are both nonsingular M-matrices. Then the following are equivalent:

(a) $D_V + D_H - (|B_V| + |B_H|)$ *is a nonsingular M-matrix;*

(b) *For any matrix of the set $\tilde{\Omega}(A)$ and for any sequence (r_k) satisfying*

$$\tau \leqq r_k \leqq \sigma < \infty , \quad k = o,1,2,...,$$

$(\sigma \geqq \tau)$ *PRM is convergent.* ■

As a special case of Theorem 1 we get

Corollary 1. Let $A = [a_{ij}] \in R^{n,n}$ be decomposed into the sum $A = H + V$ of two real matrices $H = [h_{ij}]$ and $V = [v_{ij}]$. Let

$$\tau = \max_{1 \leqq i \leqq n} \{h_{ii}, v_{ii}\}$$

and assume that

$$\tau I + H \quad \text{and} \quad \tau I + V$$

are both nonsingular M-matrices. Then the following are equivalent:

(a) A *is a nonsingular M-matrix;*

(b) *PRM is convergent for any matrix of the set $\tilde{\Omega}(A)$ and for any sequence (r_k) satisfying*

$$\tau \leqq r_k \leqq \sigma < \infty, \quad k = o,1,2,...,$$

$(\sigma \geqq \tau)$. ■

3. Applications to Disctretized Elliptic Equations

Let R be a bounded plane region with boundary ∂R . Consider the linear second-order partial differential equation

$$L[u] \equiv Au_{xx} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

with coefficients A,C,D,E,F and G which are functions of x and y and with $A \geqq m$, $C \geqq m$, $m > o$ and $F \leqq o$ in R. The function u is also required to satisfy the condition

$$u(x,y) = g(x,y)$$

on the boundary ∂R of R. Replacing the derivatives by difference quotients leads to a second-order partial difference operator

$$L_h[u] \equiv \alpha_o u(x,y) - \alpha_1 u(x+h,y) - \alpha_2 u(x,y+h) - \alpha_3 u(x-h,y) - \alpha_4 u(x,y-h) = t(x,y)$$

where

$$\alpha_1 = A + \frac{h}{2} D , \qquad \alpha_2 = C + \frac{h}{2} E ,$$

$$\alpha_3 = A - \frac{h}{2} D , \qquad \alpha_4 = C - \frac{h}{2} E ,$$

$$\alpha_o = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - h^2 F ,$$

$$t(x,y) = -h^2 G .$$

Here we have used the usual three-point central difference quotients. For simplicity we assume that it is not necessary to approximate the boundary ∂R. The equation $L_h[u] = t(x,y)$ is equivalent to a system of linear algebraic equations $Au = b$. It is well know that for

$$h < h_o = \min\{\min_{R+\partial R} \frac{2A}{|D|} , \min_{R+\partial R} \frac{2C}{|E|}\}$$

the matrix A is a nonsingular M-matrix. Expressing $L_h[u]$ as

$$L_h[u] = H_h[u] + V_h[u]$$

where

$$H_h[u] = (2A - \tfrac{1}{2}h^2F)u(x,y) - (A + \tfrac{1}{2}hD)u(x + h,y) - (A - \tfrac{1}{2}hD)u(x - h,y),$$

$$V_h[u] = (2C - \tfrac{1}{2}h^2F)u(x,y) - (C + \tfrac{1}{2}hE)u(x,y + h) - (C - \tfrac{1}{2}hE)u(x,y - h)$$

the matrix A can be written in the form $A = H + V$ where H and V are both nonsingular M-matrices. But then the same is true for $H + \tau I$ and $V + \tau I$ ([9]). Therefore by applying Corollary 1 the following holds.

<u>Theorem 2.</u> <u>Let</u> $L_h[u] = -h^2 G$ <u>where</u> $h < h_o$.

Let

$$\tau = \max\{\max_{R+\partial R} (2A - \tfrac{1}{2}h^2F),\ \max_{R+\partial R} (2C - \tfrac{1}{2}h^2F)\}$$

Then PRM is convergent for any sequence (r_k) satisfying $\tau \leqq r_k \leqq \sigma < \infty$, $k = o,1,2,\ldots,\ (\sigma \geqq \tau)$. ■

4. Remarks on Methods for $m \geqq 2$ Space Variables

We consider the problem of solving the system of linear equations

$$(A_1 + A_2 + \ldots + A_m)x = b \quad , \qquad m \geqq 2 .$$

In [5] among others the following iterative method was proposed:

$$(rI + A_i)x_i^{(k+1)} = \sum_{j=1}^{i-1} (\frac{r}{m-1} I - A_j)x_j^{(k+1)} + \sum_{j=i+1}^{m} (\frac{r}{m-1} I - A_j)x_j^{(k)} + b, \qquad (*)$$

$$i = 1(1)m \ , \qquad k = o,1,2,\ldots,\ (r > o) .$$

It was proved in [5] that provided the matrices A_i, $1 \leqq i \leqq m$, are alle Hermitian and positive definite and provided the eigenvalues $\lambda_j(i)$ of A_i satisfy $o < a \leqq \lambda_j(i) \leqq b$, $1 \leqq i \leqq n$, $1 \leqq j \leqq m$, then for $r > (m-2)b/2$ it follows $\lim_{k\to\infty} x^{(k)} = z$, $i = 1(1)m$, where z is the solution of the given system. It was pointed out in [5] that it is important that this result holds without assuming commutativity of the matrices A_i. The same is true for the result given in the next theorem.

Theorem 3. *Let* $A_i = [a_{st}^{(i)}]$, $1 \leqq i \leqq m$, *be all nonsingular M-matrices. Then if* A *is a nonsingular M-matrix method* $(*)$ *is convergent for*

$$r \geqq \tau := (m-1) \max_{\substack{1 \leqq s \leqq n \\ 1 \leqq i \leqq m}} \{a_{ss}^{(i)}\} .$$

Proof. Consider the $m\cdot n \times m\cdot n$ matrix $\tilde{A}_r$ given by

$$\tilde{A}_r = \begin{bmatrix} rI + A_1 & -(\frac{r}{m-1} I - A_2) & \ldots & -(\frac{r}{m-1} I - A_m) \\ -(\frac{r}{m-1} I - A_1) & rI + A_2 & \ldots & -(\frac{r}{m-1} I - A_m) \\ \ldots & \ldots & & \ldots \\ -(\frac{r}{m-1} I - A_1) & -(\frac{r}{m-1} I - A_2) & \ldots & rI + A_m \end{bmatrix}$$

and the splitting

$$\tilde{A}_r = \tilde{M}_r - \tilde{N}_r$$

where $\tilde{M}_r$ is given by

$$\tilde{M}_r = \begin{bmatrix} rI + A_1 & & & 0 \\ -(\frac{r}{m-1} I - A_1) & rI + A_2 & & \\ \cdots & \cdots & \ddots & \\ -(\frac{r}{m-1} I - A_1) & -(\frac{r}{m-1} I - A_2) & \cdots & rI + A_m \end{bmatrix} .$$

Defining the vectors $\overline{x} \in R^{m \cdot n}$ and $\overline{c} \in R^{m \cdot n}$ by

$$\overline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{bmatrix} , \; x_i \in R^n , \quad i = 1(1)m , \quad \overline{c} = \begin{bmatrix} b \\ b \\ \cdot \\ \cdot \\ \cdot \\ b \end{bmatrix} , \quad b \in R^n ,$$

it is easy to see that (*) can be written in the form

$$\overline{x}^{(k+1)} = \tilde{M}_r^{-1} \tilde{N}_r \overline{x}^{(k)} + \tilde{M}_r^{-1} \overline{c} , \quad k = o,1,2,\ldots .$$

First we show that $\tilde{A}_r$ is a nonsingular M-matrix. To do this it is sufficient to verify that the off-diagonal elements of $\tilde{A}_r$ are nonpositive and that there exists a positive vector $\overline{z} \in R^{m \cdot n}$ with $\tilde{A}_r \overline{z} > \overline{o}$. (See [13], p.85 and exercise 2 on p. 87). Since by hypothesis the A_i are nonsingular M-matrices then the off-diagonal elements of $\tilde{A}_r$ are nonpositive if $r \geqq (m - 1)a_{ss}^{(i)}$, $1 \leqq s \leqq n$, $1 \leqq i \leqq m$. Again, since A itself is by hypothesis a nonsingular M-matrix there exists a positive vector $z \in R^n$ with $Az > o$. If we take

$$\overline{z} = \begin{bmatrix} z \\ z \\ \cdot \\ \cdot \\ \cdot \\ z \end{bmatrix} \in R^{m \cdot n}$$

then a simple calculation shows $\tilde{A}_r \overline{z} > \overline{o}$. Hence $\tilde{A}_r$ is a nonsingular M-matrix. Especially, we have $\tilde{A}_r^{-1} \geqq 0$ ([13], p.85). Since any matrix obtained from a nonsingular M-Matrix by setting certain off-diagonal entries to zero,is also a nonsingular M-matrix we have that $\tilde{M}_r$ is a nonsingular M-matrix from which it follows again that $\tilde{M}_r^{-1} \geqq 0$. Furthermore $\tilde{N}_r \geqq 0$. Hence the splitting $\tilde{A}_r = \tilde{M}_r - \tilde{N}_r$ is a regular splitting ([13], Definition 3.5) and therefore by Theorem 3.13 in [13] the method (:) is convergent. This completes the proof of this theorem. ■

We close this paper with two remarks:

1. It is easy to give similiar convergence results for the other methods proposed in [5].
2. Theorem 3 can directly be applied to discrete versions of boundary value problems for $m \geqq 2$ space variables.

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