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An exclusion theorem for the solutions of operator equations

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Let F be a nonlinear mapping from a Banach space X into a Banach space Y. We prove a theorem which yields a ball containing no solution of the equation $F(x) = 0$. This ball is obtained after having performed one step of Newton's method.

1. Introduction

Let F be a nonlinear mapping of the Banach space X into the Banach space Y. There are many results which assure the existence of a solution of the (nonlinear) equation $F(x) = 0$. In many cases there are also error estimates for such a solution. Famous results in this direction follow from the contraction mapping principle. Another well known result is the Kantorovic theorem on Newton's method (Rall [3]). Thereby one step of Newton's method is performed and then under certain conditions we get a ball in which a solution of the equation $F(x) = 0$ exists and to which Newton's method will converge. Another interesting result in this direction was proved by W. Burmeister in [2]. Correspondingly in this note we give a ball which does not contain any solution of the equation $F(x) = 0$. This ball is also obtained after having performed one step of Newton's method. Of course every ball K of the Banach space X, not containing a solution of the equation $F(x) = 0$, also yields an inclusion set $X \setminus K$ for all solutions of $F(x) = 0$. The following theorem contains as a special case a result for real functions which was proved in [1], page 113.

2. Exclusion theorem

Theorem. *Let X and Y be Banach spaces. Let*

$$F : D \subset X \to Y$$

be Fréchet-differentiable in the open set $D_0 \subset D$. Suppose that the Fréchet-derivative *of F is Lipschitz-continuous with Lipschitz-constant* $\gamma > 0$ *in the ball*

$$\tilde{K}_0 = \{x \mid \|x - x_0\| \leqq \tilde{r}_0\} \subset D_0,$$

that $F'(x_0)$ *is invertible, that*

$$\|F'(x_0)^{-1}\| = \beta,$$

and that

$$0 < \|F'(x_0)^{-1} F(x_0)\| = \eta.$$

Then, if the inequalities

$$0 \leqq r_0 < \frac{-1 + \sqrt{1 + 2\beta\gamma\eta}}{\beta\gamma}, \qquad r_0 \leqq \tilde{r}_0$$

hold, $F(x) = 0$ *has no solution in the ball*

$$K_0 = \{x \mid \|x - x_0\| \leqq r_0\}.$$

The proof of this theorem will follow from a simple lemma.

Lemma. *Let X and Y be Banach spaces. Let*

$$F : D \subset X \to Y$$

be Fréchet-differentiable in the open set $D_0 \subset D$. *Suppose that the Fréchet-derivative of F is Lipschitz-continuous in the ball*

$$\tilde{K}_0 = \{x \mid \|x - x_0\| \leqq \tilde{r}_0\} \subset D_0$$

with Lipschitz-constant $\gamma \geqq 0$, *that* $F'(x_0)$ *is invertible and that*

$$\|F'(x_0)^{-1}\| = \beta.$$

If the equation $F(x) = 0$ *has a solution* x^* *in the ball*

$$K_0 = \{x \mid \|x - x_0\| \leqq r_0\}, \qquad r_0 \leqq \tilde{r}_0,$$

then

$$x^* \in K_1 = \{x \mid \|x - x_1\| \leqq r_1\}$$

where

$$x_1 = x_0 - F'(x_0)^{-1} F(x_0), \qquad r_1 = \frac{1}{2} \beta\gamma r_0^2.$$

Proof. By Taylor's theorem (RALL [3], page 124) we have

$$F(x_0) - F(x^*) = \int_0^1 F'\big(x^* + \theta(x_0 - x^*)\big)\,(x_0 - x^*)\,d\theta$$

$$= F'(x_0)\,(x_0 - x^*) + \int_0^1 \big[F'\big(x^* + \theta(x_0 - x^*)\big) - F'(x_0)\big]\,(x_0 - x^*)\,d\theta.$$

From this it follows that

$$\begin{aligned}\|x^* - x_1\| &= \left\|x^* - \left(x_0 - F'(x_0)^{-1} F(x_0)\right)\right\| \\ &= \left\|F'(x_0)^{-1} \int_0^1 [F'\left(x^* + \theta(x_0 - x^*)\right) - F'(x_0)]\,(x_0 - x^*)\,d\theta\right\| \\ &\leqq \beta\gamma\|x_0 - x^*\|^2 \int_0^1 (1-\theta)\,d\theta \\ &\leqq \frac{1}{2}\,\beta\gamma r_0^{\,2}\end{aligned}$$

holds, which is the assertion.

Proof of the theorem. If K_0 contains a solution x^* of $F(x) = 0$ then by the preceding lemma x^* is contained in K_1 and $K_0 \cap K_1 \neq \emptyset$. Therefore if $K_0 \cap K_1 = \emptyset$, $F(x) = 0$ has no solution in K_0. The equation $K_0 \cap K_1 = \emptyset$ holds if and only if

$$\|x_0 - x_1\| > r_0 + r_1$$

holds. This is again the case if and only if

$$\frac{1}{2}\,\beta\gamma r_0^2 + r_0 - \eta < 0$$

holds. Because of $r_0 \geqq 0$ this inequality is equivalent with

$$0 \leqq r_0 < \frac{-1 + \sqrt{1 + 2\beta\gamma\eta}}{\beta\gamma}$$

and the theorem is proved.

3. Remark

The following example shows that in general the upper bound for r_0 cannot be replaced by a greater one: Let $X = Y = \mathbf{R}$ be the real numbers with the usual modulus as a norm. Consider the real function

$$F(x) = \frac{1}{2}\,\tilde{\beta}\tilde{\gamma}x^2 + x - \tilde{\eta}$$

with positive constants $\tilde{\beta}$, $\tilde{\gamma}$ and $\tilde{\eta}$. For $x_0 = 0$ we get from the theorem that for

$$0 \leqq r_0 < \frac{-1 + \sqrt{1 + 2\tilde{\beta}\tilde{\gamma}\tilde{\eta}}}{\tilde{\beta}\tilde{\gamma}}$$

no solution of $F(x) = 0$ is contained in the interval

$$K_0 = \{x \mid |x - x_0| = |x| \leqq r_0\}.$$

However, for

$$r_0 = \frac{-1 + \sqrt{1 + 2\bar{\beta}\bar{\gamma}\bar{\eta}}}{\bar{\beta}\bar{\gamma}}$$

there is a solution of $F(x) = 0$ on the boundary of K_0.

References

[1] ALEFELD, G., und J. HERZBERGER, Einführung in die Intervallrechnung, Reihe Informatik, Band 12, Bibliographisches Institut, Mannheim 1974.
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