An exclusion theorem for the solutions of operator equations

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Let $F$ be a nonlinear mapping from a Banach space $X$ into a Banach space $Y$. We prove a theorem which yields a ball containing no solution of the equation $F(x) = 0$. This ball is obtained after having performed one step of Newton's method.

1. Introduction

Let $F$ be a nonlinear mapping of the Banach space $X$ into the Banach space $Y$. There are many results which assure the existence of a solution of the (nonlinear) equation $F(x) = 0$. In many cases there are also error estimates for such a solution. Famous results in this direction follow from the contraction mapping principle. Another well known result is the Kantorovic theorem on Newton’s method (Rall [3]). Thereby one step of Newton’s method is performed and then under certain conditions we get a ball in which a solution of the equation $F(x) = 0$ exists and to which Newton’s method will converge. Another interesting result in this direction was proved by W. Burmeister in [2]. Correspondingly in this note we give a ball which does not contain any solution of the equation $F(x) = 0$. This ball is also obtained after having performed one step of Newton’s method. Of course every ball $K$ of the Banach space $X$, not containing a solution of the equation $F(x) = 0$, also yields an inclusion set $X \setminus K$ for all solutions of $F(x) = 0$. The following theorem contains as a special case a result for real functions which was proved in [1], page 113.

2. Exclusion theorem

Theorem. Let $X$ and $Y$ be Banach spaces. Let

$$F : D \subset X \to Y$$

be Fréchet-differentiable in the open set $D_0 \subset D$. Suppose that the Fréchet-derivative
of $F$ is Lipschitz-continuous with Lipschitz-constant $\gamma > 0$ in the ball
\[ \bar{K}_0 = \{ x \mid \| x - x_0 \| \leq \bar{r}_0 \} \subset D_0, \]
that $F'(x_0)$ is invertible, that
\[ \| F'(x_0)^{-1} \| = \beta, \]
and that
\[ 0 < \| F'(x_0)^{-1} F(x_0) \| = \eta. \]
Then, if the inequalities
\[ 0 \leq r_0 < \frac{-1 + \sqrt{1 + 2\beta \gamma \eta}}{\beta \gamma}, \quad r_0 \leq \bar{r}_0 \]
hold, $F(x) = 0$ has no solution in the ball
\[ K_0 = \{ x \mid \| x - x_0 \| \leq r_0 \}. \]
The proof of this theorem will follow from a simple lemma.

Lemma. Let $X$ and $Y$ be Banach spaces. Let
\[ F : D \subset X \to Y \]
be Fréchet-differentiable in the open set $D_0 \subset D$. Suppose that the Fréchet-derivative
of $F$ is Lipschitz-continuous in the ball
\[ \bar{K}_0 = \{ x \mid \| x - x_0 \| \leq \bar{r}_0 \} \subset D_0 \]
with Lipschitz-constant $\gamma \geq 0$, that $F'(x_0)$ is invertible and that
\[ \| F'(x_0)^{-1} \| = \beta. \]
If the equation $F(x) = 0$ has a solution $x^*$ in the ball
\[ K_0 = \{ x \mid \| x - x_0 \| \leq r_0 \}, \quad r_0 \leq \bar{r}_0, \]
then
\[ x^* \in K_1 = \{ x \mid \| x - x_1 \| \leq r_1 \} \]
where
\[ x_1 = x_0 - F'(x_0)^{-1} F(x_0), \quad r_1 = \frac{1}{2} \beta \gamma \eta r_0^2. \]

Proof. By Taylor's theorem (Rall [3], page 124) we have
\[ F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*)) (x_0 - x^*) d\theta \]
\[ = F'(x_0) (x_0 - x^*) + \int_0^1 \left[ F'(x^* + \theta(x_0 - x^*)) - F'(x_0) \right] (x_0 - x^*) d\theta. \]
From this it follows that

\[ \|x^* - x_1\| = \left\| x^* - \left( x_0 - F'(x_0)^{-1} F(x_0) \right) \right\| \]

\[ = \left\| F'(x_0)^{-1} \int_0^1 \left[ F'(x + \theta(x_0 - x^*)) - F'(x_0) \right] (x_0 - x^*) \, d\theta \right\| \]

\[ \leq \beta \gamma \|x_0 - x^*\|^2 \int_0^1 (1 - \theta) \, d\theta \]

\[ \leq \frac{1}{2} \beta \gamma r_0^2 \]

holds, which is the assertion.

**Proof of the theorem.** If \( K_0 \) contains a solution \( x^* \) of \( F(x) = 0 \) then by the preceding lemma \( x^* \) is contained in \( K_1 \) and \( K_0 \cap K_1 = \emptyset \). Therefore if \( K_0 \cap K_1 = \emptyset \), \( F(x) = 0 \) has no solution in \( K_0 \). The equation \( K_0 \cap K_1 = \emptyset \) holds if and only if

\[ \|x_0 - x_1\| > r_0 + r_1 \]

holds. This is again the case if and only if

\[ \frac{1}{2} \beta \gamma r_0^2 + r_0 - \eta < 0 \]

holds. Because of \( r_0 \geq 0 \) this inequality is equivalent with

\[ 0 \leq r_0 < -\frac{1 + \sqrt{1 + 2\beta \gamma \eta}}{\beta \gamma} \]

and the theorem is proved.

3. **Remark**

The following example shows that in general the upper bound for \( r_0 \) cannot be replaced by a greater one: Let \( X = Y = \mathbb{R} \) be the real numbers with the usual modulus as a norm. Consider the real function

\[ F(x) = \frac{1}{2} \beta \gamma x^2 + x - \eta \]

with positive constants \( \beta, \gamma \) and \( \eta \). For \( x_0 = 0 \) we get from the theorem that for

\[ 0 \leq r_0 < -\frac{1 + \sqrt{1 + 2\beta \gamma \eta}}{\beta \gamma} \]
no solution of \( F(x) = 0 \) is contained in the interval
\[
K_0 = \{ x | |x - x_0| = |x| \leq r_0 \}.
\]
However, for
\[
r_0 = \frac{-1 + \sqrt{1 + 2\beta\gamma\eta}}{\beta\gamma}
\]
there is a solution of \( F(x) = 0 \) on the boundary of \( K_0 \).

References


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