Finite Element Heterogeneous Multiscale Method for Maxwell's Equation in Frequency Domain

Patrick Ciarlet Jr.[1], **Sonia Fliss**[1], **Christian Stohrer**[1,*]
[1]POEMS (CNRS/ENSTA ParisTech/INRIA), Palaiseau, France
*Email: christian.stohrer@ensta-paristech.fr

Abstract

To solve numerically Maxwell's equations, describing electromagnetic phenomena, one usually uses Nédélec's first family of edge finite elements. If the medium, where the electromagnetic wave is propagating through, oscillates on a microscopic length scale, this approximation becomes infeasible since the mesh must resolve all details of the medium. We propose a multiscale scheme to overcome this difficulty following the framework of the finite element heterogeneous multiscale method (FE-HMM). We prove convergence to the homogenized solution in the periodic case, and show some numerical experiments.

Keywords: Maxwell's Equations, heterogeneous multiscale methods, Nédélec finite element

1 Model problem

Let Ω be a bounded polyhedral domain of $\mathbb{R}^3$, and denote by $V := H_0(\mathrm{curl};\Omega)$ the Sobolev space of functions in $L^2(\Omega)$ whose curl is in $L^2(\Omega)$ with vanishing tangential trace on $\partial\Omega$. We assume that within Ω the electric permittivity ε^η and the inverse of the magnetic permeability ν^η are admissible quasi-periodic tensors, i.e. they are given almost everywhere by

$$\varepsilon^\eta(x) = \varepsilon(x, \eta^{-1}x) \text{ and } \nu^\eta(x) = \nu(x, \eta^{-1}x),$$

where η is a small parameter, and ε and ν are uniformly bounded and coercive matrix-valued functions, which are $Y = (-1/2, 1/2)^3$-periodic in their second variable.

We consider the model problem corresponding to Maxwell's equations in frequency domain

$$\begin{cases} \text{Find } u^\eta \in V, \text{ such that } \forall v \in V \\ B^\eta(u^\eta, v) - \omega^2 \left(u^\eta, v\right)_\eta = (f, v)\,, \end{cases} \tag{1}$$

where $(\cdot,\cdot)$ is the standard L^2 inner product, $(v,w)_\eta := (\varepsilon^\eta v, w)$, and

$$B^\eta(v,w) := (\nu^\eta \nabla\times v, \nabla\times w)\,.$$

To ensure the well-posedness of equation (1) for small η we assume that there is $\gamma > 0$ such that for a threshold value $\tilde\eta$ we have

$$\inf_{0<\eta\le\tilde\eta}\ \inf_{\lambda^\eta\in\Lambda^\eta} \left(\left|\omega^2 - \lambda^\eta\right|\right) \ge \gamma > 0.$$

Here, Λ^η denotes the discrete set of the eigenvalues of the eigenproblem associated to (1).

2 Homogenization theory

Under the assumptions mentioned above, the sequence of solutions of (1) converges weakly in $L^2(\Omega)$ as $\eta \to 0$ to u^0, the solution of the homogenized problem

$$\begin{cases} \text{Find } u^0 \in V, \text{ such that } \forall v \in V \\ B^0(u^0, v) - \omega^2 \left(u^0, v\right)_0 = (f, v)\,, \end{cases} \tag{2}$$

where $(v,w)_0 = (\mathcal{H}_{\mathrm{div}}(\varepsilon)v, w)$ and

$$B^0(v,w) = (\mathcal{H}_{\mathrm{curl}}(\nu)\nabla\times v, \nabla\times w)\,.$$

The homogenized permittivity $\mathcal{H}_{\mathrm{div}}(\varepsilon)$ and the homogenized inverse permittivity $\mathcal{H}_{\mathrm{curl}}(\nu)$ display no oscillation on the micro scale of order η. Both operators $\mathcal{H}_{\mathrm{div}}$ and $\mathcal{H}_{\mathrm{curl}}$ map a quasi-periodic tensor to its homogenized counterpart. The operator $\mathcal{H}_{\mathrm{div}}$ is the usual homogenization operator appearing in homogenization of a second order elliptic equation and given by

$$\mathcal{H}_{\mathrm{div}}(\varepsilon) = \int_Y \varepsilon(x,y)\big(I + D_y^T \chi^\varepsilon(x,y)\big)\,dy,$$

where $\chi^\varepsilon = (\chi_1^\varepsilon, \chi_2^\varepsilon, \chi_3^\varepsilon)^T$ and χ_i^ε solves

$$\begin{cases} \text{Find } \chi_i^\varepsilon : \Omega \to H^1_{\mathrm{per}}(Y), \text{ such that} \\ \displaystyle\int_Y \varepsilon(x,y)(e_i + \nabla\chi_i^\varepsilon)\cdot\nabla z\,dy = 0, \\ \text{for all } z \in H^1_{\mathrm{per}}(Y). \end{cases}$$

On the other hand, the operator $\mathcal{H}_{\mathrm{curl}}$ is designed for Maxwell's equation. It is given by

$$\mathcal{H}_{\mathrm{curl}}(\nu) = \int_Y \nu(x,y)\big(I + \nabla\times X^\nu(x,y)\big)\,dy,$$

where $X^\nu = (X_1^\nu, X_2^\nu, X_3^\nu)$ and X_i^ν solves

$$\begin{cases} \text{Find } X_i^\nu : \Omega \to H_{\text{per}}(\text{curl}, Y), \text{ such that} \\ \displaystyle\int_Y \nu(x,y)(e_i + \nabla \times X_i^\nu) \times (\nabla \times z)\, dy = 0, \\ \text{for all } z \in H_{\text{per}}(\text{curl}, Y). \end{cases}$$

Convergence of u^η to the homogenized solution u^0 can be proven using two-scale convergence, cf. [2]. The multiscale scheme proposed next reflects the specific structure of the homogenization operators involved.

3 Multiscale method

Let V_H be the finite dimensional space of lowest order Nédélec elements on a macroscopic triangulation $\mathcal{T}_H$ of Ω into simplices K. This triangulation does not need to resolve the fine scale structure, i.e. mesh sizes $H \gg \eta$ are allowed. Our FE-HMM scheme is given by

$$\begin{cases} \text{Find } u_H \in V_H, \text{ such that } \forall v_H \in V_H \\ B_H(u_H, v_H) - \omega^2 (u_H, v_H)_H = (f, v_H). \end{cases} \tag{3}$$

The FE-HMM bilinear form is defined as follows

$$B_H(v_H, w_H) = \sum_{K,j} \frac{\omega_{K,j}}{|Y_{K,j}^\eta|} \int_{Y_{K,j}^\eta} \nu^\eta(x)(\nabla \times v_h) \cdot (\nabla \times w_h)\, dx,$$

where $Y_{K,j}^\eta := x_{K,j} + \eta Y$, $(x_{K,j}, \omega_{K,j})_{j=1}^J$ are the nodes and weights of a quadrature formula on the simplex K and v_h (resp. w_h) are the FE solution of

$$\begin{cases} \nabla \times \big(\nu^\eta(x)(\nabla \times v_h)\big) = 0 \text{ in } Y_{K,j}^\eta, \\ \nabla \cdot v_h = 0 \text{ in } Y_{K,j}^\eta, \ \displaystyle\int_{Y_{K,j}^\eta} v_h\, dx = 0, \\ v_h(x) - v_{H,\text{curl}}(x) \text{ is } \eta Y\text{-periodic.} \end{cases}$$

We use the linear function $v_{H,\text{curl}}$ given by

$$v_{H,\text{curl}}(x) = v_H(x_{K,j}) + \frac{1}{2}\big(\nabla \times v_H(x_{K,j})\big) \times (x - x_{K,j}).$$

to couple the macro and micro scales. By construction, $v_{H,\text{curl}}(x) = v_H(x_{K,j})$ and

$$\nabla \times v_{H,\text{curl}}(x_{K,j}) = \nabla \times v_H(x_{K,j}),$$

which are the important properties.

The FE-HMM scalar product is given by

$$(v_H, w_H)_H = \sum_{K,j} \frac{\omega_{K,j}}{|Y_{K,j}^\eta|} \int_{Y_{K,j}^\eta} \varepsilon^\eta(x)(\nabla\varphi_h) \cdot (\nabla\psi_h)\, dx,$$

where φ_h (resp. ψ_h) are the FE solution of

$$\begin{cases} -\nabla \cdot \big(\varepsilon^\eta(x)(\nabla\varphi_h)\big) = 0 \text{ in } Y_{K,j}^\eta, \\ \displaystyle\int_{Y_{K,j}^\eta} \varphi_h\, dx = 0, \\ \varphi_h(x) - v_H(x_{K,j}) \cdot (x - x_{K,j}) \ \eta Y\text{-periodic.} \end{cases}$$

The FE-HMM scalar product is closely related to standard FE-HMM schemes, see e.g. [1]. In contrast, the use of curl-curl micro problems for the FE-HMM bilinear form is a novelty.

4 A priori error estimate

The following theorem can be proven combining classical arguments of numerical analysis with the discretized version of T-coercivity [3].

Theorem. *Under sufficient regularity assumptions, we have for H small enough*

$$\|u^0 - u_H\|_V \lesssim \inf_{v_H \in V_H} \Bigg(\|u^0 - v_H\|_V + \sup_{w_H \in V_H \setminus \{0\}} \frac{|B^0(v_H, w_H) - B_H(v_H, w_H)|}{\|w_H\|_V} + \sup_{w_H \in V_H \setminus \{0\}} \frac{|(v_H, w_H)_0 - (v_H, w_H)_H|}{\|w_H\|_V} \Bigg).$$

The consistency error terms on the second and third line can be bounded further.

Acknowledgement

This work is financially supported by the ANR project METAMATH (ANR-11-MONU-016) and the SNSF (Early Postdoc.Mobility 151906).

References

[1] A. Abdulle, W. E, B. Engquist, and E. Vanden-Eijnden, The Heterogeneous Multiscale Method, *Acta Numer.* **21** (2012), pp. 1–87

[2] N. Wellander and G. Kristensson, Homogenization of the Maxwell equations at fixed frequency, *SIAM J. Appl. Math.* **64** (2003), pp. 170–195.

[3] P. Ciarlet Jr., T-coercivity: Application to the discretization of Helmholtz-like problems, *Comput. Math. Appl.* **64** (2012), pp. 22–34.