Finite Element Heterogeneous Multiscale Method for the Wave Equation: Long-Time Effects

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Abstract

For limited time the propagation of waves in a highly oscillatory medium is well-described by the non-dispersive homogenized wave equation. With increasing time, however, the true solution deviates from the classical homogenization limit, as a large secondary wave train develops unexpectedly. Here, we propose a new finite element heterogeneous multiscale method (FE-HMM), which captures not only the short-time macroscale behavior of the wave field but also those secondary long-time dispersive effects.

1 Long-Time Wave Propagation

Let $\Omega \subset \mathbb{R}^d$ be a domain and $T > 0$. We consider the wave equation

$$\begin{cases} \partial_{tt} u^\varepsilon - \nabla \cdot (a^\varepsilon \nabla u^\varepsilon) = F & \text{in } \Omega \times (0,T), \\ u^\varepsilon(x,0) = f(x) & \text{in } \Omega, \\ \partial_t u^\varepsilon(x,0) = g(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $a^\varepsilon(x) \in (L^\infty(\Omega))^{d\times d}$ is symmetric, uniformly elliptic, and bounded. Here $\varepsilon > 0$ represents a small scale in the problem, which characterizes the multiscale nature of the tensor $a^\varepsilon(x)$. We set either homogeneous Dirichlet or periodic boundary conditions to uniquely determine the solution for every $\varepsilon > 0$.

1.1 Classical homogenization

According to classical homogenization theory, u^ε converges to the solution u^0 of the "homogenized" wave equation as $\varepsilon \to 0$,

$$\partial_{tt} u^0 - \nabla \cdot (a^0 \nabla u^0) = F,$$

where the homogenized tensor (or squared velocity field) a^0 can only rarely be computed explicitly. Thus, u^0 approximates u^ε but only for short times. For longer times $T \sim \varepsilon^{-2}$, the homogenized solution becomes increasingly inadequate, since it neglects microscopic dispersive effects that accumulate over time, as shown in Figure 1. Here we consider (1) in $\Omega = (-1, 1)$ with periodic boundary conditions, let $u(x, 0)$ be a Gaussian pulse with zero initial velocity

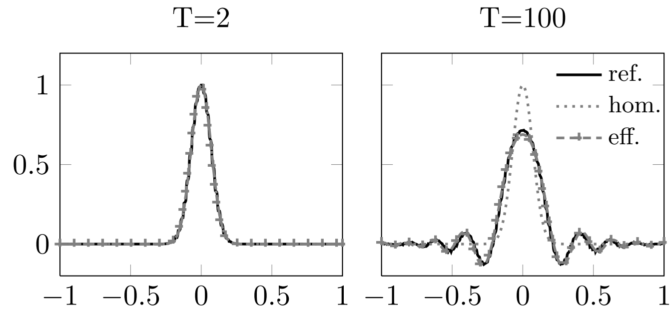

Figure 1: Reference (ref.), homogenized (hom.) and effective (eff.) solution: short-time (left) and long-time (right).

and set

$$a^\varepsilon = \sqrt{2} + \sin\Big(2\pi \frac{x}{\varepsilon}\Big) \ \text{ with } \varepsilon = \frac{1}{50}. \tag{2}$$

The reference solution of (1)–(2) corresponds to a direct numerical simulation (DNS), where the microscale is fully resolved. After one revolution ($T = 2$), the homogenized and the DNS solution coincide. After fifty revolutions ($T = 100$), however, the DNS displays dispersive effects, which the homogenized solution fails to capture.

1.2 Effective dispersive equation

Various formal asymptotic arguments were derived to elucidate that peculiar inherently dispersive long-time behavior of waves propagating through a strongly heterogeneous periodic medium [1]. An effective equation that captures those dispersive effects was recently derived in [2] for the one-dimensional case when a^ε is ε-periodic:

$$\partial_{tt}(u^{\text{eff}} - \varepsilon^2 b \partial_{xx} u^{\text{eff}}) - a^0 \partial_{xx} u^{\text{eff}} = F. \tag{3}$$

Again, a^0 denotes the homogenized effective coefficient from classical homogenization theory and $b > 0$. As shown in Figure 1, u^ε and u^{eff} essentially coincide both at early and later times.

2 FE Heterogeneous Multiscale Method

In [3], the FE-HMM for elliptic [4] was extended to the time dependent wave equation. It was shown to converge to u^0 at finite times, yet it failed to capture long-time dispersive effects in the true solution. To incorporate those dispersive effects, we not only need an effective bilinear form but we add a correction to the L^2 inner product, akin to the weak formulation of (3). Similarly to the computation of the bilinear form, the correction relies on numerical solutions of micro problems on sampling domains K_δ of size δ proportional to ε. An alternative HMM scheme, based on the finite difference approximation of an effective flux, was proposed in [5].

We now give a description of the algorithm: First, we generate a macro triangulation $\mathcal{T}_H$ and choose an appropriate macro FE space $S(\Omega, \mathcal{T}_H)$. By macro we mean that $H \gg \varepsilon$ is allowed. Within each macro element $K \in \mathcal{T}_H$ we choose a quadrature formula $\{x_{K,j}, \omega_{K,j}\}$. The FE-HMM solution u_H is given by the following variational problem:

$$\begin{cases} \text{Find } u_H : [0,T] \to S(\Omega, \mathcal{T}_H) \text{ such that} \\ (\partial_{tt} u_H, v_H)_Q + B_H(u_H, v_H) = (F, v_H) \\ \text{for all } v_H \in S(\Omega, \mathcal{T}_H) \text{ and,} \\ \quad u_H(0) = f_H,\ \partial_t u_H(0) = g_H \text{ in } \Omega, \end{cases} \tag{4}$$

where the initial data f_H and g_h are suitable approximations of f and g in $S(\Omega, \mathcal{T}_H)$. The effective bilinear form B_H and $(\cdot,\cdot)_Q$ are defined as follows:

$$B_H(v_H, w_H) = \sum_{K,j} \frac{\omega_{K,j}}{|K_\delta|} \int_{K_\delta} a^\varepsilon(x) \nabla v_h(x) \cdot \nabla w_h(x) dx,$$

and

$$(v_H, w_H)_Q = (v_H, w_H) + \sum_{K,j} \frac{\omega_{K,j}}{|K_\delta|} \int_{K_\delta} (v_h(x) - v_{H,\mathrm{lin}}(x))(w_h(x) - w_{H,\mathrm{lin}}(x)) dx.$$

In the above, the micro solution v_h (resp. w_h) is given by

$$\begin{cases} \text{Find } v_h \text{ such that } (v_h - v_{H,\mathrm{lin}}) \in S(K_\delta, \mathcal{T}_h) \text{ and} \\ \displaystyle\int_{K_\delta} a^\varepsilon(x) \nabla v_h(x) \cdot \nabla z_h(x) dx = 0, \\ \text{for all } z_h \in S(K_\delta, \mathcal{T}_h). \end{cases}$$

Here $S(K_\delta, \mathcal{T}_h)$ is a micro FE space on the sampling domain K_δ with micro triangulation $\mathcal{T}_h$, and $v_{H,\mathrm{lin}}$ denotes the linearization of v_H at the quadrature point $x_{K,j}$, Since B_H is elliptic and bounded and $(\cdot,\cdot)_Q$ is a true inner product, the FE-HMM is well-defined for all $H, h > 0$. It can be shown that the correction of the L^2 inner product is of order ε^2 in agreement with (3).

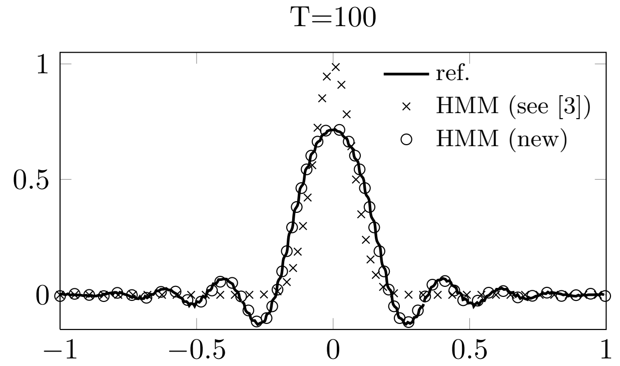

Figure 2: Reference solution (ref.), FE-HMM from [3] and new FE-HMM.

3 Numerical Experiments

We again apply our FE-HMM, defined in (4), to (1)–(2) as in Figure 1. We use cubic FE at the macro- and the micro-scale, with mesh sizes $H = 1/75$ and $h = \varepsilon/20 = 1/1000$. Note that linear or quadratic finite elements could also be used. For time-stepping we use a standard Leap-Frog scheme, with $\Delta t = H/10$. As shown in Figure 2, the new FE-HMM succeeds in capturing, the long-time effects in the true solution. In contrast, the solution of the FE-HMM of [3] without correction is unable to capture those dispersive effects, since this solution was proven to converge to the homogenized solution, u^0, as $\varepsilon \to 0$.

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