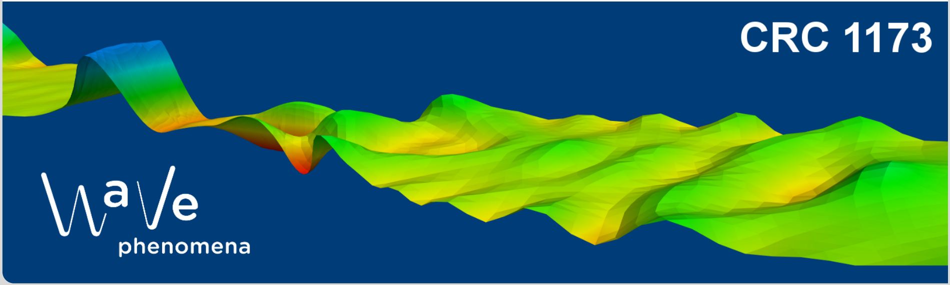
Karlsruhe Institute of Technology

Stability of leap-frog type methods

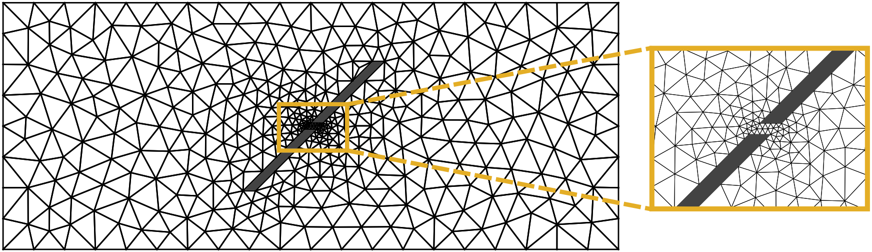
Andreas, Marlis, Michaela

Institute for Applied and Numerical Mathematics, Karlsruhe Institute of Technology



Problem setting

Solve Maxwells' equations on domains which require a **locally refined grid**



Avoid restrictive CFL condition

Semidiscrete (central fluxes dG in space)

Karlsruhe Institute of Technology

Maxwell's equations in 2nd order form:

$$\partial_t^2 \mathbf{E}(t) = -\mathbf{L}\mathbf{E}(t), \qquad \mathbf{L} = \mathbf{C_H}\mathbf{C_E}, \qquad \text{initial values,}$$

where **L symmetric, positive semi-definite**, i.e.

$$\big(\mathbf{C_H}\mathbf{H}, \mathbf{E}\big) = \big(\mathbf{H}, \mathbf{C_E}\mathbf{E}\big), \qquad \big(\mathbf{L}\mathbf{E}, \mathbf{E}\big) = \|\mathbf{C_E}\mathbf{E}\|^2 \geq 0.$$

- **Energy technique**

$$\frac{1}{2}\frac{d}{dt}\|\partial_t \mathbf{E}(t)\|^2 = \big(\partial_t^2 \mathbf{E}(t), \partial_t \mathbf{E}(t)\big) = -\big(\mathbf{L}\mathbf{E}(t), \partial_t \mathbf{E}(t)\big) = -\frac{1}{2}\frac{d}{dt}\|\mathbf{C_E}\mathbf{E}(t)\|^2$$

- **Stability in energy norm**

$$|||\mathbf{E}(t)|||^2 := \|\partial_t \mathbf{E}(t)\|^2 + \|\mathbf{C_E}\mathbf{E}(t)\|^2 \equiv \|\partial_t \mathbf{E}(0)\|^2 + \|\mathbf{C_E}\mathbf{E}(0)\|^2$$

Time integration with leap frog

Semidiscrete: $$\partial_t^2 \mathbf{E} = -\mathbf{L}\mathbf{E}$$

Leap frog: $$\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^n$$

- Difference and mean value

$$\Delta^n = \mathbf{E}^{n+1} - \mathbf{E}^n, \qquad \mu^n = \frac{1}{2}(\mathbf{E}^{n+1} + \mathbf{E}^n)$$

- LHS:

$$\begin{aligned}&\left(\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1}, \mathbf{E}^{n+1} - \mathbf{E}^{n-1}\right)\\ &= \left(\Delta^n - \Delta^{n-1}, \Delta^n + \Delta^{n-1}\right) = \|\Delta^n\|^2 - \|\Delta^{n-1}\|^2\end{aligned}$$

- RHS:

$$\begin{aligned}&\left(\mathbf{E}^n, \mathbf{E}^{n+1} - \mathbf{E}^{n-1}\right)\\ &= \left(\mu^n - \frac{1}{2}\Delta^n, \mu^n + \frac{1}{2}\Delta^n\right) - \left(\mu^{n-1} + \frac{1}{2}\Delta^{n-1}, \mu^{n-1} - \frac{1}{2}\Delta^{n-1}\right)\\ &= \|\mu^n\|^2 - \frac{1}{4}\|\Delta^n\|^2 - \|\mu^{n-1}\|^2 + \frac{1}{4}\|\Delta^{n-1}\|^2\end{aligned}$$

Conserved quantity

Karlsruhe Institute of Technology

Semidiscrete: $\partial_t^2 \mathbf{E} = -\mathbf{L}\mathbf{E}$

- Energy: $|||\mathbf{E}(t)|||^2 = \|\partial_t \mathbf{E}(t)\|^2 + \|\mathbf{C_E}\mathbf{E}(t)\|^2$

Leap frog: $\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^n, \qquad \mathbf{L} = \mathbf{C_H}\mathbf{C_E}$

- Identities:

$$\left(\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1}, \mathbf{E}^{n+1} - \mathbf{E}^{n-1}\right) = \|\Delta^n\|^2 - \|\Delta^{n-1}\|^2$$

$$\left(\mathbf{E}^n, \mathbf{E}^{n+1} - \mathbf{E}^{n-1}\right) = \|\mu^n\|^2 - \frac{1}{4}\|\Delta^n\|^2 - \|\mu^{n-1}\|^2 + \frac{1}{4}\|\Delta^{n-1}\|^2$$

- Conserved quantity:

$$\mathcal{M}_{\mathrm{LF}}^n = \|\Delta^n\|^2 + \tau^2 \|\mathbf{C_E}\mu^n\|^2 - \frac{\tau^2}{4}\|\mathbf{C_E}\Delta^n\|^2$$

$$= |||(\mathbf{E}^{n+1}, \mathbf{E}^n)|||_\tau^2 - \frac{\tau^2}{4}\|\mathbf{C_E}\Delta^n\|^2$$

$$\Delta^n = \mathbf{E}^{n+1} - \mathbf{E}^n, \qquad \mu^n = \frac{1}{2}(\mathbf{E}^{n+1} + \mathbf{E}^n)$$

Stability

Leap frog: $\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^n$

- Conserved quantity:

$$\mathcal{M}_{\mathrm{LF}}^n = \|\Delta^n\|^2 + \tau^2 \|\mathbf{C}_{\mathbf{E}}\mu^n\|^2 - \frac{\tau^2}{4}\|\mathbf{C}_{\mathbf{E}}\Delta^n\|^2 = \mathcal{M}_{\mathrm{LF}}^0$$

- **CFL condition**: $\tau^2 \|\mathbf{C}_{\mathbf{E}}\|^2 \leq 4\theta^2, \qquad \theta \in (0,1)$
- **Stability**:

$$\begin{aligned}
(1-\theta^2)|||(\mathbf{E}^{n+1}, \mathbf{E}^n)|||_\tau^2 &\leq (1-\theta^2)\|\Delta^n\|^2 + \tau^2\|\mathbf{C}_{\mathbf{E}}\mu^n\|^2 \\
&\leq \|\Delta^n\|^2 + \tau^2\|\mathbf{C}_{\mathbf{E}}\mu^n\|^2 - \frac{\tau^2}{4}\|\mathbf{C}_{\mathbf{E}}\Delta^n\|^2 \\
&= \mathcal{M}_{\mathrm{LF}}^n \\
&= \mathcal{M}_{\mathrm{LF}}^0 \\
&\leq \|\Delta^0\|^2 + \tau^2\|\mathbf{C}_{\mathbf{E}}\mu^0\|^2 = |||(\mathbf{E}^1, \mathbf{E}^0)|||_\tau^2
\end{aligned}$$

Relaxing the CFL condition

Karlsruhe Institute of Technology

Leap frog: $\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^n$

- Conserved quantity:

$$\mathcal{M}_{\mathrm{LF}}^n = |||(\mathbf{E}^{n+1}, \mathbf{E}^n)|||_\tau^2 - \frac{\tau^2}{4}\|\mathbf{C}_\mathbf{E}\Delta^n\|^2$$

- Identity:

$$\left(\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1}, \mathbf{E}^{n+1} - \mathbf{E}^{n-1}\right) = \|\Delta^n\|^2 - \|\Delta^{n-1}\|^2$$

Idea:

$$\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^n - \frac{\tau^2}{4}\mathbf{L}\left(\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1}\right)$$

- Conserved quantity:

$$\mathcal{M}_{\mathrm{CN}}^n = |||(\mathbf{E}^{n+1}, \mathbf{E}^n)|||_\tau^2 - \frac{\tau^2}{4}\|\mathbf{C}_\mathbf{E}\Delta^n\|^2 + \frac{\tau^2}{4}\|\mathbf{C}_\mathbf{E}\Delta^n\|^2$$

Relaxing the CFL condition

Karlsruhe Institute of Technology

Leap frog: $$\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^n$$

- Conserved quantity:

$$\mathcal{M}^n_{\mathrm{LF}} = |||(\mathbf{E}^{n+1}, \mathbf{E}^n)|||^2_\tau - \frac{\tau^2}{4}\|\mathbf{C}_\mathbf{E}\Delta^n\|^2$$

- Identity:

$$\left(\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1}, \mathbf{E}^{n+1} - \mathbf{E}^{n-1}\right) = \|\Delta^n\|^2 - \|\Delta^{n-1}\|^2$$

Idea:

$$\begin{aligned}\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} &= -\tau^2 \mathbf{L}\mathbf{E}^n - \frac{\tau^2}{4}\mathbf{L}\left(\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1}\right)\\ &= -\frac{\tau^2}{4}\mathbf{L}\left(\mathbf{E}^{n+1} + 2\mathbf{E}^n + \mathbf{E}^{n-1}\right) \qquad \text{Crank–Nicolson}\end{aligned}$$

- Conserved quantity:

$$\mathcal{M}^n_{\mathrm{CN}} = |||(\mathbf{E}^{n+1}, \mathbf{E}^n)|||^2_\tau$$

Relaxing the CFL condition

Leap frog: $$\mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^{n}$$

- Conserved quantity:

$$\mathcal{M}_{\mathrm{LF}}^{n} = |||(\mathbf{E}^{n+1}, \mathbf{E}^{n})|||_{\tau}^{2} - \frac{\tau^2}{4} \|\mathbf{C}_{\mathbf{E}} \Delta^{n}\|^2$$

- Identity:

$$\left(\mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1}, \mathbf{E}^{n+1} - \mathbf{E}^{n-1}\right) = \|\Delta^{n}\|^2 - \|\Delta^{n-1}\|^2$$

Idea:

$$\mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^{n} - \frac{\tau^2}{4} \mathbf{L}^{i} \left(\mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1}\right) \qquad \mathbf{L}^{i} = \mathbf{C}_{\mathbf{H}} \chi_i \mathbf{C}_{\mathbf{E}}$$

- Conserved quantity:

$$\mathcal{M}_{\mathrm{LI}}^{n} = |||(\mathbf{E}^{n+1}, \mathbf{E}^{n})|||_{\tau}^{2} - \frac{\tau^2}{4} \|\mathbf{C}_{\mathbf{E}} \Delta^{n}\|^2 + \frac{\tau^2}{4} \|\chi_i \mathbf{C}_{\mathbf{E}} \Delta^{n}\|^2$$

Relaxing the CFL condition

Leap frog: $\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^n$

- Conserved quantity:

$$\mathcal{M}_{\mathrm{LF}}^n = |||(\mathbf{E}^{n+1}, \mathbf{E}^n)|||_\tau^2 - \frac{\tau^2}{4} \|\mathbf{C_E}\Delta^n\|^2$$

- Identity:

$$\left(\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1}, \mathbf{E}^{n+1} - \mathbf{E}^{n-1}\right) = \|\Delta^n\|^2 - \|\Delta^{n-1}\|^2$$

Idea:

$$\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^n - \frac{\tau^2}{4}\mathbf{L}^i\left(\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1}\right) \qquad \mathbf{L}^i = \mathbf{C_H}\chi_i\mathbf{C_E}$$

$$= -\tau^2 \mathbf{L}^e\mathbf{E}^n - \frac{\tau^2}{4}\mathbf{L}^i\left(\mathbf{E}^{n+1} + 2\mathbf{E}^n + \mathbf{E}^{n-1}\right) \qquad \text{locally implicit}$$

- Conserved quantity:

$$\mathcal{M}_{\mathrm{LI}}^n = |||(\mathbf{E}^{n+1}, \mathbf{E}^n)|||_\tau^2 - \frac{\tau^2}{4} \|\chi_e\mathbf{C_E}\Delta^n\|^2$$

Summary implicit methods

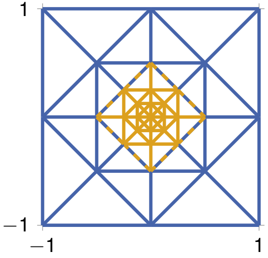
Leap frog: $\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^n$

- **implicit methods:**

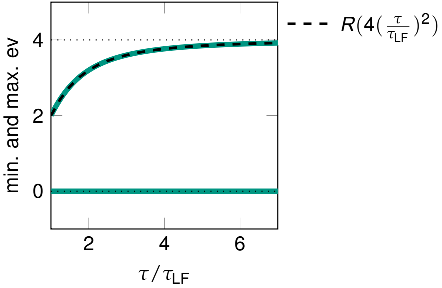
  Crank–Nicolson: $\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} = -R(\tau^2 \mathbf{L})\mathbf{E}^n,$

  locally implicit:
$$\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} = -\left(\mathbf{I} + \frac{\tau^2}{4}\mathbf{L}^i\right)^{-1}(\tau^2\mathbf{L})\mathbf{E}^n = -\tau^2\mathbf{L}^e\mathbf{E}^n - R(\tau^2\mathbf{L}^i)\mathbf{E}^n$$

with **rational** function: $R(z) = z/(1 + \frac{z}{4})$



refinement explicit-implicit: 4



Summary implicit methods

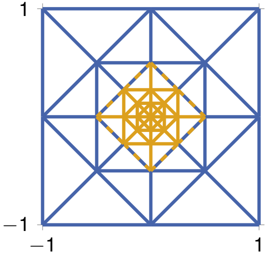
Leap frog: $\mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^{n}$

- **implicit methods:**

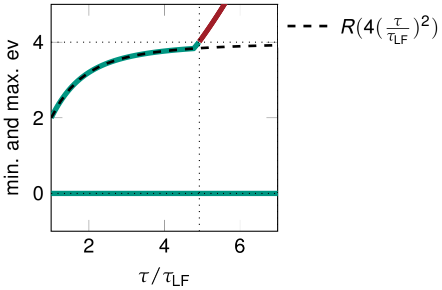
  Crank–Nicolson: $\mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1} = -R(\tau^2 \mathbf{L})\mathbf{E}^{n},$

  locally implicit:
  $$\mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1} = -\left(\mathbf{I} + \frac{\tau^2}{4}\mathbf{L}^{i}\right)^{-1}(\tau^2 \mathbf{L})\mathbf{E}^{n}$$
  $$= -\tau^2 \mathbf{L}^{e}\mathbf{E}^{n} - R(\tau^2 \mathbf{L}^{i})\mathbf{E}^{n}$$

with **rational** function: $R(z) = z/(1 + \frac{z}{4})$



refinement explicit-implicit: 4



Leap-frog-Chebyshev methods

Karlsruhe Institute of Technology

Leap frog: $\mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^{n}$

- **explicit methods:**

Polynomial P_p : $\mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1} = -P_p(\tau^2 \mathbf{L})\mathbf{E}^{n},$

$p = 2$: $\mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^{n} + \lambda\tau^4 \mathbf{L}^2\mathbf{E}^{n}$

- Conserved quantity ($p = 2$):

$$\mathcal{M}_2 = \|\Delta\|^2 + \tau^2\|\mathbf{C_E}\mu\|^2 - \frac{\tau^2}{4}\|\mathbf{C_E}\Delta\|^2 - \lambda\tau^4\|\mathbf{C_H}\mathbf{C_E}\mu\|^2 + \lambda\frac{\tau^4}{4}\|\mathbf{C_H}\mathbf{C_E}\Delta\|^2$$

Leap-frog-Chebyshev methods

Karlsruhe Institute of Technology

Leap frog: $\mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^{n}$

- **explicit methods:**

  Polynomial P_p : $\mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1} = -P_p(\tau^2\mathbf{L})\mathbf{E}^{n}$,

  $p = 2$: $\mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^{n} + \lambda\tau^4\mathbf{L}^2\mathbf{E}^{n}$

- Conserved quantity ($p = 2$):

$$\mathcal{M}_2 = \|\Delta\|^2 \quad + \tau^2\|\mathbf{C_E}\mu\|^2 - \frac{\tau^2}{4}\|\mathbf{C_E}\Delta\|^2 \quad - \lambda\tau^4\|\mathbf{C_H}\mathbf{C_E}\mu\|^2 + \lambda\frac{\tau^4}{4}\|\mathbf{C_H}\mathbf{C_E}\Delta\|^2$$

$$= \|\Delta - \frac{\tau^2}{8}\mathbf{C_H}\mathbf{C_E}\Delta\|^2 + \left(\frac{\lambda}{4} - \frac{1}{64}\right)\tau^4\|\mathbf{C_H}\mathbf{C_E}\Delta\|^2$$

$$+\tau^2\|\mathbf{C_E}\mu\|^2 - \lambda\tau^4\|\mathbf{C_H}\mathbf{C_E}\mu\|^2$$

- **CFL condition:** $\lambda\tau^2\|\mathbf{C_H}\|^2 \leq \theta^2$, for $\lambda = \frac{1}{16}$: $\tau^2\|\mathbf{C_H}\|^2 \leq 4 \cdot 4\theta^2$

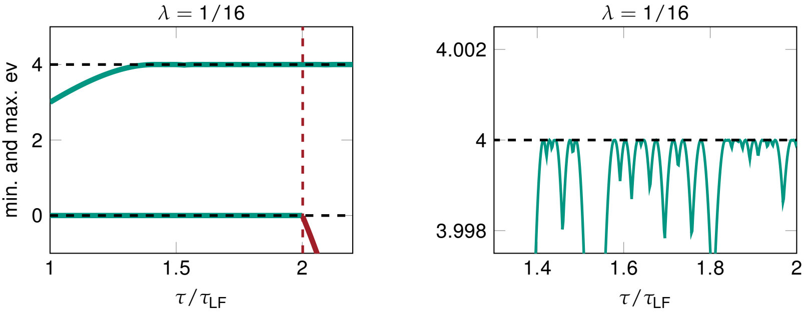
Leap-frog-Chebyshev methods

$p = 2$: $\mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^{n} + \lambda\tau^4\mathbf{L}^2\mathbf{E}^{n}$

- Conserved quantity ($p = 2$):

$$\mathcal{M}_2 = \|\Delta - \frac{\tau^2}{8}\mathbf{C_E}\Delta\|^2 + \left(\frac{\lambda}{4} - \frac{1}{64}\right)\tau^4\|\mathbf{C_H}\mathbf{C_E}\Delta\|^2 + \tau^2\|\mathbf{C_E}\mu\|^2 - \lambda\tau^4\|\mathbf{C_H}\mathbf{C_E}\mu\|^2$$

- **CFL condition:** $\lambda\tau^2\|\mathbf{C_H}\|^2 \leq \theta^2$, for $\lambda = \frac{1}{16}$: $\tau^2\|\mathbf{C_H}\|^2 \leq 4 \cdot 4\theta^2$



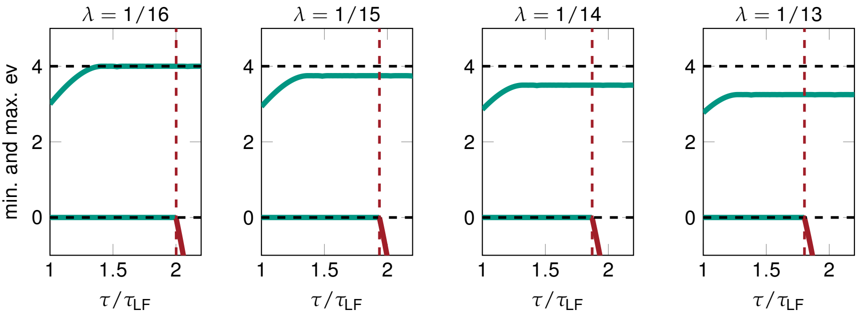
Leap-frog-Chebyshev methods

$p = 2$: $$\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^n + \lambda\tau^4\mathbf{L}^2\mathbf{E}^n$$

- Conserved quantity ($p = 2$):

$$\mathcal{M}_2 = \|\Delta - \frac{\tau^2}{8}\mathbf{C_E}\Delta\|^2 + \left(\frac{\lambda}{4} - \frac{1}{64}\right)\tau^4\|\mathbf{C_H}\mathbf{C_E}\Delta\|^2 + \tau^2\|\mathbf{C_E}\mu\|^2 - \lambda\tau^4\|\mathbf{C_H}\mathbf{C_E}\mu\|^2$$

- **CFL condition:** $\lambda\tau^2\|\mathbf{C_H}\|^2 \leq \theta^2$



Local time stepping

Leap frog: $\mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^{n}$

LTS ($p = 2$): $\mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^{n} + \frac{\tau^4}{16}\mathbf{L}\chi_i\mathbf{L}\mathbf{E}^{n}$

- Conserved quantity:

$$\mathcal{M}_2 = \|\Delta\|^2 + \tau^2\|\mathbf{C}_\mathbf{E}\mu\|^2 - \frac{\tau^2}{4}\|\mathbf{C}_\mathbf{E}\Delta\|^2 - \frac{\tau^4}{16}\|\chi_i\mathbf{C}_\mathbf{H}\mathbf{C}_\mathbf{E}\mu\|^2 + \frac{\tau^4}{64}\|\chi_i\mathbf{C}_\mathbf{H}\mathbf{C}_\mathbf{E}\Delta\|^2$$

$$= \|\chi_e\Delta\|^2 - \frac{\tau^2}{4}\|\chi_e\mathbf{C}_\mathbf{E}\Delta\|^2 \quad ✗$$

$$+ \|\chi_i\Delta\|^2 - \frac{\tau^2}{4}\|\chi_i\mathbf{C}_\mathbf{E}\Delta\|^2 + \frac{\tau^4}{64}\|\chi_i\mathbf{C}_\mathbf{H}\mathbf{C}_\mathbf{E}\Delta\|^2 \quad ✓$$

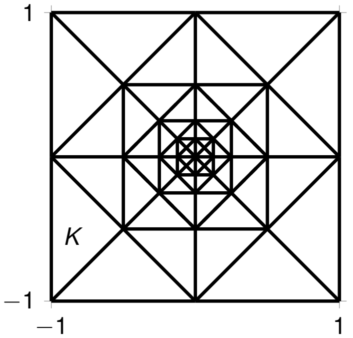
$$+ \tau^2\|\mathbf{C}_\mathbf{E}\mu\|^2 - \frac{\tau^4}{16}\|\chi_i\mathbf{C}_\mathbf{H}\mathbf{C}_\mathbf{E}\mu\|^2 \quad ✓$$

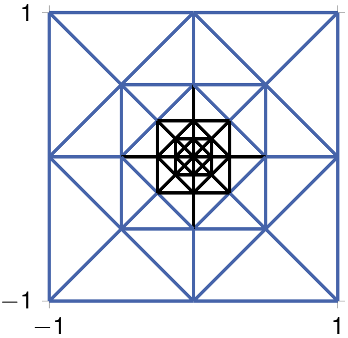
Discrete curl-operator

We have

$$(\mathbf{C}_\mathbf{E}\mathbf{E}, \varphi) = \sum_K (\operatorname{curl}\mathbf{E}, \varphi)_K + \sum_F ([\![\mathbf{E}]\!]^t, \{\!\{\varphi\}\!\})_F + \text{boundary terms}$$

$$(\chi_e\mathbf{C}_\mathbf{E}\mathbf{E}, \varphi) = \sum_{K_e} (\operatorname{curl}\mathbf{E}, \varphi)_K + \sum_F ([\![\mathbf{E}]\!]^t, \{\!\{\chi_e\varphi\}\!\})_F + \text{boundary terms}$$



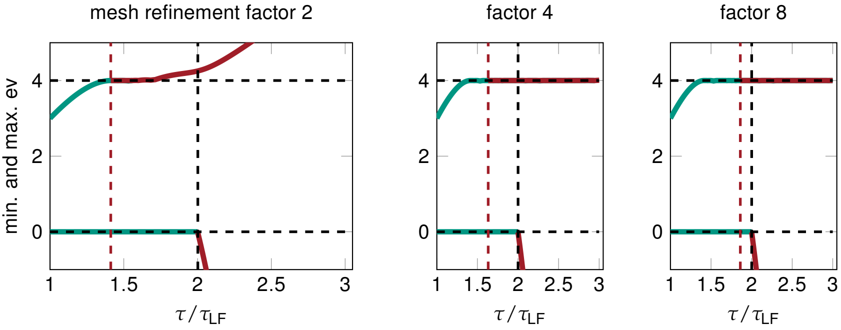


Local time stepping

LTS ($p = 2$): $$\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^n + \frac{\tau^4}{16}\mathbf{L}\chi_i\mathbf{L}\mathbf{E}^n$$

- Conserved quantity:

$$\mathcal{M}_2 = \|\chi_e \Delta\|^2 - \frac{\tau^2}{4}\|\chi_e \mathbf{C}_\mathbf{E}\Delta\|^2 + \dots \checkmark$$



Damping

Karlsruhe Institute of Technology

Leap frog: $$\mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^{n}$$

$p = 2$, damped ($\nu > 1$): $$\mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^{n} + \nu\frac{\tau^4}{16}\mathbf{L}^2\mathbf{E}^{n}$$

- Conserved quantity:

$$\mathcal{M}_2 = \|\Delta\|^2 + \tau^2\|\mathbf{C_E}\mu\|^2 - \frac{\tau^2}{4}\|\mathbf{C_E}\Delta\|^2 - \frac{\tau^4}{16}\nu\|\mathbf{C_H}\mathbf{C_E}\mu\|^2 + \frac{\tau^4}{64}\nu\|\mathbf{C_H}\mathbf{C_E}\Delta\|^2$$

$$= \left(1 - \frac{1}{\nu}\right)\|\Delta\|^2 + \left\|\frac{1}{\sqrt{\nu}}\Delta - \frac{\sqrt{\nu}\tau^2}{8}\mathbf{C_E}\Delta\right\|^2 + \tau^2\|\mathbf{C_E}\mu\|^2 - \frac{\tau^4}{16}\nu\|\mathbf{C_H}\mathbf{C_E}\mu\|^2$$

- **CFL condition:** $$\tau^2\|\mathbf{C_H}\|^2 \le \frac{4}{\nu} \cdot 4\theta^2$$

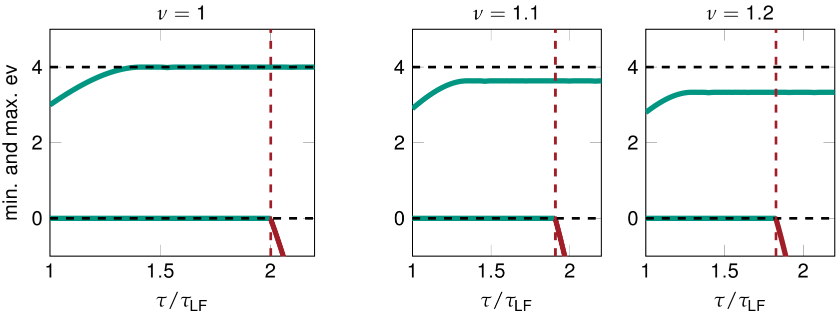
Damping

$p = 2$, damped ($\nu > 1$): $\quad \mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^n + \nu \frac{\tau^4}{16} \mathbf{L}^2 \mathbf{E}^n$

- Conserved quantity:

$$\mathcal{M}_2 = \left(1 - \frac{1}{\nu}\right) \|\Delta\|^2 + \cdots + \tau^2 \|\mathbf{C_E}\mu\|^2 - \frac{\tau^4}{16} \nu \|\mathbf{C_H}\mathbf{C_E}\mu\|^2$$

- **CFL condition:** $\quad \tau^2 \|\mathbf{C_H}\|^2 \leq \frac{4}{\nu} \cdot 4\theta^2$



Damped local time stepping

Leap frog: $$\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^n$$

Damped LTS ($p = 2$): $$\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^n + \nu \frac{\tau^4}{16} \mathbf{L}\chi_i \mathbf{L}\mathbf{E}^n$$

- Conserved quantity:

$$\begin{aligned}
\mathcal{M}_2 = & \|\chi_e \Delta\|^2 - \frac{\tau^2}{4} \|\chi_e \mathbf{C_E} \Delta\|^2 \quad ✗ \\
& + \|\chi_i \Delta\|^2 - \frac{\tau^2}{4} \|\chi_i \mathbf{C_E} \Delta\|^2 + \nu \frac{\tau^4}{64} \|\chi_i \mathbf{C_H} \mathbf{C_E} \Delta\|^2 \quad ✓ \\
& + \tau^2 \|\mathbf{C_E} \mu\|^2 - \nu \frac{\tau^4}{16} \|\chi_i \mathbf{C_H} \mathbf{C_E} \mu\|^2 \quad ✓ \\
= & \|\chi_e \Delta\|^2 - \frac{\tau^2}{4} \|\chi_e \mathbf{C_E} \Delta\|^2 \quad ✓ \\
& + \left(1 - \frac{1}{\nu}\right) \|\chi_i \Delta\|^2 + \|\ldots\|^2 \quad ✓ \\
& + \ldots \quad ✓
\end{aligned}$$

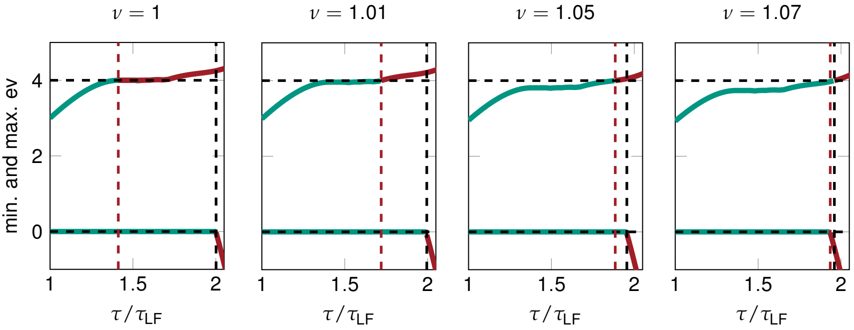
Damped local time stepping

Damped LTS ($p = 2$):

$$\mathbf{E}^{n+1} - 2\mathbf{E}^{n} + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^n + \nu \frac{\tau^4}{16} \mathbf{L}\chi_i \mathbf{L}\mathbf{E}^n$$

- Conserved quantity:

$$\mathcal{M}_2 = \|\chi_e \Delta\|^2 - \frac{\tau^2}{4}\|\chi_e \mathbf{C}_\mathbf{E} \Delta\|^2 + \left(1 - \frac{1}{\nu}\right)\|\chi_i \Delta\|^2 + \dots$$



Outlook

Leap frog:

$$\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} = -\tau^2 \mathbf{L}\mathbf{E}^n$$

leap-frog-Chebyshev:

$$\begin{aligned}\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} &= -P_p(\tau^2 \mathbf{L})\mathbf{E}^n \\ &= -\big(\mathbf{I} + \widetilde{Q}_{p-1}(\tau^2\mathbf{L})\big)\tau^2\mathbf{L}\mathbf{E}^n \\ &= -\tau^2\mathbf{L}\mathbf{E}^n - Q_p(\tau^2\mathbf{L})\mathbf{E}^n\end{aligned}$$

LTS:

$$\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} = -\big(\mathbf{I} + \widetilde{Q}_{p-1}(\tau^2\mathbf{L}\chi_i)\big)\tau^2\mathbf{L}\mathbf{E}^n$$

stability ? – ✗, consistency ✓

multi-rate leap-frog-Chebyshev:

$$\begin{aligned}\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1} &= -\tau^2\mathbf{L}\mathbf{E}^n - Q_p(\tau^2\mathbf{L}^i)\mathbf{E}^n \\ &= -\tau^2\mathbf{L}^e\mathbf{E}^n - P_p(\tau^2\mathbf{L}^i)\mathbf{E}^n\end{aligned}$$

where $\mathbf{L}^e = \mathbf{C}_\mathbf{H}\chi_e\mathbf{C}_\mathbf{E}$, $\mathbf{L}^i = \mathbf{C}_\mathbf{H}\chi_i\mathbf{C}_\mathbf{E}$

stability ✓, consistency ? – ✗