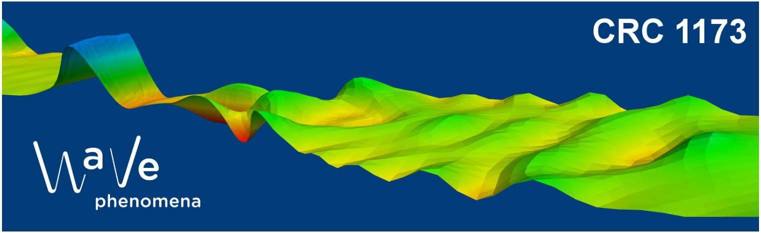
Low-regularity exponential-type integrators

Katharina Schratz (Karlsruhe)
joint work with
M. Hofmanova (Berlin) & A. Ostermann (Innsbruck)



Kompaktseminar: Time integration, 2016, Anweiler

“Miracle equations”

▷ KdV equation ($x \in \mathbb{T}$)

$$\partial_t u + \partial_x^3 u + \frac{1}{2}\partial_x (u)^2 = 0$$

[Splitting: Holden, Lubich, Risebro $(*)$, Karlsen, Tao, Tappert, ...] $(*)$ Strang: order two in H^r for solutions in H^{r+5} $(r \geq 1)$

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▷ Quadratic NLS equations $(x \in \mathbb{T})$

$$i\partial_t u = \partial_x^2 u + \mu u^2, \quad i\partial_t u = \partial_x^2 u + \mu |u|^2$$

[Analysis: Bejenaru-Tao, Germain-Masmoudi-Shatah, $\ldots$]

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"Others", e.g.

▷ NLS equation $(x \in \mathbb{T}^d)$

$$i\partial_t u = \Delta u + \mu |u|^{2p} u$$

[Splitting: Descombes, Gauckler, Faou, Lubich, ... Exponential integrators: Cano, Celledoni, Cohen, Gauckler, Owren, ...]

Low-regularity exp. int. vs. splitting & classical exp. int.

▷ Quadratic NLS: $i\partial_t u = \partial_x^2 u + u^2 \qquad (x \in \mathbb{T})$

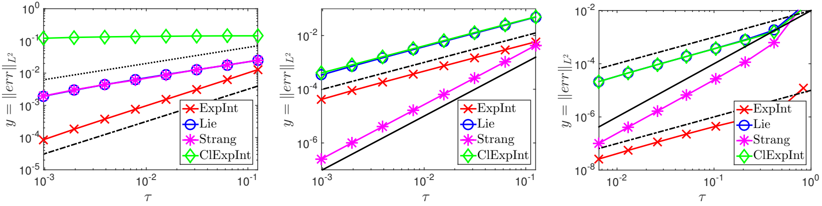

Figure: Random / Smooth / Smooth & small nonlinearity εu^2

Low-regularity exp. int. vs. splitting & classical exp. int.

▷ **Quadratic NLS:** $i\partial_t u = \partial_x^2 u + u^2 \qquad (x \in \mathbb{T})$

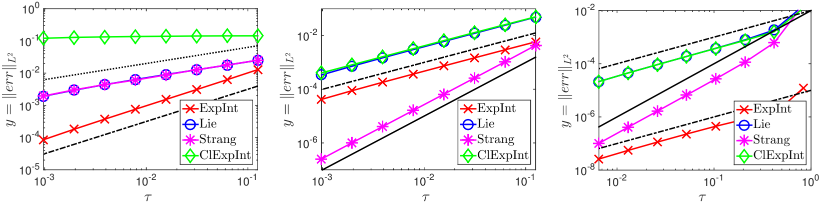

Figure: Random / Smooth / Smooth & small nonlinearity εu^2

▷ **Cubic NLS:** $i\partial_t u = \partial_x^2 u + |u|^2 u \qquad (x \in \mathbb{T})$

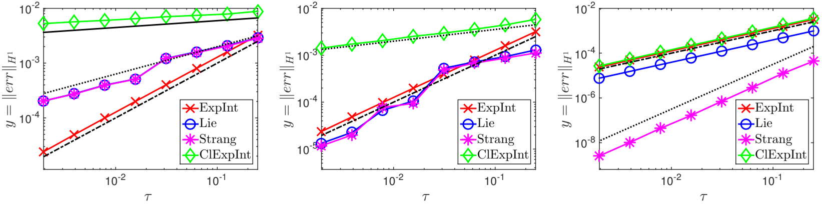

Figure: $H^{3/2}$ / H^2 / H^5 solutions

▹ KdV equation (joint work with M. Hofmanova)

$$\partial_t u + \partial_x^3 u + \frac{1}{2}\partial_x (u)^2 = 0 \qquad (x \in \mathbb{T})$$

Idea: Approximate in "$v(t) := \mathrm{e}^{t\partial_x^3} u(t)$" [cf. Lawson methods]

Duhamel:

$$u(t_n + \tau) = \mathrm{e}^{-\tau\partial_x^3} u(t_n) + \frac{1}{2} \int_0^\tau \mathrm{e}^{-(\tau-\xi)\partial_x^3} \partial_x \big(u(t_n + \xi)\big)^2 \mathrm{d}\xi$$

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$$u(t_n+\tau) = \mathrm{e}^{-\tau\partial_x^3}u(t_n) + \frac{1}{2}\mathrm{e}^{-\tau\partial_x^3}\int_0^\tau \mathrm{e}^{+\xi\partial_x^3}\partial_x\left(\mathrm{e}^{-\xi\partial_x^3}u(t_n)\right)^2 \mathrm{d}\xi \tag{KdV}$$

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▷ Scheme: (for $\hat{u}_0(0)=0$)

$$u^{n+1} = \mathrm{e}^{-\tau\partial_x^3}u^n + \frac{1}{6}\left(\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}u^n\right)^2 - \frac{1}{6}\mathrm{e}^{-\tau\partial_x^3}\left(\partial_x^{-1}u^n\right)^2.$$

[Convergence: First-order in H^1 for solutions in H^{1+2}.]

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$$\begin{aligned} u^{n+1} &= \left(1-2i\tau\hat{u}_0^n\right)\mathrm{e}^{-i\tau\partial_x^2}u^n + i\tau\left(\hat{u}_0^n\right)^2 \\ &\quad -\frac{1}{2}\left(\mathrm{e}^{-i\tau\partial_x^2}\partial_x^{-1}u^n\right)^2 + \frac{1}{2}\mathrm{e}^{-i\tau\partial_x^2}\left(\partial_x^{-1}u^n\right)^2. \end{aligned}$$

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Local error [Lubich (08')] ($p = 1$)

$$\frac{1}{2}[T, V](u) = (\nabla u \cdot \nabla \overline{u})\, u + (\nabla u \overline{u}) \cdot \nabla u + (u \nabla \overline{u}) \cdot \nabla u + \left(u \overline{\Delta u}\right) u$$

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▹ "Low-regularity" exponential-type integrator for NLS

$$u^{n+1} = \mathrm{e}^{-i\tau\Delta}\left[u^n - i\tau (u^n)^{p+1} \Big(\varphi_1(2i\tau\Delta)(\overline{u^n})^p\Big)\right]$$

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Look at (cNLS) in Fourier + use that

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$$-\, i\mathrm{e}^{-i\tau\partial_x^2}\int_0^\tau \mathrm{e}^{+i\xi\partial_x^2}\left(\mathrm{e}^{-i\xi\partial_x^2}u(t_n)\right)^2 \mathrm{e}^{i\xi\partial_x^2}\overline{u(t_n)}\mathrm{d}\xi + \mathcal{O}_s\Big(\tau^2\|u(\xi)\|_s\Big)$$

Look at (cNLS) in Fourier + use that

$$\mathrm{e}^{-i(k_1+k_2-k_3)^2\xi}\mathrm{e}^{+ik_1^2\xi}\mathrm{e}^{+ik_2^2\xi}\mathrm{e}^{-ik_3^2\xi} = \underbrace{\mathrm{e}^{-2ik_3^2\xi}}_{\text{“dominant”}} \cdot \underbrace{\mathrm{e}^{2i\xi[(k_1+k_2)k_3-k_1k_2]}}_{\approx\, 1+\mathcal{O}(\xi\sum_{i\neq j}k_ik_j)}$$

▷ Cubic NLS equation

$$u(t_n+\tau) = \mathrm{e}^{-i\tau\partial_x^2}u(t_n) \qquad \text{(cNLS)}$$
$$- i\mathrm{e}^{-i\tau\partial_x^2}\int_0^\tau \mathrm{e}^{+i\xi\partial_x^2}\left(\mathrm{e}^{-i\xi\partial_x^2}u(t_n)\right)^2 \mathrm{e}^{i\xi\partial_x^2}\overline{u(t_n)}\mathrm{d}\xi + \mathcal{O}_s\left(\tau^2\|u(\xi)\|_s\right)$$

Look at (cNLS) in Fourier + use that

$$\mathrm{e}^{-i(k_1+k_2-k_3)^2\xi}\mathrm{e}^{+ik_1^2\xi}\mathrm{e}^{+ik_2^2\xi}\mathrm{e}^{-ik_3^2\xi} = \underbrace{\mathrm{e}^{-2ik_3^2\xi}}_{\text{“dominant”}} \cdot \underbrace{\mathrm{e}^{2i\xi[(k_1+k_2)k_3-k_1k_2]}}_{\approx\, 1+\mathcal{O}(\xi\sum_{i\neq j}k_ik_j)}$$

▷ Scheme:

$$u^{n+1} = \mathrm{e}^{-i\tau\Delta}\left[u^n - i\tau\left(u^n\right)^2\left(\varphi_1(2i\tau\Delta)\overline{u^n}\right)\right], \quad \varphi_1(z) := \frac{\mathrm{e}^z-1}{z}.$$

[Convergence: First-order in H^s for solutions in H^{s+1} $(s > d/2)$]

Numerical experiments:

▷ Quadratic NLS: $\quad i\partial_t u = \partial_x^2 u + u^2 \qquad (x \in \mathbb{T})$

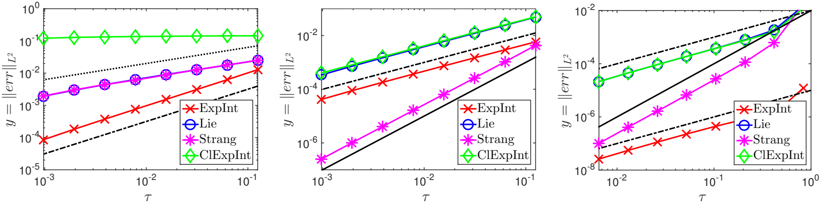


Figure: Random / Smooth / Smooth & small nonlinearity εu^2

▷ Cubic NLS: $\quad i\partial_t u = \partial_x^2 u + |u|^2 u \qquad (x \in \mathbb{T})$

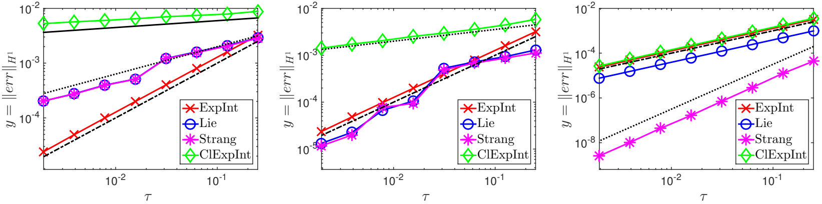


Figure: $H^{3/2}$ / H^2 / H^5 solutions

Numerical experiments: Resolution of solitons for KdV

$$\partial_t u + \partial_x^3 u + \frac{1}{2}\partial_x \left(u^2\right) = 0$$

Solitary wave

$$\phi(t,x) = 3c\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct - a)\right)$$

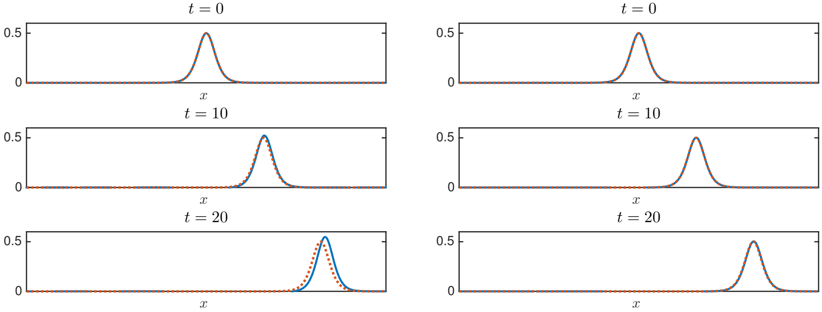
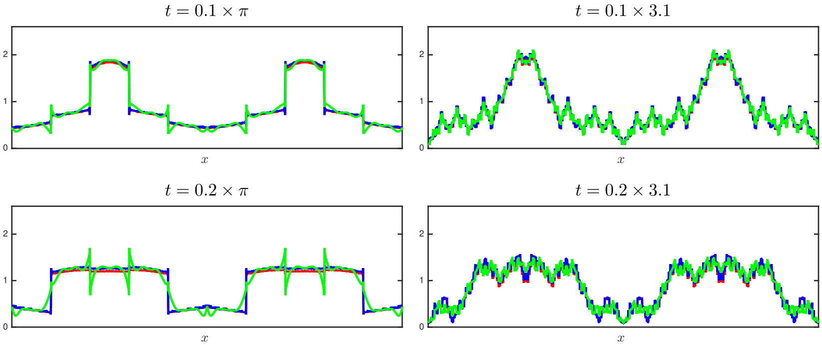


Figure: Solitary wave (dotted), Low-regularity Exponential Integrator: Left $\tau = 10^{-2}$, right $\tau = 10^{-3}$

Numerical experiments: cNLS with step-function initial data

$$i\partial_t u = \partial_x^2 u + |u|^2 u = 0, \quad u(0,x) = \text{sign}(x) \qquad (x \in \mathbb{T})$$



Strang Splitting[1], Low-regularity Exponential-type integrator, Classical Exponential integrator (time-step size $\tau = 0.002 \times \pi$)

[1] G. Chen, P. Olver: *Numerical simulation of nonlinear dispersive quantization* (14')

Thanks for your attention !

Preprints:

- M. Hofmanová, K. Schratz: *An exponential-type integrator for the KdV equation* (2016) http://arxiv.org/abs/1601.05311
- A. Ostermann, K. Schratz: *Low regularity exponential-type integrators for semilinear Schrödinger equations in the energy space* (2016) http://arxiv.org/abs/1603.07746