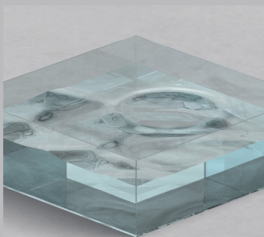
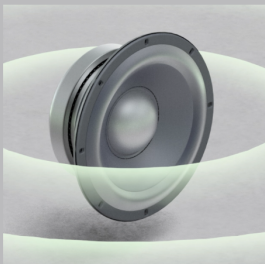


Numerical analysis of the wave equation with numerous boundary conditions

David Hipp joint work with Marlis Hochbruck

Numerical Analysis Group



Consider bdd. domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\Gamma = \partial\Omega$

Model problem

$$u_{tt}(t, x) = \Delta u(t, x), \quad (t, x) \in [0, T] \times \Omega \quad + \quad \text{initial cond.}$$

supplemented by boundary conditions on Γ :

acoustic

$$m\delta_{tt} + d\delta_t + k\delta = -\rho u_t$$

$$\delta_t = \partial_\nu u$$

Consider bdd. domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\Gamma = \partial\Omega$

Model problem

$$u_{tt}(t, x) = \Delta u(t, x), \quad (t, x) \in [0, T] \times \Omega \quad + \quad \text{initial cond.}$$

supplemented by boundary conditions on Γ :

acoustic

$$m\delta_{tt} + d\delta_t + k\delta = -\rho u_t$$

$$\delta_t = \partial_\nu u$$

kinetic

$$u_{tt} + k^2 u + \partial_\nu u = \Delta_\Gamma u$$

Robin

$$\partial_\nu u + k^2 u = 0$$

Consider bdd. domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\Gamma = \partial\Omega$

Model problem

$$u_{tt}(t, x) = \Delta u(t, x), \quad (t, x) \in [0, T] \times \Omega \quad + \quad \text{initial cond.}$$

supplemented by boundary conditions on Γ :

acoustic

$$m\delta_{tt} + d\delta_t + k\delta = -\rho u_t$$

$$\delta_t = \partial_\nu u$$

kinetic

$$u_{tt} + k^2 u + \partial_\nu u = \Delta_\Gamma u$$

Robin

$$\partial_\nu u + k^2 u = 0$$

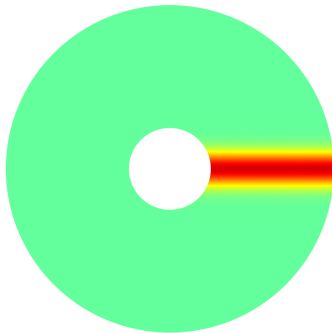
Goal: unified framework $\left\{ \begin{array}{l} \text{which delivers well-posedness result} \\ \text{has variational background for numerics} \end{array} \right.$

Example: Kinetic boundary conditions

Find $u: [0, T] \times \Omega \rightarrow \mathbb{R}$ s.t.

$$u_{tt} = \Delta u, \quad \text{in } \Omega$$

$$u_{tt} + k^2 u = \Delta_\Gamma u - \partial_\nu u, \quad \text{on } \Gamma \text{ suff. / pw. smooth}$$



Example: Kinetic boundary conditions

Find $u: [0, T] \times \Omega \rightarrow \mathbb{R}$ s.t.

$$u_{tt} = \Delta u, \quad \text{in } \Omega$$

$$u_{tt} + k^2 u = \Delta_\Gamma u - \partial_\nu u, \quad \text{on } \Gamma \text{ suff. / pw. smooth}$$

Derivation of variational formulation:

1. multiply pde with $\varphi \in C^\infty(\overline{\Omega})$ and integrate over Ω

$$\int_{\Omega} u_{tt} \varphi - \Delta u \varphi \, dx = 0 \quad \forall \varphi$$

Example: Kinetic boundary conditions

Find $u: [0, T] \times \Omega \rightarrow \mathbb{R}$ s.t.

$$u_{tt} = \Delta u, \quad \text{in } \Omega$$

$$u_{tt} + k^2 u = \Delta_\Gamma u - \partial_\nu u, \quad \text{on } \Gamma \text{ suff. / pw. smooth}$$

Derivation of variational formulation:

1. multiply pde with $\varphi \in C^\infty(\overline{\Omega})$ and integrate over Ω

$$\int_{\Omega} u_{tt} \varphi - \Delta u \varphi \, dx = 0 \quad \forall \varphi$$

2. use Green's formula

$$\int_{\Omega} u_{tt} \varphi + \nabla u \cdot \nabla \varphi \, dx - \int_{\Gamma} \partial_\nu u \varphi \, d\sigma = 0 \quad \forall \varphi$$

Example: Kinetic boundary conditions

Find $u: [0, T] \times \Omega \rightarrow \mathbb{R}$ s.t.

$$u_{tt} = \Delta u, \quad \text{in } \Omega$$

$$u_{tt} + k^2 u = \Delta_\Gamma u - \partial_\nu u, \quad \text{on } \Gamma \text{ suff. / pw. smooth}$$

Derivation of variational formulation:

1. multiply pde with $\varphi \in C^\infty(\overline{\Omega})$ and integrate over Ω

$$\int_{\Omega} u_{tt} \varphi - \Delta u \varphi \, dx = 0 \quad \forall \varphi$$

2. use Green's formula

$$\int_{\Omega} u_{tt} \varphi + \nabla u \cdot \nabla \varphi \, dx - \int_{\Gamma} \partial_\nu u \varphi \, d\sigma = 0 \quad \forall \varphi$$

3. replace $\partial_\nu u$ by boundary conditions

$$\int_{\Omega} u_{tt} \varphi + \nabla u \cdot \nabla \varphi \, dx + \int_{\Gamma} (u_{tt} + k^2 u - \Delta_\Gamma u) \varphi \, d\sigma = 0 \quad \forall \varphi$$

Example: Kinetic boundary conditions

4. Green's formula on Γ

$$\begin{aligned}a(u, \varphi) &= \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx + \int_{\Gamma} k^2 u \varphi \, d\sigma - \int_{\Gamma} \Delta_{\Gamma} u \varphi \, d\sigma \\&= \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx + \int_{\Gamma} k^2 u \varphi \, d\sigma + \int_{\Gamma} \nabla_{\Gamma} u \cdot \nabla_{\Gamma} \varphi \, d\sigma\end{aligned}$$

Example: Kinetic boundary conditions

4. Green's formula on Γ

$$\begin{aligned} a(u, \varphi) &= \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx + \int_{\Gamma} k^2 u \varphi \, d\sigma - \int_{\Gamma} \Delta_{\Gamma} u \varphi \, d\sigma \\ &= \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx + \int_{\Gamma} k^2 u \varphi \, d\sigma + \int_{\Gamma} \nabla_{\Gamma} u \cdot \nabla_{\Gamma} \varphi \, d\sigma \end{aligned}$$

5. each (classical) solution satisfies

$$\int_{\Omega} u_{tt} \varphi \, dx + \int_{\Gamma} u_{tt} \varphi \, d\sigma + a(u, \varphi) = 0 \quad \forall \varphi \in C^{\infty}(\Omega)$$

Example: Kinetic boundary conditions

4. Green's formula on Γ

$$\begin{aligned} a(u, \varphi) &= \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx + \int_{\Gamma} k^2 u \varphi \, d\sigma - \int_{\Gamma} \Delta_{\Gamma} u \varphi \, d\sigma \\ &= \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx + \int_{\Gamma} k^2 u \varphi \, d\sigma + \int_{\Gamma} \nabla_{\Gamma} u \cdot \nabla_{\Gamma} \varphi \, d\sigma \end{aligned}$$

5. each (classical) solution satisfies

$$\int_{\Omega} u_{tt} \varphi \, dx + \int_{\Gamma} u_{tt} \varphi \, d\sigma + a(u, \varphi) = 0 \quad \forall \varphi \in C^{\infty}(\Omega)$$

6. with $H = \mathbb{L}^2 \stackrel{\text{def}}{=} L^2(\Omega) \times L^2(\Gamma)$

$$(u_{tt} \mid \varphi)_H + a(u, \varphi) = 0 \quad \forall \varphi \in V$$

and a V -elliptic

$$V = H^1(\Omega; \Gamma) \stackrel{\text{def}}{=} \left\{ u \in H^1(\Omega) \mid \gamma u \in H^1(\Gamma) \right\}$$

for $\gamma: H^1(\Omega) \rightarrow H^{1/2}(\Gamma)$ trace op.

Let $V \hookrightarrow H$ densely; Gelfand triple $V \hookrightarrow H = H^* \hookrightarrow V^*$

Variational problem

Find $u: [0, T] \rightarrow V$ s.t.

$$\langle u''(t), \varphi \rangle_V + e(u'(t), \varphi) + a(u(t), \varphi) = 0 \quad \forall \varphi \in V \quad + \text{ i.c.}$$

- V -elliptic symmetric bilinear form $a: V \times V \rightarrow \mathbb{R}$, i.e.,

$$|a(u, \varphi)| \leq C_a \|u\|_V \|\varphi\|_V \quad (\text{boundedness})$$

$$a(u, u) \geq c_a \|u\|_V^2 \quad (\text{coercivity})$$

- positive semidefinite, continuous bilinear form $e: V \times V \rightarrow \mathbb{R}$, i.e.,

$$|e(v, \varphi)| \leq C_e \|v\|_V \|\varphi\|_V \quad (\text{boundedness})$$

$$e(v, v) \geq 0 \quad (\text{psd})$$

Unified framework: well-posedness

- with $\mathcal{A}u = a(u, \cdot)$ and $\mathcal{E}v = e(v, \cdot)$

$$u''(t) + \mathcal{E}u'(t) + \mathcal{A}u(t) = 0 \quad \text{in } V^*$$

where $\mathcal{A}, \mathcal{E} \in \mathcal{B}(V, V^*)$

Unified framework: well-posedness

- with $\mathcal{A}u = a(u, \cdot)$ and $\mathcal{E}v = e(v, \cdot)$

$$u''(t) + \mathcal{E}u'(t) + \mathcal{A}u(t) = 0 \quad \text{in } V^*$$

where $\mathcal{A}, \mathcal{E} \in \mathcal{B}(V, V^*)$

Unified framework: well-posedness

- with $\mathcal{A}u = a(u, \cdot)$ and $\mathcal{E}v = e(v, \cdot)$

$$u''(t) + \mathcal{E}u'(t) + \mathcal{A}u(t) = 0 \quad \text{in } V^*$$

where $\mathcal{A}, \mathcal{E} \in \mathcal{B}(V, V^*)$

- first order problem with $v = u'$

$$\mathbf{x}'(t) + \mathcal{L}\mathbf{x}(t) = 0 \quad \text{for } \mathcal{L} = \begin{bmatrix} 0 & -I \\ \mathcal{A} & \mathcal{E} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} u \\ v \end{bmatrix}$$

then $\mathcal{L} \in \mathcal{B}(V \times V, V \times V^*)$

- with $\mathcal{A}u = a(u, \cdot)$ and $\mathcal{E}v = e(v, \cdot)$

$$u''(t) + \mathcal{E}u'(t) + \mathcal{A}u(t) = 0 \quad \text{in } V^*$$

where $\mathcal{A}, \mathcal{E} \in \mathcal{B}(V, V^*)$

- first order problem with $v = u'$

$$\mathbf{x}'(t) + \mathcal{L}\mathbf{x}(t) = 0 \quad \text{for } \mathcal{L} = \begin{bmatrix} 0 & -I \\ \mathcal{A} & \mathcal{E} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} u \\ v \end{bmatrix}$$

then $\mathcal{L} \in \mathcal{B}(V \times V, V \times V^*)$

The part of $\mathcal{A} \in \mathcal{B}(V, V^*)$ in H is defined as $A = \mathcal{A}|_{D(A)}$ with

$$A: D(A) \rightarrow H, \quad D(A) = \{u \in V \mid \mathcal{A}u \in H\}.$$

- with $\mathcal{A}u = a(u, \cdot)$ and $\mathcal{E}v = e(v, \cdot)$

$$u''(t) + \mathcal{E}u'(t) + \mathcal{A}u(t) = 0 \quad \text{in } V^*$$

where $\mathcal{A}, \mathcal{E} \in \mathcal{B}(V, V^*)$

- first order problem with $v = u'$

$$\mathbf{x}'(t) + \mathcal{L}\mathbf{x}(t) = 0 \quad \text{for } \mathcal{L} = \begin{bmatrix} 0 & -I \\ \mathcal{A} & \mathcal{E} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} u \\ v \end{bmatrix}$$

then $\mathcal{L} \in \mathcal{B}(V \times V, V \times V^*)$

- $\mathbf{L} = \mathcal{L}|_{D(\mathbf{L})}$ part of $\mathcal{L}$ in $\mathbf{X} = V \times H$ with

$$D(\mathbf{L}) = \left\{ [u, v]^T \in V \times V \mid \mathcal{A}u + \mathcal{E}v \in H \right\} \subset \mathbf{X}$$

The part of $\mathcal{A} \in \mathcal{B}(V, V^*)$ in H is defined as $A = \mathcal{A}|_{D(A)}$ with

$$A: D(A) \rightarrow H, \quad D(A) = \{u \in V \mid \mathcal{A}u \in H\}.$$

Unified framework: well-posedness

Check assumptions of Lumer-Phillips for $-\mathbf{L}: D(\mathbf{L}) \rightarrow \mathbf{X}$

1. **dissipativity** in energy norm $\|\mathbf{x}\|_{\mathbf{X}}^2 = \|u\|_a^2 + \|v\|_H^2$

$$\begin{aligned} (-\mathbf{L}\mathbf{x} \mid \mathbf{x})_{\mathbf{X}} &= \left(\begin{bmatrix} v \\ -\mathcal{A}u - \mathcal{E}v \end{bmatrix} \mid \begin{bmatrix} u \\ v \end{bmatrix} \right)_{\mathbf{X}} \\ &= a(v, u) - \langle \mathcal{A}u, v \rangle - \langle \mathcal{E}v, v \rangle \leq 0 \end{aligned}$$

Unified framework: well-posedness

Check assumptions of Lumer-Phillips for $-\mathbf{L}: D(\mathbf{L}) \rightarrow \mathbf{X}$

1. **dissipativity** in energy norm $\|\mathbf{x}\|_{\mathbf{X}}^2 = \|u\|_a^2 + \|v\|_H^2$

$$\begin{aligned}
 (-\mathbf{L}\mathbf{x} \mid \mathbf{x})_{\mathbf{X}} &= \left(\begin{bmatrix} v \\ -\mathcal{A}u - \mathcal{E}v \end{bmatrix} \mid \begin{bmatrix} u \\ v \end{bmatrix} \right)_{\mathbf{X}} \\
 &= a(v, u) - \langle \mathcal{A}u, v \rangle - \langle \mathcal{E}v, v \rangle \leq 0
 \end{aligned}$$

2. $(\lambda \mathbf{I} + \mathbf{L})D(\mathbf{L}) = \mathbf{X}$ for $\lambda > 0$

$$(\lambda + \mathbf{L})\mathbf{x} = \mathbf{f}$$

Check assumptions of Lumer-Phillips for $-\mathbf{L}: D(\mathbf{L}) \rightarrow \mathbf{X}$

1. **dissipativity** in energy norm $\|\mathbf{x}\|_{\mathbf{X}}^2 = \|u\|_a^2 + \|v\|_H^2$

$$\begin{aligned} (-\mathbf{L}\mathbf{x} \mid \mathbf{x})_{\mathbf{X}} &= \left(\begin{bmatrix} v \\ -\mathcal{A}u - \mathcal{E}v \end{bmatrix} \mid \begin{bmatrix} u \\ v \end{bmatrix} \right)_{\mathbf{X}} \\ &= a(v, u) - \langle \mathcal{A}u, v \rangle - \langle \mathcal{E}v, v \rangle \leq 0 \end{aligned}$$

2. $(\lambda \mathbf{I} + \mathbf{L})D(\mathbf{L}) = \mathbf{X}$ for $\lambda > 0$

$$(\lambda + \mathbf{L})\mathbf{x} = \mathbf{f} \quad \Longleftrightarrow \quad \begin{bmatrix} \lambda u - v \\ \lambda v + \mathcal{E}v + \mathcal{A}u \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

Unified framework: well-posedness

Check assumptions of Lumer-Phillips for $-\mathbf{L}: D(\mathbf{L}) \rightarrow \mathbf{X}$

1. **dissipativity** in energy norm $\|\mathbf{x}\|_{\mathbf{X}}^2 = \|u\|_a^2 + \|v\|_H^2$

$$\begin{aligned} (-\mathbf{L}\mathbf{x} \mid \mathbf{x})_{\mathbf{X}} &= \left(\begin{bmatrix} v \\ -\mathcal{A}u - \mathcal{E}v \end{bmatrix} \mid \begin{bmatrix} u \\ v \end{bmatrix} \right)_{\mathbf{X}} \\ &= a(v, u) - \langle \mathcal{A}u, v \rangle - \langle \mathcal{E}v, v \rangle \leq 0 \end{aligned}$$

2. $(\lambda \mathbf{I} + \mathbf{L})D(\mathbf{L}) = \mathbf{X}$ for $\lambda > 0$

$$\begin{aligned} (\lambda + \mathbf{L})\mathbf{x} = \mathbf{f} &\iff \begin{bmatrix} \lambda u - v \\ \lambda v + \mathcal{E}v + \mathcal{A}u \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \\ &\iff \underbrace{\lambda^2 (u \mid \varphi)_H + \lambda e(u, \varphi) + a(u, \varphi)}_{V\text{-elliptic}} = \langle h, \varphi \rangle (*) \end{aligned}$$

$$(*) : \quad \forall \varphi \in V, h = g + \lambda f + \mathcal{E}f \in V^*$$

Check assumptions of Lumer-Phillips for $-\mathbf{L}: D(\mathbf{L}) \rightarrow \mathbf{X}$

1. **dissipativity** in energy norm $\|\mathbf{x}\|_{\mathbf{X}}^2 = \|u\|_a^2 + \|v\|_H^2$

$$\begin{aligned} (-\mathbf{L}\mathbf{x} \mid \mathbf{x})_{\mathbf{X}} &= \left(\begin{bmatrix} v \\ -\mathcal{A}u - \mathcal{E}v \end{bmatrix} \mid \begin{bmatrix} u \\ v \end{bmatrix} \right)_{\mathbf{X}} \\ &= a(v, u) - \langle \mathcal{A}u, v \rangle - \langle \mathcal{E}v, v \rangle \leq 0 \end{aligned}$$

2. $(\lambda \mathbf{I} + \mathbf{L})D(\mathbf{L}) = \mathbf{X}$ for $\lambda > 0$

$$\begin{aligned} (\lambda + \mathbf{L})\mathbf{x} = \mathbf{f} &\iff \begin{bmatrix} \lambda u - v \\ \lambda v + \mathcal{E}v + \mathcal{A}u \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \\ &\iff \underbrace{\lambda^2 (u \mid \varphi)_H + \lambda e(u, \varphi) + a(u, \varphi)}_{V\text{-elliptic}} = \langle h, \varphi \rangle (*) \end{aligned}$$

3. **density** of $D(\mathbf{L})$

$$(*) : \quad \forall \varphi \in V, h = g + \lambda f + \mathcal{E}f \in V^*$$

Theorem ([Showalter '10, Thm. VI.2.1.]

– $\mathbf{L}$ generates a semigroup of contractions $\mathbf{S}(\cdot)$ on $\mathbf{X}$.

Corollary

- If $\mathbf{x}_0 \in D(\mathbf{L})$ then $\mathbf{x}(t) = \mathbf{S}(t)\mathbf{x}_0 \in C^1([0, T]; \mathbf{X})$ solves

$$\mathbf{x}'(t) + \mathbf{L}\mathbf{x}(t) = 0, \quad \mathbf{x}(0) = \mathbf{x}_0.$$

- If $D(\mathbf{L}) = D(A) \times V$ then

$$u \in C^2([0, T]; H) \cap C^1([0, T]; V) \cap C([0, T]; [D(A)]).$$

Faedo-Galerkin: $u \in L^2(0, T; V)$, $u_t \in L^2(0, T; H)$, $u_{tt} \in L^2(0, T; V^*)$

Example: kinetic boundary condition

Have

$$H = \mathbb{L}^2 \stackrel{\text{def}}{=} L^2(\Omega) \times L^2(\Gamma) \quad (u = [u_\Omega, u_\Gamma]^T \in \mathbb{L}^2)$$

$$V = H^1(\Omega; \Gamma) \stackrel{\text{def}}{=} \left\{ u \in H^1(\Omega) \mid \gamma u \in H^1(\Gamma) \right\}$$

$$D(\mathbf{L}) = \left\{ [u, v]^T \in V \times V \mid \mathcal{A}u + \mathcal{E}v \in H \right\} = D(\mathbf{A}) \times V$$

Example: kinetic boundary condition

Have

$$H = \mathbb{L}^2 \stackrel{\text{def}}{=} L^2(\Omega) \times L^2(\Gamma) \quad (u = [u_\Omega, u_\Gamma]^T \in \mathbb{L}^2)$$

$$V = H^1(\Omega; \Gamma) \stackrel{\text{def}}{=} \left\{ u \in H^1(\Omega) \mid \gamma u \in H^1(\Gamma) \right\}$$

$$D(\mathbf{L}) = \left\{ [u, v]^T \in V \times V \mid \mathcal{A}u + \mathcal{E}v \in H \right\} = D(\mathbf{A}) \times V$$

$$\text{Characterize } D(\mathbf{A}) = \left\{ u \in V \mid \mathcal{A}u \in \mathbb{L}^2 \Leftrightarrow a(u, \varphi) \leq C_u \|\varphi\|_{0, \Omega; \Gamma} \right\}$$

$$\begin{aligned} a(u, \varphi) &= \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx + \int_{\Gamma} k^2 \gamma u \gamma \varphi + \nabla_{\Gamma} \gamma u \cdot \nabla_{\Gamma} \gamma \varphi \, d\sigma \\ &= - \int_{\Omega} \Delta u \varphi \, dx + \int_{\Gamma} (k^2 \gamma u - \Delta_{\Gamma} \gamma u + \partial_{\nu} u) \gamma \varphi \, d\sigma \end{aligned}$$

Example: kinetic boundary condition

Have

$$H = \mathbb{L}^2 \stackrel{\text{def}}{=} L^2(\Omega) \times L^2(\Gamma) \quad (u = [u_\Omega, u_\Gamma]^T \in \mathbb{L}^2)$$

$$V = H^1(\Omega; \Gamma) \stackrel{\text{def}}{=} \left\{ u \in H^1(\Omega) \mid \gamma u \in H^1(\Gamma) \right\}$$

$$D(\mathbf{L}) = \left\{ [u, v]^T \in V \times V \mid \mathcal{A}u + \mathcal{E}v \in H \right\} = D(\mathbf{A}) \times V$$

$$\text{Characterize } D(\mathbf{A}) = \left\{ u \in V \mid \mathcal{A}u \in \mathbb{L}^2 \Leftrightarrow a(u, \varphi) \leq C_u \|\varphi\|_{0, \Omega; \Gamma} \right\}$$

$$\begin{aligned} a(u, \varphi) &= \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx + \int_{\Gamma} k^2 \gamma u \gamma \varphi + \nabla_{\Gamma} \gamma u \cdot \nabla_{\Gamma} \gamma \varphi \, d\sigma \\ &= - \int_{\Omega} \Delta u \varphi \, dx + \int_{\Gamma} (k^2 \gamma u - \Delta_{\Gamma} \gamma u + \partial_{\nu} u) \gamma \varphi \, d\sigma \end{aligned}$$

$$\text{thus } D(\mathbf{A}) = \left\{ u \in H^1(\Omega; \Gamma) \mid \Delta u \in L^2(\Omega), \Delta_{\Gamma} \gamma u \in L^2(\Gamma) \right\} \text{ with}$$

$$\mathbf{A}u = \left[\begin{array}{c} -\Delta \\ (k^2 - \Delta_{\Gamma}) \gamma + \partial_{\nu} \end{array} \right] u \in \mathbb{L}^2$$

Example: kinetic boundary condition

Aftermath of abstract framework:

■ $\mathbf{x}'(t) + \mathbf{L}\mathbf{x}(t) = 0$ is

$$\begin{bmatrix} u'(t) \\ v'_\Omega(t) \\ v'_\Gamma(t) \end{bmatrix} + \begin{bmatrix} 0 & -I \\ -\Delta & 0 \\ (k^2 - \Delta_\Gamma) \gamma + \partial_\nu & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = 0$$

Example: kinetic boundary condition

Aftermath of abstract framework:

- $\mathbf{x}'(t) + \mathbf{L}\mathbf{x}(t) = 0$ is

$$\begin{bmatrix} u'(t) \\ v'_\Omega(t) \\ v'_\Gamma(t) \end{bmatrix} + \begin{bmatrix} 0 & -I \\ -\Delta & 0 \\ (k^2 - \Delta_\Gamma) \gamma + \partial_\nu & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = 0$$

- unique solution for $\mathbf{x}(0) \in D(\mathbf{L})$ where

$$D(\mathbf{L}) = \left\{ u \in H^1(\Omega; \Gamma) \mid \Delta u \in L^2(\Omega), \Delta_\Gamma \gamma u \in L^2(\Gamma) \right\} \times H^1(\Omega; \Gamma)$$

Example: kinetic boundary condition

Aftermath of abstract framework:

- $\mathbf{x}'(t) + \mathbf{L}\mathbf{x}(t) = 0$ is

$$\begin{bmatrix} u'(t) \\ v'_\Omega(t) \\ v'_\Gamma(t) \end{bmatrix} + \begin{bmatrix} 0 & -I \\ -\Delta & 0 \\ (k^2 - \Delta_\Gamma) \gamma + \partial_\nu & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = 0$$

- unique solution for $\mathbf{x}(0) \in D(\mathbf{L})$ where

$$D(\mathbf{L}) = \left\{ u \in H^1(\Omega; \Gamma) \mid \Delta u \in L^2(\Omega), \Delta_\Gamma \gamma u \in L^2(\Gamma) \right\} \times H^1(\Omega; \Gamma)$$

- stability estimate in $\mathbf{X} = V_a \times H = H^1(\Omega; \Gamma) \times \mathbb{L}^2$

$$\begin{aligned} \|\nabla u(t)\|_{0,\Omega}^2 + \|k \gamma u(t)\|_{0,\Gamma}^2 + \|\nabla_\Gamma \gamma u(t)\|_{0,\Gamma}^2 + \|u_t(t)\|_{0,\Omega}^2 + \|\gamma u_t(t)\|_{0,\Gamma}^2 \\ \leq \|\mathbf{x}(0)\|_{\mathbf{X}}^2 \end{aligned}$$

Example: kinetic boundary condition

Aftermath of abstract framework:

- $\mathbf{x}'(t) + \mathbf{L}\mathbf{x}(t) = 0$ is

$$\begin{bmatrix} u'(t) \\ v'_\Omega(t) \\ v'_\Gamma(t) \end{bmatrix} + \begin{bmatrix} 0 & -I \\ -\Delta & 0 \\ (k^2 - \Delta_\Gamma) \gamma + \partial_\nu & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = 0$$

- unique solution for $\mathbf{x}(0) \in D(\mathbf{L})$ where

$$D(\mathbf{L}) = \left\{ u \in H^1(\Omega; \Gamma) \mid \Delta u \in L^2(\Omega), \Delta_\Gamma \gamma u \in L^2(\Gamma) \right\} \times H^1(\Omega; \Gamma)$$

- stability estimate in $\mathbf{X} = V_a \times H = H^1(\Omega; \Gamma) \times \mathbb{L}^2$

$$\begin{aligned} \|\nabla u(t)\|_{0,\Omega}^2 + \|k \gamma u(t)\|_{0,\Gamma}^2 + \|\nabla_\Gamma \gamma u(t)\|_{0,\Gamma}^2 + \|u_t(t)\|_{0,\Omega}^2 + \|\gamma u_t(t)\|_{0,\Gamma}^2 \\ \leq \|\mathbf{x}(0)\|_{\mathbf{X}}^2 \end{aligned}$$

Remark: $u_{tt} = [u_{tt}^\Omega, u_{tt}^\Gamma]^T \in \mathbb{L}^2$ in general $u_{tt}^\Omega|_\Gamma \neq u_{tt}^\Gamma \iff$ dynamic bc

Example: acoustic boundary conditions

Find $u: [0, T] \times \Omega \rightarrow \mathbb{R}$, $\delta: [0, T] \times \Gamma \rightarrow \mathbb{R}$ s.t.

$$\begin{aligned} u_{tt} &= c^2 \Delta u && \text{in } \Omega \\ m\delta_{tt} + d\delta_t + k\delta &= -\rho u_t && \text{on } \Gamma \\ \delta_t &= \partial_\nu u && \text{on } \Gamma \end{aligned}$$

Variational formulation for $\hat{u} = \begin{bmatrix} u \\ \delta \end{bmatrix}$ and $\hat{\varphi} = \begin{bmatrix} \varphi \\ \psi \end{bmatrix}$ is

$$(\hat{u}''(t) \mid \hat{\varphi})_H + a(\hat{u}(t), \hat{\varphi}) + e(\hat{u}'(t), \hat{\varphi}) = 0 \quad \forall \hat{\varphi}$$

where

$$\begin{aligned} a(\hat{u}, \hat{\varphi}) &= \int_{\Omega} \rho \nabla u \cdot \nabla \varphi \, dx + \int_{\Gamma} k \delta \psi \\ e(\hat{u}, \hat{\varphi}) &= \int_{\Gamma} \rho (\gamma u \psi - \delta \gamma \varphi) \, d\sigma + \int_{\Gamma} d \delta \psi \, d\sigma \end{aligned}$$

with $H = \mathbb{L}^2 = L^2(\Omega) \times L^2(\Gamma) \hookrightarrow V = H^1(\Omega) \times L^2(\Gamma)$ densely

Already known:

- well-posedness of examples, cf. [Beale '76], [Vitillaro '13]
- well-posedness of framework, cf. [Showalter '10]
- Trotter-Kato / energy techniques for error estimates

Our contribution:

- acoustic, kinetic boundary conditions in framework
- FEM for (these) dynamic boundary conditions
- spatial / temporal / full error bounds

Approximation setting:

- $\Omega \subset \mathbb{R}^2$ polygonal domain
- standard assumptions on mesh
- approximation in $V_h^k = \{ \text{pw. polynomials of degree } \leq k \}$

Then

Approximation setting:

- $\Omega \subset \mathbb{R}^2$ polygonal domain
- standard assumptions on mesh
- approximation in $V_h^k = \{ \text{pw. polynomials of degree } \leq k \}$

Then

- **acoustic boundary conditions** error is $\mathcal{O}(h^k)$ (as last year)

Approximation setting:

- $\Omega \subset \mathbb{R}^2$ polygonal domain
- standard assumptions on mesh
- approximation in $V_h^k = \{ \text{pw. polynomials of degree } \leq k \}$

Then

- acoustic boundary conditions error is $\mathcal{O}(h^k)$ (as last year)
- kinetic boundary conditions

$$\|e(t)\|_{1,\Gamma} + \|e(t)\|_{1,\Omega} + \|e_t(t)\|_{0,\Omega} + \|e_t(t)\|_{0,\Gamma} \leq C(1+t)h^k$$

Approximation setting:

- $\Omega \subset \mathbb{R}^2$ polygonal domain
- standard assumptions on mesh
- approximation in $V_h^k = \{\text{pw. polynomials of degree } \leq k\}$

Then

- acoustic boundary conditions error is $\mathcal{O}(h^k)$ (as last year)
- kinetic boundary conditions

$$\|e(t)\|_{1,\Gamma} + \|e(t)\|_{1,\Omega} + \|e_t(t)\|_{0,\Omega} + \|e_t(t)\|_{0,\Gamma} \leq C(1+t)h^k$$

but not quite complete since

- Δ_Γ requires smooth Γ
- then $V_h^k \not\subset V$

use techniques from [Kovacs, Lubich '16]

Conclusion

Summary:

- unified framework
- well-posedness
- abstract proof of convergence of space discretization

Summary:

- unified framework
- well-posedness
- abstract proof of convergence of space discretization

Remarks:

- convergence of exponential int. / alg. stable RKM with

$$\mathbf{x}'(t) + \mathbf{L}\mathbf{x}(t) = \mathbf{f}(t)$$

and $(-\mathbf{L}\mathbf{x} \mid \mathbf{x})_{\mathbf{x}} \leq 0$

- full discretization analogously

Conclusion

Summary:

- unified framework
- well-posedness
- abstract proof of convergence of space discretization

Remarks:

- convergence of exponential int. / alg. stable RKM with

$$\mathbf{x}'(t) + \mathbf{L}\mathbf{x}(t) = \mathbf{f}(t)$$

and $(-\mathbf{L}\mathbf{x} \mid \mathbf{x})_{\mathbf{x}} \leq 0$

- full discretization analogously

Outlook:

- numerical tests
- splitting bulk surface dynamics
- approximation results for smooth boundaries (with Kovacs and Lubich)

References



R. E. Showalter

Hilbert Space Methods in Partial Differential Equations
Dover Publications, 2010.



E. Vitillaro

Strong solutions for the wave equation with a kinetic boundary condition

Contemporary Mathematics 594: Recent Trends in Nonlinear Partial Differential Equations I: Evolution Problems, 295–307, 2013.



J.T. Beale

Spectral properties of an acoustic boundary condition

Indiana University Mathematics Journal, 9:895–917, 1976



B. Kovacs, C. Lubich

Numerical analysis of parabolic problems with dynamic boundary conditions,

IMA Journal of Numerical Analysis, 2016.