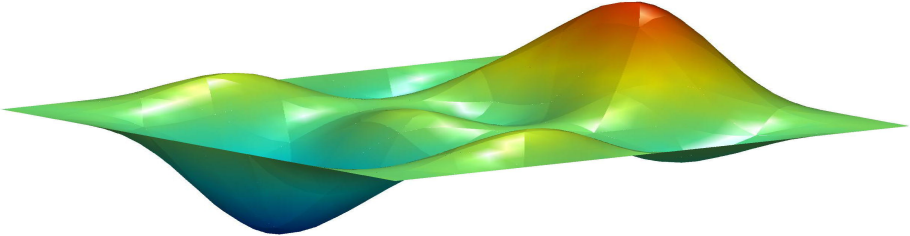
DG Methoden und explizite RK-Verfahren für Maxwell-Gleichungen

Andreas Sturm | 21. Februar 2014

ARBEITSGRUPPE NUMERIK



Lineare Maxwellgleichungen

Suche $\mathcal{H}, \mathcal{E} : [0, T] \times \Omega \to \mathbb{R}^3$, s.d.

$$\boxed{\begin{aligned} \mu\partial_t\mathcal{H} + \nabla \times \mathcal{E} &= 0, \\ \varepsilon\partial_t\mathcal{E} - \nabla \times \mathcal{H} &= -\mathcal{J}, \\ \nabla \cdot (\varepsilon\mathcal{E}) &= \rho, \\ \nabla \cdot (\mu\mathcal{H}) &= 0, \end{aligned}}$$

$$\boxed{\begin{aligned} (n \times \mathcal{E})|_{\partial\Omega} &= 0, \\ (n \cdot (\mu\mathcal{H}))|_{\partial\Omega} &= 0, \end{aligned}}$$

$$\boxed{\begin{aligned} \mathcal{E}(t = 0) &= \mathcal{E}_0, \\ \mathcal{H}(t = 0) &= \mathcal{H}_0. \end{aligned}}$$

Maxwell-Operator $A : \mathcal{D}(A) \to V$

$$\begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \mapsto \begin{bmatrix} \mu^{-1}\nabla \times \mathcal{E} \\ -\varepsilon^{-1}\nabla \times \mathcal{H} \end{bmatrix},$$

mit

$$\mathcal{D}(A) = H(\mathrm{curl}, \Omega) \times H_0(\mathrm{curl}, \Omega), \qquad V = L^2(\Omega)^3 \times L^2(\Omega)^3.$$

$H(\mathrm{curl}, \Omega) = \{v \in L^2(\Omega)^3 \mid \nabla \times v \in L^2(\Omega)^3\}$

Lineare Maxwellgleichungen

KIT Karlsruher Institut für Technologie

Suche $\mathcal{H}, \mathcal{E} : [0, T] \times \Omega \to \mathbb{R}^3$, s.d.

$$\boxed{\begin{aligned} \mu\partial_t\mathcal{H} + \nabla \times \mathcal{E} &= 0, \\ \varepsilon\partial_t\mathcal{E} - \nabla \times \mathcal{H} &= -\mathcal{J}, \\ \nabla \cdot (\varepsilon\mathcal{E}) &= \rho, \\ \nabla \cdot (\mu\mathcal{H}) &= 0, \end{aligned}}$$

$$\boxed{\begin{aligned} (n \times \mathcal{E})|_{\partial\Omega} &= 0, \\ (n \cdot (\mu\mathcal{H}))|_{\partial\Omega} &= 0, \end{aligned}}$$

$$\boxed{\begin{aligned} \mathcal{E}(t = 0) &= \mathcal{E}_0, \\ \mathcal{H}(t = 0) &= \mathcal{H}_0. \end{aligned}}$$

Maxwell-Operator $A : \mathcal{D}(A) \to V$

$$\begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \mapsto \begin{bmatrix} \mu^{-1}\nabla \times \mathcal{E} \\ -\varepsilon^{-1}\nabla \times \mathcal{H} \end{bmatrix},$$

mit

$$\mathcal{D}(A) = H(\mathrm{curl}, \Omega) \times H_0(\mathrm{curl}, \Omega), \qquad V = L^2(\Omega)^3 \times L^2(\Omega)^3.$$

$H(\mathrm{curl}, \Omega) = \{v \in L^2(\Omega)^3 \mid \nabla \times v \in L^2(\Omega)^3\}$

Lineare Maxwellgleichungen

Suche $\mathcal{H}, \mathcal{E} : [0, T] \times \Omega \to \mathbb{R}^3$, s.d.

$$\begin{aligned} \mu \partial_t \mathcal{H} + \nabla \times \mathcal{E} &= 0, \\ \varepsilon \partial_t \mathcal{E} - \nabla \times \mathcal{H} &= -\mathcal{J}, \\ \nabla \cdot (\varepsilon \mathcal{E}) &= \rho, \\ \nabla \cdot (\mu \mathcal{H}) &= 0, \end{aligned}$$

$$\begin{aligned} (n \times \mathcal{E})|_{\partial\Omega} &= 0, \\ (n \cdot (\mu \mathcal{H}))|_{\partial\Omega} &= 0, \end{aligned}$$

$$\begin{aligned} \mathcal{E}(t = 0) &= \mathcal{E}_0, \\ \mathcal{H}(t = 0) &= \mathcal{H}_0. \end{aligned}$$

Maxwell-Operator $A : \mathcal{D}(A) \to V$

$$\begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \mapsto \begin{bmatrix} \mu^{-1} \nabla \times \mathcal{E} \\ -\varepsilon^{-1} \nabla \times \mathcal{H} \end{bmatrix},$$

mit

$$\mathcal{D}(A) = H(\mathrm{curl}, \Omega) \times H_0(\mathrm{curl}, \Omega), \qquad V = L^2(\Omega)^3 \times L^2(\Omega)^3.$$

$H(\mathrm{curl}, \Omega) = \{ v \in L^2(\Omega)^3 \mid \nabla \times v \in L^2(\Omega)^3 \}$

Evolutionsgleichung: Suche $u = [\mathcal{H}, \mathcal{E}]^T \in \mathcal{D}(A)$, s. d.

$$\boxed{\begin{aligned} \partial_t u + Au &= g, \\ u(0) &= u_0, \end{aligned}}$$

mit $g = [0, -\varepsilon \mathcal{J}]^T$ und $u_0 = [\mathcal{H}_0, \mathcal{E}_0]^T$.

Wohlgestelltheit: Es gibt eindeutige Lösung.

Schief-symmetrisch: Für $v_1, v_2 \in \mathcal{D}(A)$,

$$(Av_1, v_2)_V = -(v_1, Av_2)_V.$$

Energieerhaltung: Für $g = 0$,

$$\|u(t)\|_V = \|u(0)\|_V.$$

Stabilität:

$$\|u(t)\|_V \leq \|u_0\|_V + \int_0^t \|g(s)\|_V \, ds.$$

Für $v_1 = [H_1, E_1]^T$, $v_2 = [H_2, E_2]^T$,

$$(v_1, v_2)_V = (\mu^{1/2} H_1, \mu^{1/2} H_2)_{L^2(\Omega)^3} + (\varepsilon^{1/2} E_1, \varepsilon^{1/2} E_2)_{L^2(\Omega)^3}$$

Evolutionsgleichung: Suche $u = [\mathcal{H}, \mathcal{E}]^T \in \mathcal{D}(A)$, s. d.

$$\boxed{\begin{aligned} \partial_t u + Au &= g, \\ u(0) &= u_0, \end{aligned}}$$

mit $g = [0, -\varepsilon \mathcal{J}]^T$ und $u_0 = [\mathcal{H}_0, \mathcal{E}_0]^T$.

Wohlgestelltheit: Es gibt eindeutige Lösung.

Schief-symmetrisch: Für $v_1, v_2 \in \mathcal{D}(A)$,

$$(Av_1, v_2)_V = -(v_1, Av_2)_V.$$

Energieerhaltung: Für $g = 0$,

$$\|u(t)\|_V = \|u(0)\|_V.$$

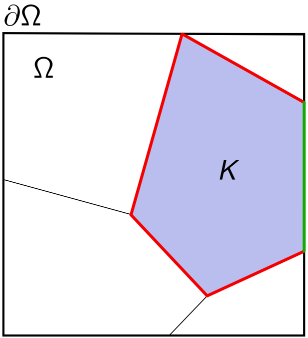
Stabilität:

$$\|u(t)\|_V \leq \|u_0\|_V + \int_0^t \|g(s)\|_V \, ds.$$

Für $v_1 = [H_1, E_1]^T$, $v_2 = [H_2, E_2]^T$,

$$(v_1, v_2)_V = (\mu^{1/2} H_1, \mu^{1/2} H_2)_{L^2(\Omega)^3} + (\varepsilon^{1/2} E_1, \varepsilon^{1/2} E_2)_{L^2(\Omega)^3}$$

DG Ortsdiskretisierung



$\mathcal{T}_h$: Gitter,
$\mathcal{F}_h^i$: innere Kanten,
$\mathcal{F}_h^b$: äußere Kanten.

Gebrochener Polynomraum

$$\mathbb{P}_3^k(\mathcal{T}_h) := \left\{ v \,:\, v|_K \in \mathbb{P}_3^k(K) \right\}.$$

Gebrochener Sobolevraum

$$H^m(\mathcal{T}_h) := \left\{ v \,:\, v|_K \in H^m(K) \right\}.$$

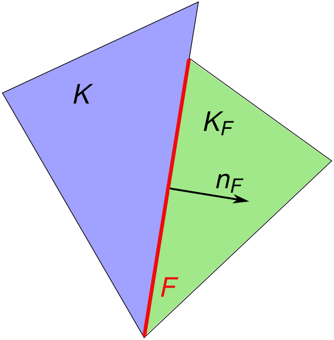
Gebrochener *curl*-Operator

$$(\nabla_h \times v)|_K := \nabla \times (v|_K).$$

DG Räume

$$V_h := \mathbb{P}_3^k(\mathcal{T}_h)^3 \times \mathbb{P}_3^k(\mathcal{T}_h)^3,$$
$$V_\star := \mathcal{D}(A) \cap (H^1(\mathcal{T}_h)^3 \times H^1(\mathcal{T}_h)^3).$$

Weitere Notationen



Mittelwert

$$\{\!\{v\}\!\}_F = \frac{1}{2}\big((v_K)|_F + (v_{K_F})|_F\big),$$

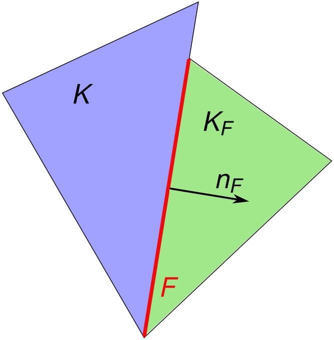
Sprung

$$[\![v]\!]_F = (v_{K_F})|_F - (v_K)|_F.$$

Für $v = [H, E]^T \in V_\star$ gilt

$$\boxed{\begin{aligned} n_F \times [\![H]\!]_F = n_F \times [\![E]\!]_F = \quad 0 \qquad & \forall F \in \mathcal{F}_h^i, \\ n \times E = \quad 0 \qquad & \forall F \in \mathcal{F}_h^b. \end{aligned}}$$

Weitere Notationen



Mittelwert

$$\{\!\{v\}\!\}_F = \frac{1}{2}\big((v_K)|_F + (v_{K_F})|_F\big),$$

Sprung

$$[\![v]\!]_F = (v_{K_F})|_F - (v_K)|_F.$$

Für $v = [H, E]^T \in V_\star$ gilt

$$\boxed{\begin{aligned} n_F \times [\![H]\!]_F = n_F \times [\![E]\!]_F = \quad & 0 \qquad \forall F \in \mathcal{F}_h^i, \\ n \times E = \quad & 0 \qquad \forall F \in \mathcal{F}_h^b. \end{aligned}}$$

Konstruktion

Kont. Bilinearform: Für $v = [H, E]^T \in \mathcal{D}(A)$, $\varphi = [\phi, \psi]^T \in V$,

$$a(v, \varphi) = (\nabla \times E, \phi)_{L^2(\Omega)^3} - (\nabla \times H, \psi)_{L^2(\Omega)^3},$$

mit $v \in V_{\star h} = V_\star + V_h$. Für $v_h = [H_h, E_h]^T \in V_h$,

$$0 \stackrel{?}{=} a_h^{(0)}(v_h, v_h) = \sum_{K \in \mathcal{T}_h} \left[(\nabla \times E_K, H_K)_{L^2(K)^3} - (\nabla \times H_K, E_K)_{L^2(K)^3} \right]$$

$$= \sum_{K \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_K} (n_K \times E_K, H_K)_{L^2(F)^3}$$

$$= \sum_{F \in \mathcal{F}_h^i} \left[-(n_F \times [\![E_h]\!]_F, \{\!\{H_h\}\!\}_F)_{L^2(F)^3} + (n_F \times [\![H_h]\!]_F, \{\!\{E_h\}\!\}_F)_{L^2(F)^3} \right]$$

$$+ \sum_{F \in \mathcal{F}_h^b} (n \times E_h, H_h)_{L^2(F)^3}$$

Konstruktion

Diskr. Bilinearform: Für $v = [H, E]^T$, $\varphi_h = [\phi_h, \psi_h]^T \in V_h$,

$$a_h^{(0)}(v, \varphi_h) = (\nabla_h \times E, \phi_h)_{L^2(\Omega)^3} - (\nabla_h \times H, \psi_h)_{L^2(\Omega)^3},$$

mit $v \in V_{\star h} = V_\star + V_h$. Für $v_h = [H_h, E_h]^T \in V_h$,

$$\begin{aligned} 0 \overset{?}{=} a_h^{(0)}(v_h, v_h) &= \sum_{K \in \mathcal{T}_h} \left[(\nabla \times E_K, H_K)_{L^2(K)^3} - (\nabla \times H_K, E_K)_{L^2(K)^3} \right] \\ &= \sum_{K \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_K} (n_K \times E_K, H_K)_{L^2(F)^3} \\ &= \sum_{F \in \mathcal{F}_h^i} \left[-(n_F \times [\![E_h]\!]_F, \{\!\{ H_h \}\!\}_F)_{L^2(F)^3} + (n_F \times [\![H_h]\!]_F, \{\!\{ E_h \}\!\}_F)_{L^2(F)^3} \right] \\ &\quad + \sum_{F \in \mathcal{F}_h^b} (n \times E_h, H_h)_{L^2(F)^3} \end{aligned}$$

Konstruktion

Diskr. Bilinearform: Für $v = [H, E]^T$, $\varphi_h = [\phi_h, \psi_h]^T \in V_h$,

$$a_h^{(0)}(v, \varphi_h) = (\nabla \times E, \phi_h)_{L^2(\Omega)^3} - (\nabla \times H, \psi_h)_{L^2(\Omega)^3},$$

mit $v \in V_{\star h} = V_\star + V_h$. Für $v_h = [H_h, E_h]^T \in V_h$,

$$0 \stackrel{?}{=} a_h^{(0)}(v_h, v_h) = \sum_{K \in \mathcal{T}_h} \left[(\nabla \times E_K, H_K)_{L^2(K)^3} - (\nabla \times H_K, E_K)_{L^2(K)^3} \right]$$

$$= \sum_{K \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_K} (n_K \times E_K, H_K)_{L^2(F)^3}$$

$$= \sum_{F \in \mathcal{F}_h^i} \left[-(n_F \times [\![E_h]\!]_F, \{\!\{H_h\}\!\}_F)_{L^2(F)^3} + (n_F \times [\![H_h]\!]_F, \{\!\{E_h\}\!\}_F)_{L^2(F)^3} \right]$$

$$+ \sum_{F \in \mathcal{F}_h^b} (n \times E_h, H_h)_{L^2(F)^3}$$

Konstruktion

Diskr. Bilinearform: Für $v = [H, E]^T$, $\varphi_h = [\phi_h, \psi_h]^T \in V_h$,

$$a_h^{(0)}(v, \varphi_h) = (\nabla_h \times E, \phi_h)_{L^2(\Omega)^3} - (\nabla_h \times H, \psi_h)_{L^2(\Omega)^3},$$

mit $v \in V_{\star h} = V_\star + V_h$. Für $v_h = [H_h, E_h]^T \in V_h$,

$$0 \stackrel{?}{=} a_h^{(0)}(v_h, v_h) = \sum_{K \in \mathcal{T}_h} \left[(\nabla \times E_K, H_K)_{L^2(K)^3} - (\nabla \times H_K, E_K)_{L^2(K)^3} \right]$$

$$= \sum_{K \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_K} (n_K \times E_K, H_K)_{L^2(F)^3}$$

$$= \sum_{F \in \mathcal{F}_h^i} \left[-(n_F \times [\![E_h]\!]_F, \{\!\{H_h\}\!\}_F)_{L^2(F)^3} + (n_F \times [\![H_h]\!]_F, \{\!\{E_h\}\!\}_F)_{L^2(F)^3} \right]$$

$$+ \sum_{F \in \mathcal{F}_h^b} (n \times E_h, H_h)_{L^2(F)^3}$$

Konstruktion

Diskr. Bilinearform: Für $v = [H, E]^T$, $\varphi_h = [\phi_h, \psi_h]^T \in V_h$,

$$a_h^{(0)}(v, \varphi_h) = (\nabla_h \times E, \phi_h)_{L^2(\Omega)^3} - (\nabla_h \times H, \psi_h)_{L^2(\Omega)^3},$$

mit $v \in V_{\star h} = V_\star + V_h$. Für $v_h = [H_h, E_h]^T \in V_h$,

$$0 \stackrel{?}{=} a_h^{(0)}(v_h, v_h) = \sum_{K \in \mathcal{T}_h} \left[(\nabla \times E_K, H_K)_{L^2(K)^3} - (\nabla \times H_K, E_K)_{L^2(K)^3} \right]$$

$$= \sum_{K \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_K} (n_K \times E_K, H_K)_{L^2(F)^3}$$

$$= \sum_{F \in \mathcal{F}_h^i} \left[-(n_F \times [\![E_h]\!]_F, \{\!\{H_h\}\!\}_F)_{L^2(F)^3} + (n_F \times [\![H_h]\!]_F, \{\!\{E_h\}\!\}_F)_{L^2(F)^3} \right]$$

$$+ \sum_{F \in \mathcal{F}_h^b} (n \times E_h, H_h)_{L^2(F)^3}$$

Konstruktion

Diskr. Bilinearform: Für $v = [H, E]^T$, $\varphi_h = [\phi_h, \psi_h]^T \in V_h$,

$$a_h^{(0)}(v, \varphi_h) = (\nabla_h \times E, \phi_h)_{L^2(\Omega)^3} - (\nabla_h \times H, \psi_h)_{L^2(\Omega)^3},$$

mit $v \in V_{\star h} = V_\star + V_h$. Für $v_h = [H_h, E_h]^T \in V_h$,

$$0 \stackrel{?}{=} a_h^{(0)}(v_h, v_h) = \sum_{K \in \mathcal{T}_h} \left[(\nabla \times E_K, H_K)_{L^2(K)^3} - (\nabla \times H_K, E_K)_{L^2(K)^3} \right]$$

$$= \sum_{K \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_K} (n_K \times E_K, H_K)_{L^2(F)^3}$$

$$= \sum_{F \in \mathcal{F}_h^i} \left[-(n_F \times [\![E_h]\!]_F, \{\!\{H_h\}\!\}_F)_{L^2(F)^3} + (n_F \times [\![H_h]\!]_F, \{\!\{E_h\}\!\}_F)_{L^2(F)^3} \right]$$

$$+ \sum_{F \in \mathcal{F}_h^b} (n \times E_h, H_h)_{L^2(F)^3}$$

Konstruktion

Diskr. Bilinearform: Für $v = [H, E]^T$, $\varphi_h = [\phi_h, \psi_h]^T \in V_h$,

$$a_h^{(0)}(v, \varphi_h) = (\nabla_h \times E, \phi_h)_{L^2(\Omega)^3} - (\nabla_h \times H, \psi_h)_{L^2(\Omega)^3},$$

mit $v \in V_{\star h} = V_\star + V_h$. Für $v_h = [H_h, E_h]^T \in V_h$,

$$0 \stackrel{?}{=} a_h^{(0)}(v_h, v_h) = \sum_{K \in \mathcal{T}_h} \left[(\nabla \times E_K, H_K)_{L^2(K)^3} - (\nabla \times H_K, E_K)_{L^2(K)^3} \right]$$

$$= \sum_{K \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_K} (n_K \times E_K, H_K)_{L^2(F)^3}$$

$$= \sum_{F \in \mathcal{F}_h^i} \left[(n_F \times E_K, H_K)_{L^2(F)^3} - (n_F \times E_{K_F}, H_{K_F})_{L^2(F)^3} \right]$$

$$+ \sum_{F \in \mathcal{F}_h^b} (n \times E_h, H_h)_{L^2(F)^3}$$

Konstruktion

Diskr. Bilinearform: Für $v = [H, E]^T$, $\varphi_h = [\phi_h, \psi_h]^T \in V_h$,

$$a_h^{(0)}(v, \varphi_h) = (\nabla_h \times E, \phi_h)_{L^2(\Omega)^3} - (\nabla_h \times H, \psi_h)_{L^2(\Omega)^3},$$

mit $v \in V_{\star h} = V_\star + V_h$. Für $v_h = [H_h, E_h]^T \in V_h$,

$$\begin{aligned}
0 \stackrel{?}{=} a_h^{(0)}(v_h, v_h) &= \sum_{K \in \mathcal{T}_h} \left[(\nabla \times E_K, H_K)_{L^2(K)^3} - (\nabla \times H_K, E_K)_{L^2(K)^3} \right] \\
&= \sum_{K \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_K} (n_K \times E_K, H_K)_{L^2(F)^3} \\
&= \sum_{F \in \mathcal{F}_h^i} \left[-(n_F \times [\![E_h]\!]_F, \{\!\{H_h\}\!\}_F)_{L^2(F)^3} + (n_F \times [\![H_h]\!]_F, \{\!\{E_h\}\!\}_F)_{L^2(F)^3} \right] \\
&\quad + \sum_{F \in \mathcal{F}_h^b} (n \times E_h, H_h)_{L^2(F)^3}
\end{aligned}$$

Konstruktion

Diskr. Bilinearform: Für $v = [H, E]^T$, $\varphi_h = [\phi_h, \psi_h]^T \in V_h$,

$$a_h^{(0)}(v, \varphi_h) = (\nabla_h \times E, \phi_h)_{L^2(\Omega)^3} - (\nabla_h \times H, \psi_h)_{L^2(\Omega)^3},$$

mit $v \in V_{\star h} = V_\star + V_h$. Für $v_h = [H_h, E_h]^T \in V_h$,

$$\begin{aligned} 0 = a_h^{(0)}(v, v_h) &= \sum_{K \in \mathcal{T}_h} \left[(\nabla \times E_K, H_K)_{L^2(K)^3} - (\nabla \times H_K, E_K)_{L^2(K)^3} \right] \\ &= \sum_{K \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_K} (n_K \times E_K, H_K)_{L^2(F)^3} \\ &= \sum_{F \in \mathcal{F}_h^i} \left[-(n_F \times [\![E]\!]_F, \{\!\{ H_h \}\!\}_F)_{L^2(F)^3} + (n_F \times [\![H]\!]_F, \{\!\{ E_h \}\!\}_F)_{L^2(F)^3} \right] \\ &\quad + \sum_{F \in \mathcal{F}_h^b} (n \times E, H_h)_{L^2(F)^3} \end{aligned}$$

Centered Fluxes Bilinearform: Für $v = [H, E]^T$ und $\varphi_h = [\phi_h, \psi_h]^T$,

$$
\begin{aligned}
a_h^{\mathrm{cf}}(v, \varphi_h) = {} & (\nabla_h \times E, \phi_h)_{L^2(\Omega)^3} - (\nabla_h \times H, \psi_h)_{L^2(\Omega)^3} \\
& + \sum_{F \in \mathcal{F}_h^i} \left[(n_F \times [\![E]\!]_F, \{\!\{\phi_h\}\!\}_F)_{L^2(F)^3} - (n_F \times [\![H]\!]_F, \{\!\{\psi_h\}\!\}_F)_{L^2(F)^3} \right] \\
& + \sum_{F \in \mathcal{F}_h^b} \left[-(n \times E, \phi_h)_{L^2(F)^3} \right].
\end{aligned}
$$

Stabilisierungsbilinearform: Für $v = [H, E]^T$ und $\varphi_h = [\phi_h, \psi_h]^T$,

$$
\begin{aligned}
s_h(v, \varphi_h) = {} & \frac{1}{2} \sum_{F \in \mathcal{F}_h^i} (n_F \times [\![H]\!]_F, n_F \times [\![\phi_h]\!]_F)_{L^2(F)^3} \\
& + \frac{1}{2} \sum_{F \in \mathcal{F}_h^i} (n_F \times [\![E]\!]_F, n_F \times [\![\psi_h]\!]_F)_{L^2(F)^3} + \sum_{F \in \mathcal{F}_h^b} (n \times E, n \times \psi_h)_{L^2(F)^3}.
\end{aligned}
$$

Upwind Fluxes Bilinearform: Für $v = [H, E]^T$ und $\varphi_h = [\phi_h, \psi_h]^T$,

$$
a_h^{\mathrm{upw}}(v, \varphi_h) = a_h^{\mathrm{cf}}(v, \varphi_h) + s_h(v, \varphi_h).
$$

Centered Fluxes Bilinearform: Für $v = [H, E]^T$ und $\varphi_h = [\phi_h, \psi_h]^T$,

$$a_h^{\mathrm{cf}}(v, \varphi_h) = (\nabla_h \times E, \phi_h)_{L^2(\Omega)^3} - (\nabla_h \times H, \psi_h)_{L^2(\Omega)^3} + \sum_{F \in \mathcal{F}_h^i} \left[(n_F \times [\![E]\!]_F, \{\!\{\phi_h\}\!\}_F)_{L^2(F)^3} - (n_F \times [\![H]\!]_F, \{\!\{\psi_h\}\!\}_F)_{L^2(F)^3} \right] + \sum_{F \in \mathcal{F}_h^b} \left[-(n \times E, \phi_h)_{L^2(F)^3} \right].$$

Stabilisierungsbilinearform: Für $v = [H, E]^T$ und $\varphi_h = [\phi_h, \psi_h]^T$,

$$s_h(v, \varphi_h) = \frac{1}{2} \sum_{F \in \mathcal{F}_h^i} (n_F \times [\![H]\!]_F, n_F \times [\![\phi_h]\!]_F)_{L^2(F)^3} + \frac{1}{2} \sum_{F \in \mathcal{F}_h^i} (n_F \times [\![E]\!]_F, n_F \times [\![\psi_h]\!]_F)_{L^2(F)^3} + \sum_{F \in \mathcal{F}_h^b} (n \times E, n \times \psi_h)_{L^2(F)^3}.$$

Upwind Fluxes Bilinearform: Für $v = [H, E]^T$ und $\varphi_h = [\phi_h, \psi_h]^T$,

$$a_h^{\mathrm{upw}}(v, \varphi_h) = a_h^{\mathrm{cf}}(v, \varphi_h) + s_h(v, \varphi_h).$$

Centered Fluxes Bilinearform: Für $v = [H, E]^T$ und $\varphi_h = [\phi_h, \psi_h]^T$,

$$a_h^{\mathrm{cf}}(v, \varphi_h) = (\nabla_h \times E, \phi_h)_{L^2(\Omega)^3} - (\nabla_h \times H, \psi_h)_{L^2(\Omega)^3}$$
$$+ \sum_{F \in \mathcal{F}_h^i} \left[(n_F \times [\![E]\!]_F, \{\!\{\phi_h\}\!\}_F)_{L^2(F)^3} - (n_F \times [\![H]\!]_F, \{\!\{\psi_h\}\!\}_F)_{L^2(F)^3} \right]$$
$$+ \sum_{F \in \mathcal{F}_h^b} \left[-(n \times E, \phi_h)_{L^2(F)^3} \right].$$

Stabilisierungsbilinearform: Für $v = [H, E]^T$ und $\varphi_h = [\phi_h, \psi_h]^T$,

$$s_h(v, \varphi_h) = \frac{1}{2} \sum_{F \in \mathcal{F}_h^i} (n_F \times [\![H]\!]_F, n_F \times [\![\phi_h]\!]_F)_{L^2(F)^3}$$
$$+ \frac{1}{2} \sum_{F \in \mathcal{F}_h^i} (n_F \times [\![E]\!]_F, n_F \times [\![\psi_h]\!]_F)_{L^2(F)^3} + \sum_{F \in \mathcal{F}_h^b} (n \times E, n \times \psi_h)_{L^2(F)^3}.$$

Upwind Fluxes Bilinearform: Für $v = [H, E]^T$ und $\varphi_h = [\phi_h, \psi_h]^T$,

$$a_h^{\mathrm{upw}}(v, \varphi_h) = a_h^{\mathrm{cf}}(v, \varphi_h) + s_h(v, \varphi_h).$$

Diskrete Operatoren: Sei $a_h \in \{a_h^{\mathrm{cf}}, a_h^{\mathrm{upw}}\}$. Für $v \in V_{\star h}$,

$$(A_h v, \varphi_h) := a_h(v, \varphi_h) \qquad \forall \varphi_h \in V_h.$$

Eigenschaften

i) Konsistent: Für alle $v \in V_\star$: $\quad A_h v = \pi_h A v$.

ii) Schief-symmetrisch: Für alle $v_h \in V_h$,

$$(A_h^{\mathrm{cf}} v_h, v_h)_V = 0.$$

Dissipativ: Für alle $v_h \in V_h$,

$$(-A_h^{\mathrm{upw}} v_h, v_h)_V = -|v_h|_S^2 \leq 0.$$

iii) Projektionsfehler: Für alle $v \in V_{\star h}$ und alle $\varphi_h \in V_h$,

$$|(A_h(v - \pi_h v), \varphi_h)_V| \leq C h^{-1/2} |\varphi_h|_S \|v - \pi_h v\|_V.$$

Für $v \in V_{\star h}$: $|v|_S^2 := s_h(v, v)$.

Zeitdiskretisierung

Evolutionsgleichung in V_h: Suche $u_h \in V_h$, s. d. $u_h(0) = \pi_h u_0$,

$$\dot{u}_h(t) = - A_h u_h(t) + g_h(t).$$

Stabilität A_h auf V_h: Für alle $v_h \in V_h$,

$$\|A_h v_h\|_V \leq C h^{-1} \|v_h\|_V.$$

Explizite RK-Verfahren (homogen)

i) RK1:

$$u_h^{n+1} = u_h^n - \tau A_h u_h^n.$$

ii) RK2:

$$\widetilde{U}_h^{n2} = \widetilde{U}_h^{n1} + \frac{1}{2}\tau A_h(u_h^n - \widetilde{U}_h^{n1}).$$

iii) RK3:

$$u_h^{n+1} = \widetilde{U}_h^{n2} + \frac{1}{3}\tau A_h(\widetilde{U}_h^{n1} - \widetilde{U}_h^{n2}).$$

Zeitdiskretisierung

Evolutionsgleichung in V_h: Suche $u_h \in V_h$, s. d. $u_h(0) = \pi_h u_0$,

$$\dot{u}_h(t) = -A_h u_h(t).$$

Stabilität A_h auf V_h: Für alle $v_h \in V_h$,

$$\|A_h v_h\|_V \leq C h^{-1} \|v_h\|_V.$$

Explizite RK-Verfahren (homogen)

i) RK1:

$$u_h^{n+1} = u_h^n - \tau A_h u_h^n.$$

ii) RK2:

$$\widetilde{U}_h^{n2} = \widetilde{U}_h^{n1} + \frac{1}{2}\tau A_h(u_h^n - \widetilde{U}_h^{n1}).$$

iii) RK3:

$$u_h^{n+1} = \widetilde{U}_h^{n2} + \frac{1}{3}\tau A_h(\widetilde{U}_h^{n1} - \widetilde{U}_h^{n2}).$$

Zeitdiskretisierung

Evolutionsgleichung in V_h: Suche $u_h \in V_h$, s. d. $u_h(0) = \pi_h u_0$,

$$\dot{u}_h(t) = -A_h u_h(t).$$

Stabilität A_h auf V_h: Für alle $v_h \in V_h$,

$$\|A_h v_h\|_V \leq C h^{-1} \|v_h\|_V.$$

Explizite RK-Verfahren (homogen)

i) RK1:

$$u_h^{n+1} = u_h^n - \tau A_h u_h^n.$$

ii) RK2:

$$u_h^{n+1} = u_h^n - \tau A_h u_h^n + \frac{1}{2}\tau^2 A_h^2 u_h^n.$$

iii) RK3:

$$u_h^{n+1} = u_h^n - \tau A_h u_h^n + \frac{1}{2}\tau^2 A_h^2 u_h^n - \frac{1}{6}\tau^3 A_h^3 u_h^n.$$

Zeitdiskretisierung

Evolutionsgleichung in V_h: Suche $u_h \in V_h$, s. d. $u_h(0) = \pi_h u_0$,

$$\dot{u}_h(t) = -A_h u_h(t).$$

Stabilität A_h auf V_h: Für alle $v_h \in V_h$,

$$\|A_h v_h\|_V \leq C h^{-1} \|v_h\|_V.$$

Explizite RK-Verfahren (homogen)

i) RK1:

$$u_h^{n+1} = u_h^n - \tau A_h u_h^n.$$

ii) RK2:

$$u_h^{n+1} = u_h^n - \tau A_h u_h^n + \frac{1}{2}\tau^2 A_h^2 u_h^n.$$

iii) RK3:

$$u_h^{n+1} = u_h^n - \tau A_h u_h^n + \frac{1}{2}\tau^2 A_h^2 u_h^n - \frac{1}{6}\tau^3 A_h^3 u_h^n.$$

Zeitdiskretisierung

Evolutionsgleichung in V_h: Suche $u_h \in V_h$, s. d. $u_h(0) = \pi_h u_0$,

$$\dot{u}_h(t) = -A_h u_h(t).$$

Stabilität A_h auf V_h: Für alle $v_h \in V_h$,

$$\|A_h v_h\|_V \leq C h^{-1} \|v_h\|_V.$$

Explizite RK-Verfahren (homogen)

i) RK1:

$$\widetilde{U}_h^{n1} = u_h^n - \tau A_h u_h^n.$$

ii) RK2:

$$u_h^{n+1} = \widetilde{U}_h^{n1} + \frac{1}{2}\tau^2 A_h^2 u_h^n.$$

iii) RK3:

$$u_h^{n+1} = u_h^n - \tau A_h u_h^n + \frac{1}{2}\tau^2 A_h^2 u_h^n - \frac{1}{6}\tau^3 A_h^3 u_h^n.$$

Zeitdiskretisierung

Evolutionsgleichung in V_h: Suche $u_h \in V_h$, s. d. $u_h(0) = \pi_h u_0$,

$$\dot{u}_h(t) = -\, A_h u_h(t).$$

Stabilität A_h auf V_h: Für alle $v_h \in V_h$,

$$\|A_h v_h\|_V \leq C h^{-1} \|v_h\|_V.$$

Explizite RK-Verfahren (homogen)

i) RK1:

$$\widetilde{U}_h^{n1} = u_h^n - \tau A_h u_h^n.$$

ii) RK2:

$$u_h^{n+1} = \widetilde{U}_h^{n1} + \frac{1}{2}\tau A_h (u_h^n - \widetilde{U}_h^{n1}).$$

iii) RK3:

$$u_h^{n+1} = u_h^n - \tau A_h u_h^n + \frac{1}{2}\tau^2 A_h^2 u_h^n - \frac{1}{6}\tau^3 A_h^3 u_h^n.$$

Zeitdiskretisierung

Evolutionsgleichung in V_h: Suche $u_h \in V_h$, s. d. $u_h(0) = \pi_h u_0$,

$$\dot{u}_h(t) = -A_h u_h(t).$$

Stabilität A_h auf V_h: Für alle $v_h \in V_h$,

$$\|A_h v_h\|_V \leq Ch^{-1} \|v_h\|_V.$$

Explizite RK-Verfahren (homogen)

i) RK1:

$$\widetilde{U}_h^{n1} = u_h^n - \tau A_h u_h^n.$$

ii) RK2:

$$\widetilde{U}_h^{n2} = \widetilde{U}_h^{n1} + \frac{1}{2} \tau A_h (u_h^n - \widetilde{U}_h^{n1}).$$

iii) RK3:

$$u_h^{n+1} = \widetilde{U}_h^{n2} + \frac{1}{3} \tau A_h (\widetilde{U}_h^{n1} - \widetilde{U}_h^{n2}).$$

Energieidentitäten (homogen)

RK1:

$$\|u_h^{n+1}\|_V^2 = \|u_h^n\|_V^2 - 2\tau(A_h u_h^n, u_h^n)_V + \|\tau A_h u_h^n\|_V^2.$$

RK2:

$$\|u_h^{n+1}\|_V^2 = \|u_h^n\|_V^2 - \tau(A_h u_h^n, u_h^n)_V - \tau(A_h \widetilde{U}_h^{n1}, \widetilde{U}_h^{n1})_V + \frac{1}{4}\|\tau^2 A_h^2 u_h^n\|_V^2.$$

RK3:

$$\begin{aligned}\|u_h^{n+1}\|_V^2 =& \|u_h^n\|_V^2 - \tau(A_h u_h^n, u_h^n)_V - \frac{1}{3}\tau(A_h \widetilde{U}_h^{n1}, \widetilde{U}_h^{n1})_V - \frac{2}{3}\tau(A_h \widetilde{U}_h^{n2}, \widetilde{U}_h^{n2})_V \\ &+ \frac{1}{3}\tau(\tau^2 A_h^2 u_h^n, \tau A_h u_h^n)_V - \frac{1}{12}\|\tau^2 A_h^2 u_h^n\|_V^2 + \frac{1}{36}\|\tau^3 A_h^3 u_h^n\|_V^2.\end{aligned}$$

Energieidentitäten (homogen)

RK1:

$$\|u_h^{n+1}\|_V^2 = \|u_h^n\|_V^2 + \|\tau A_h^{\mathrm{cf}} u_h^n\|_V^2, \qquad \text{(CF)}$$

RK2:

$$\|u_h^{n+1}\|_V^2 = \|u_h^n\|_V^2 + \frac{1}{4}\|\tau^2 (A_h^{\mathrm{cf}})^2 u_h^n\|_V^2, \qquad \text{(CF)}$$

RK3:

$$\|u_h^{n+1}\|_V^2 + \frac{1}{12}\|\tau^2 (A_h^{\mathrm{cf}})^2 u_h^n\|_V^2 = \|u_h^n\|_V^2 + \frac{1}{36}\|\tau^3 (A_h^{\mathrm{cf}})^3 u_h^n\|_V^2, \qquad \text{(CF)}$$

Energieidentitäten (homogen)

RK1:

$$\|u_h^{n+1}\|_V^2 = \|u_h^n\|_V^2 + \|\tau A_h^{\mathrm{cf}} u_h^n\|_V^2, \qquad \text{(CF)}$$

$$\|u_h^{n+1}\|_V^2 + 2\tau|u_h^n|_S^2 = \|u_h^n\|_V^2 + \|\tau A_h^{\mathrm{upw}} u_h^n\|_V^2. \qquad \text{(UPW)}$$

RK2:

$$\|u_h^{n+1}\|_V^2 = \|u_h^n\|_V^2 + \frac{1}{4}\|\tau^2 (A_h^{\mathrm{cf}})^2 u_h^n\|_V^2, \qquad \text{(CF)}$$

$$\|u_h^{n+1}\|_V^2 + \tau|u_h^n|_S^2 + \tau|\widetilde{U}_h^{n1}|_S^2 = \|u_h^n\|_V^2 + \frac{1}{4}\|\tau^2 (A_h^{\mathrm{upw}})^2 u_h^n\|_V^2. \qquad \text{(UPW)}$$

RK3:

$$\|u_h^{n+1}\|_V^2 + \frac{1}{12}\|\tau^2 (A_h^{\mathrm{cf}})^2 u_h^n\|_V^2 = \|u_h^n\|_V^2 + \frac{1}{36}\|\tau^3 (A_h^{\mathrm{cf}})^3 u_h^n\|_V^2, \qquad \text{(CF)}$$

$$\|u_h^{n+1}\|_V^2 + \tau|u_h^n|_S^2 + \frac{1}{3}\tau|\widetilde{U}_h^{n1}|_S^2 + \frac{2}{3}\tau|\widetilde{U}_h^{n2}|_S^2 + \frac{1}{12}\|\tau^2 (A_h^{\mathrm{upw}})^2 u_h^n\|_V^2$$

$$= \|u_h^n\|_V^2 + \frac{1}{3}\tau|\tau A_h^{\mathrm{upw}} u_h^n|_S^2 + \frac{1}{36}\|\tau^3 (A_h^{\mathrm{upw}})^3 u_h^n\|_V^2. \qquad \text{(UPW)}$$

Explizite RK-Verfahren (inhomogen)

Für RK1

$$u_h^{n+1} = u_h^n - \tau A_h u_h^n + \tau g_h^n.$$

Für RK2

$$\begin{aligned} u_h^{n+1} &= u_h^n - \tau A_h u_h^n + \frac{1}{2}\tau^2 A_h^2 u_h^n + \tau b_1 g_h^n + \tau b_2 g_h^{n2} - \frac{1}{2}\tau^2 A_h g_h^n \\ &= u_h^n - \tau A_h u_h^n + \frac{1}{2}\tau^2 A_h^2 u_h^n + \tau g_h^n + \frac{1}{2}\tau^2 \partial_t g_h^n - \frac{1}{2}\tau^2 A_h g_h^n + \tau R_2^n \\ &= U_h^{n1} + \frac{1}{2}\tau A_h (u_h^n - U_h^{n1}) + \frac{1}{2}\tau^2 \partial_t g_h^n + \tau R_2^n \end{aligned}$$

Für RK3

$$u_h^{n+1} = U_h^{n2} + \frac{1}{3}\tau A_h (U_h^{n1} - U_h^{n2}) + \frac{1}{6}\tau^3 \partial_{tt} g_h^n + \tau R_3^n.$$

Notation: $g_h^{n2} = g_h(t_n + c_2\tau)$.

Explizite RK-Verfahren (inhomogen)

Für RK1

$$u_h^{n+1} = u_h^n - \tau A_h u_h^n + \tau g_h^n.$$

Für RK2

$$\begin{aligned} u_h^{n+1} &= u_h^n - \tau A_h u_h^n + \frac{1}{2}\tau^2 A_h^2 u_h^n + \tau b_1 g_h^n + \tau b_2 g_h^{n2} - \frac{1}{2}\tau^2 A_h g_h^n \\ &= u_h^n - \tau A_h u_h^n + \frac{1}{2}\tau^2 A_h^2 u_h^n + \tau g_h^n + \frac{1}{2}\tau^2 \partial_t g_h^n - \frac{1}{2}\tau^2 A_h g_h^n + \tau R_2^n \\ &= U_h^{n1} + \frac{1}{2}\tau A_h (u_h^n - U_h^{n1}) + \frac{1}{2}\tau^2 \partial_t g_h^n + \tau R_2^n \end{aligned}$$

Für RK3

$$u_h^{n+1} = U_h^{n2} + \frac{1}{3}\tau A_h (U_h^{n1} - U_h^{n2}) + \frac{1}{6}\tau^3 \partial_{tt} g_h^n + \tau R_3^n.$$

Notation: $g_h^{n2} = g_h(t_n + c_2\tau)$.

Explizite RK-Verfahren (inhomogen)

Für RK1

$$U_h^{n1} = u_h^n - \tau A_h u_h^n + \tau g_h^n.$$

Für RK2

$$\begin{aligned} u_h^{n+1} &= u_h^n - \tau A_h u_h^n + \frac{1}{2}\tau^2 A_h^2 u_h^n + \tau b_1 g_h^n + \tau b_2 g_h^{n2} - \frac{1}{2}\tau^2 A_h g_h^n \\ &= u_h^n - \tau A_h u_h^n + \frac{1}{2}\tau^2 A_h^2 u_h^n + \tau g_h^n + \frac{1}{2}\tau^2 \partial_t g_h^n - \frac{1}{2}\tau^2 A_h g_h^n + \tau R_2^n \\ &= U_h^{n1} + \frac{1}{2}\tau A_h(u_h^n - U_h^{n1}) + \frac{1}{2}\tau^2 \partial_t g_h^n + \tau R_2^n \end{aligned}$$

Für RK3

$$u_h^{n+1} = U_h^{n2} + \frac{1}{3}\tau A_h(U_h^{n1} - U_h^{n2}) + \frac{1}{6}\tau^3 \partial_{tt} g_h^n + \tau R_3^n.$$

Notation: $g_h^{n2} = g_h(t_n + c_2\tau)$.

Explizite RK-Verfahren (inhomogen)

Für RK1

$$U_h^{n1} = u_h^n - \tau A_h u_h^n + \tau g_h^n.$$

Für RK2

$$\begin{aligned} U_h^{n2} =& u_h^n - \tau A_h u_h^n + \frac{1}{2}\tau^2 A_h^2 u_h^n + \tau b_1 g_h^n + \tau b_2 g_h^{n2} - \frac{1}{2}\tau^2 A_h g_h^n \\ &= u_h^n - \tau A_h u_h^n + \frac{1}{2}\tau^2 A_h^2 u_h^n + \tau g_h^n + \frac{1}{2}\tau^2 \partial_t g_h^n - \frac{1}{2}\tau^2 A_h g_h^n + \tau R_2^n \\ &= U_h^{n1} + \frac{1}{2}\tau A_h (u_h^n - U_h^{n1}) + \frac{1}{2}\tau^2 \partial_t g_h^n + \tau R_2^n \end{aligned}$$

Für RK3

$$u_h^{n+1} = U_h^{n2} + \frac{1}{3}\tau A_h (U_h^{n1} - U_h^{n2}) + \frac{1}{6}\tau^3 \partial_{tt} g_h^n + \tau R_3^n.$$

Notation: $g_h^{n2} = g_h(t_n + c_2\tau)$.

Fehlerrekursion

Fehlersplitting: $e^n = u(t_n) - u_h^n = \underbrace{(u(t_n) - \pi_h u(t_n))}_{e_\pi^n} - \underbrace{(u_h^n - \pi_h u(t_n))}_{e_h^n}$.

Für RK1

$$e_h^{n+1} = e_h^n - \tau A_h e_h^n + \tau A_h e_\pi^n + \tau D_1^n.$$

Für RK2

$$e_h^{n+1} = E_h^{n1} + \frac{1}{2}\tau A_h(e_h^n - E_h^{n1}) + \frac{1}{2}\tau^2 A_h(\partial_t e_\pi^n) + \tau D_2^n + \tau R_2^n.$$

Für RK3

$$e_h^{n+1} = E_h^{n2} + \frac{1}{3}\tau A_h(E_h^{n1} - E_h^{n2}) + \frac{1}{6}\tau^3 A_h(\partial_{tt} e_\pi^n) + \tau D_3^n + \tau R_3^n.$$

Fehlerrekursion

Fehlersplitting: $e^n = u(t_n) - u_h^n = \underbrace{(u(t_n) - \pi_h u(t_n))}_{e_\pi^n} - \underbrace{(u_h^n - \pi_h u(t_n))}_{e_h^n}$.

Für RK1

$$e_h^{n+1} = e_h^n - \tau A_h e_h^n + \tau A_h e_\pi^n + \tau D_1^n.$$

Für RK2

$$e_h^{n+1} = E_h^{n1} + \frac{1}{2}\tau A_h(e_h^n - E_h^{n1}) + \frac{1}{2}\tau^2 A_h(\partial_t e_\pi^n) + \tau D_2^n + \tau R_2^n.$$

Für RK3

$$e_h^{n+1} = E_h^{n2} + \frac{1}{3}\tau A_h(E_h^{n1} - E_h^{n2}) + \frac{1}{6}\tau^3 A_h(\partial_{tt} e_\pi^n) + \tau D_3^n + \tau R_3^n.$$

Fehlerrekursion

Fehlersplitting: $e^n = u(t_n) - u_h^n = \underbrace{(u(t_n) - \pi_h u(t_n))}_{e_\pi^n} - \underbrace{(u_h^n - \pi_h u(t_n))}_{e_h^n}$.

Für RK1

$$e_h^{n+1} = e_h^n - \tau A_h e_h^n + \tau A_h e_\pi^n + \tau D_1^n.$$

Für RK2

$$e_h^{n+1} = E_h^{n1} + \frac{1}{2}\tau A_h(e_h^n - E_h^{n1}) + \frac{1}{2}\tau^2 A_h(\partial_t e_\pi^n) + \tau D_2^n + \tau R_2^n.$$

Für RK3

$$e_h^{n+1} = E_h^{n2} + \frac{1}{3}\tau A_h(E_h^{n1} - E_h^{n2}) + \frac{1}{6}\tau^3 A_h(\partial_{tt} e_\pi^n) + \tau D_3^n + \tau R_3^n.$$

Fehlerrekursion

Fehlersplitting: $e^n = u(t_n) - u_h^n = \underbrace{(u(t_n) - \pi_h u(t_n))}_{e_\pi^n} - \underbrace{(u_h^n - \pi_h u(t_n))}_{e_h^n}$.

Für RK1

$$E_h^{n1} = e_h^n - \tau A_h e_h^n + \tau A_h e_\pi^n + \tau D_1^n.$$

Für RK2

$$e_h^{n+1} = E_h^{n1} + \frac{1}{2}\tau A_h(e_h^n - E_h^{n1}) + \frac{1}{2}\tau^2 A_h(\partial_t e_\pi^n) + \tau D_2^n + \tau R_2^n.$$

Für RK3

$$e_h^{n+1} = E_h^{n2} + \frac{1}{3}\tau A_h(E_h^{n1} - E_h^{n2}) + \frac{1}{6}\tau^3 A_h(\partial_{tt} e_\pi^n) + \tau D_3^n + \tau R_3^n.$$

Fehlerrekursion

Fehlersplitting: $e^n = u(t_n) - u_h^n = \underbrace{(u(t_n) - \pi_h u(t_n))}_{e_\pi^n} - \underbrace{(u_h^n - \pi_h u(t_n))}_{e_h^n}$.

Für RK1

$$E_h^{n1} = e_h^n - \tau A_h e_h^n + \tau A_h e_\pi^n + \tau D_1^n.$$

Für RK2

$$E_h^{n2} = E_h^{n1} + \frac{1}{2}\tau A_h(e_h^n - E_h^{n1}) + \frac{1}{2}\tau^2 A_h(\partial_t e_\pi^n) + \tau D_2^n + \tau R_2^n.$$

Für RK3

$$e_h^{n+1} = E_h^{n2} + \frac{1}{3}\tau A_h(E_h^{n1} - E_h^{n2}) + \frac{1}{6}\tau^3 A_h(\partial_{tt} e_\pi^n) + \tau D_3^n + \tau R_3^n.$$

Konvergenz RK3 mit Upwind Fluxes

$$\|e_h^{n+1}\|_V^2 + \tau|e_h^n|_S^2 + \frac{1}{3}\tau|E_h^{n1}|_S^2 + \frac{2}{3}\tau|E_h^{n2}|_S^2 + \frac{1}{12}\|\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n)\|_V^2$$

$$= \|e_h^n\|_V^2 + \tau(e_h^n, A_h E_\pi^{31})_V + \frac{1}{3}\tau(E_h^{n1}, A_h E_\pi^{32})_V + \frac{2}{3}\tau(E_h^{n2}, A_h E_\pi^{33} + \mathcal{R})_V$$

$$+ \frac{1}{3}\tau|\tau A_h e^n|_S^2 + \frac{1}{36}\|\tau^3 A_h^3 e^n + \tau^3 A_h^2(\partial_t e_\pi^n) - \tau^3 A_h(\partial_{tt} e_\pi^n) + \mathcal{R}\|_V^2.$$

CFL-Bedingung: $\boxed{\text{Für } \varrho > 0\text{: } \tau \leq \varrho h}$

Aus DG-Konstruktion:

$$\tau(e_h, A_h E_\pi)_V \leq C\tau h^{-1/2}|e_h|_S\|E_\pi\|_V \leq \gamma\tau|e_h|_S^2 + C\tau h^{2k+1}$$

Es gilt:

$$\frac{1}{3}\tau|\tau A_h e^n|_S^2 = \frac{1}{3}\tau|(e_h^n - E_h^{n2}) + \frac{1}{2}(\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n))|_S^2$$

$$\leq \frac{11}{12}\tau|e_h^n|_S^2 + \frac{7}{12}\tau|E_h^{n2}|_S^2 + \frac{1}{24}\|\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n)\|_V^2.$$

Konvergenz RK3 mit Upwind Fluxes

$$
\begin{aligned}
&\|e_h^{n+1}\|_V^2 + \tau|e_h^n|_S^2 + \frac{1}{3}\tau|E_h^{n1}|_S^2 + \frac{2}{3}\tau|E_h^{n2}|_S^2 + \frac{1}{12}\|\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n)\|_V^2 \\
&\le \|e_h^n\|_V^2 + \tau(e_h^n, A_h E_\pi^{31})_V + \frac{1}{3}\tau(E_h^{n1}, A_h E_\pi^{32})_V + \frac{2}{3}\tau(E_h^{n2}, A_h E_\pi^{33} + \mathcal{R})_V \\
&\quad + \frac{1}{3}\tau|\tau A_h e^n|_S^2 + \frac{1}{18}\|\tau^3 A_h^3 e^n + \tau^3 A_h^2(\partial_t e_\pi^n)\|_V^2 + \frac{1}{18}\|\tau^3 A_h(\partial_{tt} e_\pi^n) + \mathcal{R}\|_V^2.
\end{aligned}
$$

CFL-Bedingung: $\boxed{\text{Für } \varrho > 0\text{: } \tau \le \varrho h}$

Aus DG-Konstruktion:

$$\tau(e_h, A_h E_\pi)_V \le C\tau h^{-1/2}|e_h|_S\|E_\pi\|_V \le \gamma\tau|e_h|_S^2 + C\tau h^{2k+1}$$

Es gilt:

$$
\begin{aligned}
\frac{1}{3}\tau|\tau A_h e^n|_S^2 &= \frac{1}{3}\tau|(e_h^n - E_h^{n2}) + \frac{1}{2}(\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n))|_S^2 \\
&\le \frac{11}{12}\tau|e_h^n|_S^2 + \frac{7}{12}\tau|E_h^{n2}|_S^2 + \frac{1}{24}\|\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n)\|_V^2.
\end{aligned}
$$

Konvergenz RK3 mit Upwind Fluxes

$$\|e_h^{n+1}\|_V^2 + \tau|e_h^n|_S^2 + \frac{1}{3}\tau|E_h^{n1}|_S^2 + \frac{2}{3}\tau|E_h^{n2}|_S^2 + \frac{1}{12}\|\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n)\|_V^2$$

$$\leq \|e_h^n\|_V^2 + \tau(e_h^n, A_h E_\pi^{31})_V + \frac{1}{3}\tau(E_h^{n1}, A_h E_\pi^{32})_V + \frac{2}{3}\tau(E_h^{n2}, A_h E_\pi^{33} + \mathcal{R})_V$$

$$+ \frac{1}{3}\tau|\tau A_h e^n|_S^2 + \frac{1}{18}C\tau^2 h^{-2}\|\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n)\|_V^2 + C\tau(h^{2k+1} + \tau^6).$$

CFL-Bedingung: $\boxed{\text{Für } \varrho > 0\text{: } \tau \leq \varrho h}$

Aus DG-Konstruktion:

$$\tau(e_h, A_h E_\pi)_V \leq C\tau h^{-1/2}|e_h|_S\|E_\pi\|_V \leq \gamma\tau|e_h|_S^2 + C\tau h^{2k+1}$$

Es gilt:

$$\frac{1}{3}\tau|\tau A_h e^n|_S^2 = \frac{1}{3}\tau|(e_h^n - E_h^{n2}) + \frac{1}{2}(\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n))|_S^2$$

$$\leq \frac{11}{12}\tau|e_h^n|_S^2 + \frac{7}{12}\tau|E_h^{n2}|_S^2 + \frac{1}{24}\|\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n)\|_V^2.$$

Konvergenz RK3 mit Upwind Fluxes

$$\|e_h^{n+1}\|_V^2 + \tau|e_h^n|_S^2 + \frac{1}{3}\tau|E_h^{n1}|_S^2 + \frac{2}{3}\tau|E_h^{n2}|_S^2 + \frac{1}{12}\|\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n)\|_V^2$$
$$\leq \|e_h^n\|_V^2 + \tau(e_h^n, A_h E_\pi^{31})_V + \frac{1}{3}\tau(E_h^{n1}, A_h E_\pi^{32})_V + \frac{2}{3}\tau(E_h^{n2}, A_h E_\pi^{33} + \mathcal{R})_V$$
$$+ \frac{1}{3}\tau|\tau A_h e^n|_S^2 + \frac{1}{24}\|\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n)\|_V^2 + C\tau(h^{2k+1} + \tau^6).$$

CFL-Bedingung: $\boxed{\text{Für } \varrho > 0\text{: } \tau \leq \varrho h}$

Aus DG-Konstruktion:

$$\tau(e_h, A_h E_\pi)_V \leq C\tau h^{-1/2}|e_h|_S\|E_\pi\|_V \leq \gamma\tau|e_h|_S^2 + C\tau h^{2k+1}$$

Es gilt:

$$\frac{1}{3}\tau|\tau A_h e^n|_S^2 = \frac{1}{3}\tau|(e_h^n - E_h^{n2}) + \frac{1}{2}(\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n))|_S^2$$
$$\leq \frac{11}{12}\tau|e_h^n|_S^2 + \frac{7}{12}\tau|E_h^{n2}|_S^2 + \frac{1}{24}\|\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n)\|_V^2.$$

Konvergenz RK3 mit Upwind Fluxes

$$\|e_h^{n+1}\|_V^2 + \tau|e_h^n|_S^2 + \frac{1}{3}\tau|E_h^{n1}|_S^2 + \frac{2}{3}\tau|E_h^{n2}|_S^2 + \frac{1}{12}\|\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n)\|_V^2$$

$$\leq \|e_h^n\|_V^2 + \tau(e_h^n, A_h E_\pi^{31})_V + \frac{1}{3}\tau(E_h^{n1}, A_h E_\pi^{32})_V + \frac{2}{3}\tau(E_h^{n2}, A_h E_\pi^{33} + \mathcal{R})_V$$

$$+ \frac{1}{3}\tau|\tau A_h e^n|_S^2 + \frac{1}{24}\|\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n)\|_V^2 + C\tau(h^{2k+1} + \tau^6).$$

CFL-Bedingung: $\boxed{\text{Für } \varrho > 0\text{: } \tau \leq \varrho h}$

Aus DG-Konstruktion:

$$\tau(e_h, A_h E_\pi)_V \leq C\tau h^{-1/2}|e_h|_S\|E_\pi\|_V \leq \gamma\tau|e_h|_S^2 + C\tau h^{2k+1}$$

Es gilt:

$$\frac{1}{3}\tau|\tau A_h e^n|_S^2 = \frac{1}{3}\tau|(e_h^n - E_h^{n2}) + \frac{1}{2}(\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n))|_S^2$$

$$\leq \frac{11}{12}\tau|e_h^n|_S^2 + \frac{7}{12}\tau|E_h^{n2}|_S^2 + \frac{1}{24}\|\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n)\|_V^2.$$

Konvergenz RK3 mit Upwind Fluxes

$$\|e_h^{n+1}\|_V^2 + \tau|e_h^n|_S^2 + \frac{1}{3}\tau|E_h^{n1}|_S^2 + \frac{2}{3}\tau|E_h^{n2}|_S^2 + \frac{1}{12}\|\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n)\|_V^2$$

$$\le \|e_h^n\|_V^2 + \frac{1}{24}\tau|e_h^n|_S^2 + \frac{1}{6}\tau|E_h^{n1}|_S^2 + \frac{1}{24}\tau|E_h^{n2}|_S^2 + C\tau\|E_h^{n2}\|_V^2$$

$$+ \frac{1}{3}\tau|\tau A_h e^n|_S^2 + \frac{1}{24}\|\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n)\|_V^2 + C\tau(h^{2k+1} + \tau^6).$$

CFL-Bedingung: $\boxed{\text{Für } \varrho > 0\text{: } \tau \le \varrho h}$

Aus DG-Konstruktion:

$$\tau(e_h, A_h E_\pi)_V \le C\tau h^{-1/2}|e_h|_S\|E_\pi\|_V \le \gamma\tau|e_h|_S^2 + C\tau h^{2k+1}$$

Es gilt:

$$\frac{1}{3}\tau|\tau A_h e^n|_S^2 = \frac{1}{3}\tau|(e_h^n - E_h^{n2}) + \frac{1}{2}(\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n))|_S^2$$

$$\le \frac{11}{12}\tau|e_h^n|_S^2 + \frac{7}{12}\tau|E_h^{n2}|_S^2 + \frac{1}{24}\|\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n)\|_V^2.$$

Konvergenz RK3 mit Upwind Fluxes

$$\|e_h^{n+1}\|_V^2 + \tau|e_h^n|_S^2 + \frac{1}{3}\tau|E_h^{n1}|_S^2 + \frac{2}{3}\tau|E_h^{n2}|_S^2 + \frac{1}{12}\|\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n)\|_V^2$$

$$\leq \|e_h^n\|_V^2 + \frac{1}{24}\tau|e_h^n|_S^2 + \frac{1}{6}\tau|E_h^{n1}|_S^2 + \frac{1}{24}\tau|E_h^{n2}|_S^2 + C\tau\|E_h^{n2}\|_V^2$$

$$+ \frac{1}{3}\tau|\tau A_h e^n|_S^2 + \frac{1}{24}\|\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n)\|_V^2 + C\tau(h^{2k+1} + \tau^6).$$

CFL-Bedingung: $\boxed{\text{Für } \varrho > 0\text{: } \tau \leq \varrho h}$

Aus DG-Konstruktion:

$$\tau(e_h, A_h E_\pi)_V \leq C\tau h^{-1/2}|e_h|_S\|E_\pi\|_V \leq \gamma\tau|e_h|_S^2 + C\tau h^{2k+1}$$

Es gilt:

$$\frac{1}{3}\tau|\tau A_h e^n|_S^2 = \frac{1}{3}\tau|(e_h^n - E_h^{n2}) + \frac{1}{2}(\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n))|_S^2$$

$$\leq \frac{11}{12}\tau|e_h^n|_S^2 + \frac{7}{12}\tau|E_h^{n2}|_S^2 + \frac{1}{24}\|\tau^2 A_h^2 e^n + \tau^2 A_h(\partial_t e_\pi^n)\|_V^2.$$

Konvergenz RK3 mit Upwind Fluxes

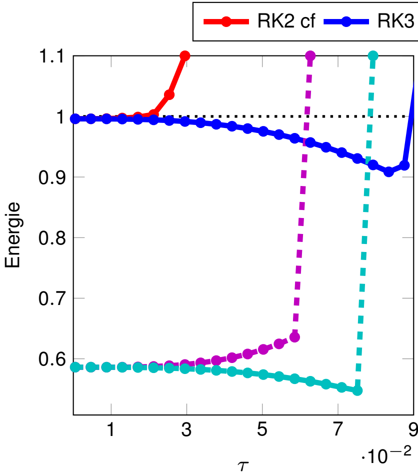
$$\|e_h^{n+1}\|_V^2 + \frac{1}{24}\tau|e_h^n|_S^2 + \frac{1}{6}\tau|E_h^{n1}|_S^2 + \frac{1}{24}\tau|E_h^{n2}|_S^2$$
$$\leq \|e_h^n\|_V^2 + C\tau\|e_h^n\|_V^2 + C\tau(h^{2k+1} + \tau^6).$$

Gronwall

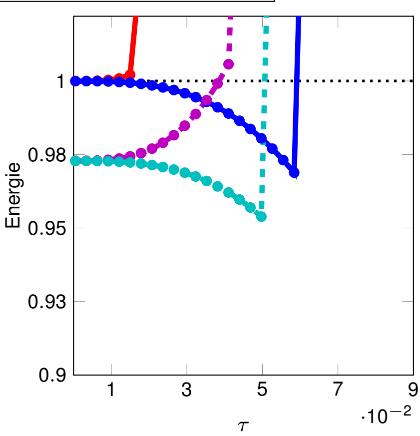
$$\|e_h^n\|_V^2 + \sum_{m=0}^{n-1}\left[\frac{1}{24}|e_h^m|_S^2 + \frac{1}{6}|E_h^{m1}|_S^2 + \frac{1}{24}|E_h^{m2}|_S^2\right] \leq C(h^{2k+1} + \tau^6).$$

Numerische Experimente

Energieerhaltung, Dissipation

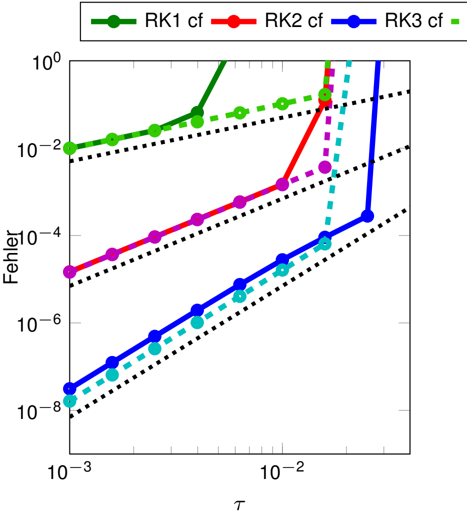


Gitterweite $h = 1$, Polynomgrad $k = 3$



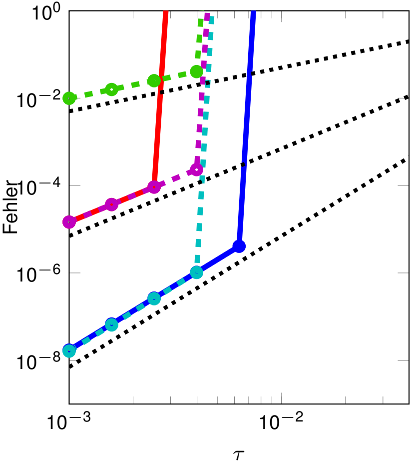
Gitterweite $h = 0.5$, Polynomgrad $k = 4$

Ordnung



Gitterweite $h = 0.5$,

Polynomgrad $k = 4$



Gitterweite $h = 0.125$,

Polynomgrad $k = 4$