Karlsruhe Institute of Technology

Convergence of an ADI splitting for Maxwell's equations

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Outline

1. Linear Maxwell's equations

2. Namiki; Zheng, Chen, Zhang method

- FDTD method
- NZCZ / ADI / Peaceman–Rachford splitting
- Efficient implementation
- Numerical example

3. Error analysis of NZCZ method

- Analytical framework, well-posedness
- Unconditional stability
- Accuracy

4. Summary and outlook

Outline

Linear Maxwell's equations

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Differential equations

$$\partial_t \mathbf{E} = \frac{1}{\varepsilon} \operatorname{rot} \mathbf{H} \qquad (x \in \Omega, t > 0)$$
$$\partial_t \mathbf{H} = -\frac{1}{\mu} \operatorname{rot} \mathbf{E}$$

Divergence conditions

$$\operatorname{div}(\varepsilon \mathbf{E}) = 0 \quad (x \in \Omega, t > 0)$$
$$\operatorname{div}(\mu \mathbf{H}) = 0$$

Initial conditions

$$\mathbf{E}(0, x) = \mathbf{E}^0(x) \qquad (x \in \Omega)$$
$$\mathbf{H}(0, x) = \mathbf{H}^0(x)$$

Boundary conditions

$$\mathbf{E} \times \nu = 0 \quad (x \in \partial\Omega, t > 0)$$
$$\mathbf{H} \cdot \nu = 0$$

$\mathbf{E}(t, x) \in \mathbb{R}^3$ electric field

$\mathbf{H}(t, x) \in \mathbb{R}^3$ magnetic field

$\nu \in \mathbb{R}^3$ outer unit normal vector

$\varepsilon(x) \in \mathbb{R}$ electrical permittivity

$\mu(x) \in \mathbb{R}$ magnetic permeability

Assumptions: $\varepsilon, \mu \in L^\infty(\Omega)$ and $\varepsilon, \mu \geq \delta > 0$.

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FDTD: finite-difference time-domain method

- Yee (1966), IEEE Trans. Antennas and Propagation
  Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media
- ISI web of science: 6 474 citations on February 20, 2014

in this talk:

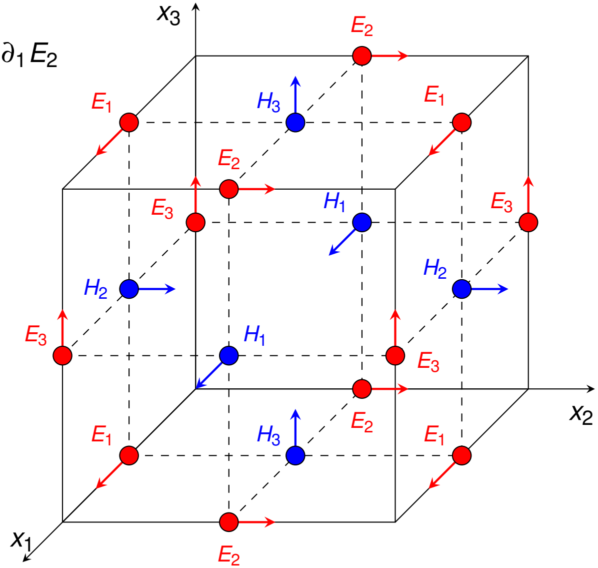
T. Namiki,
IEEE Trans. Microwave Theory and Techniques, 47 (1999)

F. Zheng, Z. Chen, and J. Zhang,
IEEE Trans. Microwave Theory and Techniques, 48 (2000)

short NZCZ method

FDTD: Yee cell

$$\mu \partial_t H_3 = \partial_2 E_1 - \partial_1 E_2$$



NZCZ method: splitting

$$\begin{pmatrix} \partial_t \mathbf{E} \\ \partial_t \mathbf{H} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\varepsilon}\,\mathrm{rot} \\ -\frac{1}{\mu}\,\mathrm{rot} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \qquad \text{on a cuboid } \Omega$$

plus divergence conditions, boundary conditions, initial data

splitting of rot operator (ADI-type)

$$\mathrm{rot} = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} = C_1 - C_2$$

splitting of Maxwell operator

$$\begin{pmatrix} 0 & \frac{1}{\varepsilon}\,\mathrm{rot} \\ -\frac{1}{\mu}\,\mathrm{rot} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\varepsilon} C_1 \\ \frac{1}{\mu} C_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{\varepsilon} C_2 \\ -\frac{1}{\mu} C_1 & 0 \end{pmatrix} = A + B$$

NZCZ / ADI / Peaceman–Rachford method

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Maxwell's equations on a cuboid Ω

$$\begin{pmatrix} \partial_t \mathbf{E} \\ \partial_t \mathbf{H} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\varepsilon}\,\mathrm{rot} \\ -\frac{1}{\mu}\,\mathrm{rot} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = (A+B) \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$$

Peaceman–Rachford method (alternating direction implicit method)

$$\begin{pmatrix} \mathbf{E}^{n+1} \\ \mathbf{H}^{n+1} \end{pmatrix} = \left(I - \tfrac{\tau}{2}B\right)^{-1}\left(I + \tfrac{\tau}{2}A\right)\left(I - \tfrac{\tau}{2}A\right)^{-1}\left(I + \tfrac{\tau}{2}B\right) \begin{pmatrix} \mathbf{E}^{n} \\ \mathbf{H}^{n} \end{pmatrix}$$

advantages

- unconditionally stable
- accuracy: order two error bound (for a fixed spatial grid)
- computationally efficient

Efficiency

$$\begin{pmatrix} \mathbf{E}^{n+1} \\ \mathbf{H}^{n+1} \end{pmatrix} = (I - \tfrac{\tau}{2}B)^{-1}(I + \tfrac{\tau}{2}A)(I - \tfrac{\tau}{2}A)^{-1}(I + \tfrac{\tau}{2}B) \begin{pmatrix} \mathbf{E}^{n} \\ \mathbf{H}^{n} \end{pmatrix}$$

two half-steps:

$$(I - \tfrac{\tau}{2}A) \begin{pmatrix} \mathbf{E}^{n+\frac{1}{2}} \\ \mathbf{H}^{n+\frac{1}{2}} \end{pmatrix} = (I + \tfrac{\tau}{2}B) \begin{pmatrix} \mathbf{E}^{n} \\ \mathbf{H}^{n} \end{pmatrix}$$

$$(I - \tfrac{\tau}{2}B) \begin{pmatrix} \mathbf{E}^{n+1} \\ \mathbf{H}^{n+1} \end{pmatrix} = (I + \tfrac{\tau}{2}A) \begin{pmatrix} \mathbf{E}^{n+\frac{1}{2}} \\ \mathbf{H}^{n+\frac{1}{2}} \end{pmatrix}$$

- have to solve two linear systems per time step
- naïve way on a grid with $m \times m \times m$ grid points requires

$$\mathcal{O}\left(\left(6 \cdot m^3\right)^3\right) \qquad \text{operations}$$

Example: $216 \cdot 10^{18}$ operations for $m = 100$

Efficiency

Formulation of the Peaceman–Rachford method in two half-steps:

$$\left(I - \frac{\tau}{2}A\right)\begin{pmatrix} \mathbf{E}^{n+\frac{1}{2}} \\ \mathbf{H}^{n+\frac{1}{2}} \end{pmatrix} = \left(I + \frac{\tau}{2}B\right)\begin{pmatrix} \mathbf{E}^{n} \\ \mathbf{H}^{n} \end{pmatrix}$$

$$\left(I - \frac{\tau}{2}B\right)\begin{pmatrix} \mathbf{E}^{n+1} \\ \mathbf{H}^{n+1} \end{pmatrix} = \left(I + \frac{\tau}{2}A\right)\begin{pmatrix} \mathbf{E}^{n+\frac{1}{2}} \\ \mathbf{H}^{n+\frac{1}{2}} \end{pmatrix}$$

Idea of Namiki; Zheng, Chen, Zhang:

- exploit special structure of operators A and B

presentation for $\varepsilon = \mu = 1$ for simplicity

Efficiency

First half-step:

$$\begin{pmatrix} I & -\frac{\tau}{2}C_1 \\ -\frac{\tau}{2}C_2 & I \end{pmatrix} \begin{pmatrix} \mathbf{E}^{n+\frac{1}{2}} \\ \mathbf{H}^{n+\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} I & -\frac{\tau}{2}C_2 \\ -\frac{\tau}{2}C_1 & I \end{pmatrix} \begin{pmatrix} \mathbf{E}^{n} \\ \mathbf{H}^{n} \end{pmatrix}$$

equivalent:

$$\begin{aligned} \mathbf{E}^{n+\frac{1}{2}} &= \mathbf{E}^n - \frac{\tau}{2}C_2\mathbf{H}^n + \frac{\tau}{2}C_1\mathbf{H}^{n+\frac{1}{2}} \\ \mathbf{H}^{n+\frac{1}{2}} &= \mathbf{H}^n - \frac{\tau}{2}C_1\mathbf{E}^n + \frac{\tau}{2}C_2\mathbf{E}^{n+\frac{1}{2}} \end{aligned}$$

insert second line into first line (recalling $\mathrm{rot} = C_1 - C_2$)

$$\begin{aligned} \mathbf{E}^{n+\frac{1}{2}} &= \mathbf{E}^n - \frac{\tau}{2}C_2\mathbf{H}^n + \frac{\tau}{2}C_1\left(\mathbf{H}^n - \frac{\tau}{2}C_1\mathbf{E}^n + \frac{\tau}{2}C_2\mathbf{E}^{n+\frac{1}{2}}\right) \\ &= \mathbf{E}^n + \frac{\tau}{2}\,\mathrm{rot}\,\mathbf{H}^n - \frac{\tau^2}{4}C_1^2\mathbf{E}^n + \frac{\tau^2}{4}C_1C_2\mathbf{E}^{n+\frac{1}{2}} \end{aligned}$$

Efficiency

First half step:

$$\left(I - \frac{\tau^2}{4} C_1 C_2\right) \mathbf{E}^{n+\frac{1}{2}} = \mathbf{E}^n + \frac{\tau}{2} \operatorname{rot} \mathbf{H}^n - \frac{\tau^2}{4} C_1^2 \mathbf{E}^n,$$
$$\mathbf{H}^{n+\frac{1}{2}} = \mathbf{H}^n - \frac{\tau}{2} C_1 \mathbf{E}^n + \frac{\tau}{2} C_2 \mathbf{E}^{n+\frac{1}{2}}$$

with

$$C_1 C_2 = \begin{pmatrix} \partial_2\partial_2 & 0 & 0 \\ 0 & \partial_3\partial_3 & 0 \\ 0 & 0 & \partial_1\partial_1 \end{pmatrix}$$

▶ linear systems decoupled and 1d

- complexity: $\mathcal{O}(m^3)$ operations to solve linear system

$$\left(I - \frac{\tau^2}{4} \partial_k \partial_k\right) v = b, \qquad k \in \{1, 2, 3\}, \quad b \in \mathbb{R}^{m \times m \times m} \text{ given}$$

- second half-step analogously

Numerical example

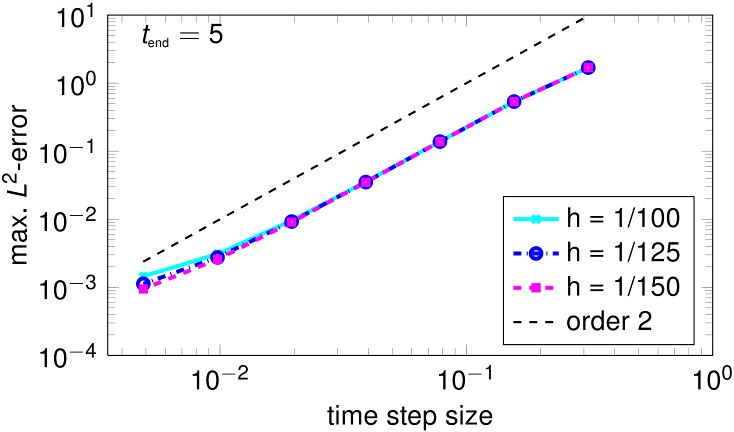
- Maxwell's equations on $\Omega = [0,1]^3$
  $+$ b.c. $+$ i.c. $+$ divergence conditions
- special exact solution for $\varepsilon \equiv \mu \equiv 1$, $(\kappa, \lambda) \in \mathbb{Z}^2 \setminus \{0, 0\}$

$$u^1_{\kappa\lambda}(t,x) = \begin{pmatrix} \sin(\kappa\pi x_2)\sin(\lambda\pi x_3)\cos(\sqrt{\kappa^2+\lambda^2}\pi t) \\ 0 \\ 0 \\ 0 \\ -\frac{\lambda}{\sqrt{\kappa^2+\lambda^2}}\sin(\kappa\pi x_2)\cos(\lambda\pi x_3)\sin(\sqrt{\kappa^2+\lambda^2}\pi t) \\ \frac{\kappa}{\sqrt{\kappa^2+\lambda^2}}\cos(\kappa\pi x_2)\sin(\lambda\pi x_3)\sin(\sqrt{\kappa^2+\lambda^2}\pi t) \end{pmatrix}$$

  $u^{2,3}_{\kappa\lambda}$ analogously (zeros at other positions, x_i permuted)
- superposition: $a^\ell_{\kappa\lambda} \in \mathbb{R}$, $a^\ell_{00} = 0$, $\ell = 1, 2, 3$

$$u(t,x) = \sum_{\kappa=0}^{\kappa_{\max}} \sum_{\lambda=0}^{\lambda_{\max}} \left(a^1_{\kappa\lambda} u^1_{\kappa\lambda}(t,x) + a^2_{\kappa\lambda} u^2_{\kappa\lambda}(t,x) + a^3_{\kappa\lambda} u^3_{\kappa\lambda}(t,x) \right)$$

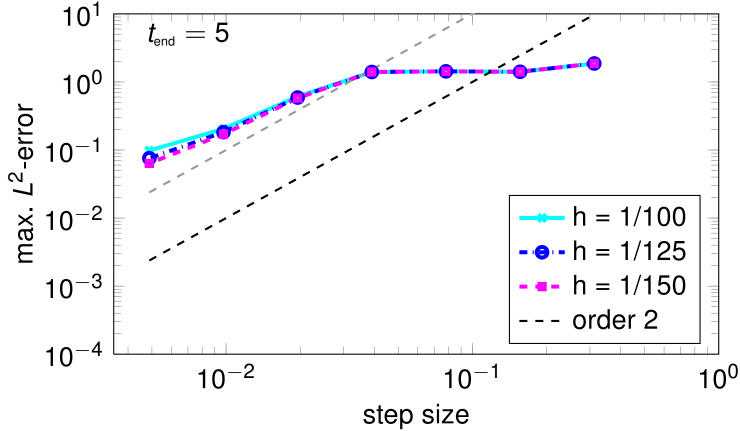
Smooth data: $a_{11}^{j} \neq 0$, rest 0



Runtime: $\approx$ 105 min for $h = \frac{1}{150}$ and $\tau = \frac{5}{1024}$

1024 time steps with 20 250 000 dof

Nonsmooth data: a^j_{11}, a^1_{54}, a^2_{35}, $a^3_{55} \neq 0$, rest 0



Goal of this talk:

explain this behavior

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   - Analytical framework, well-posedness
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4. Summary and outlook

Error analysis

- explain grid independent convergence:
  prove error bounds which do not deteriorate for $h \to 0$
  (in contrast to remainders of Taylor expansions)

proceed in the following steps

- analytical framework
  - correct function spaces
  - traces
  - well-posedness and regularity in Lipschitz domain
- unconditional stability
- error bounds for abstract Cauchy problem, i.e., for unbounded operators M, A, B

Function spaces

Assumption: $\Omega \subset \mathbb{R}^3$ open, bounded, with Lipschitz boundary

domains of rot and div:

$$H(\mathrm{rot}) = \{u \in L^2(\Omega)^3 \mid \mathrm{rot}\, u \in L^2(\Omega)^3\}$$
$$H(\mathrm{div}) = \{u \in L^2(\Omega)^3 \mid \mathrm{div}\, u \in L^2(\Omega)\}$$

Lemma $(\mathrm{rot}, H(\mathrm{rot}))$ and $(\mathrm{div}, H(\mathrm{div}))$ are closed in $L^2(\Omega)^3$

- consequence: $H(\mathrm{rot})$ and $H(\mathrm{div})$ are Hilbert spaces with corresponding graph norms
- warning: $u \in H(\mathrm{rot})$ means that, e.g., $\partial_2 u_3 - \partial_3 u_2 \in L^2(\Omega)$ but $\partial_2 u_3$, $\partial_3 u_2$ need not be L^2-functions

Known results on traces

- $C^\infty(\overline{\Omega})^3$ is dense in $H(\mathrm{rot})$ and $H(\mathrm{div})$
- $C_c^\infty(\Omega)^3$ is dense in $H_0(\mathrm{rot}) := \{u \in H(\mathrm{rot}) \mid u \times \nu = 0 \text{ on } \Gamma\}$

Lemma

- Tangential trace $u \mapsto u \times \nu$ on $C^\infty(\overline{\Omega})^3$ has a bounded extension

$$H(\mathrm{rot}) \to H^{-1/2}(\Gamma)^3, \qquad u \mapsto u \times \nu.$$

- Normal trace $u \mapsto u \cdot \nu$ on $C^\infty(\overline{\Omega})^3$ has a bounded extension

$$H(\mathrm{div}) \to H^{-1/2}(\Gamma), \qquad u \mapsto u \cdot \nu$$

Integration by parts formula: for all $u \in H(\mathrm{rot})$ and $\varphi \in H^1(\Omega)^3$

$$\int_\Omega u \cdot \mathrm{rot}\,\varphi \,\mathrm{d}x = \int_\Omega \varphi \cdot \mathrm{rot}\,u \,\mathrm{d}x + \langle u \times \nu, \varphi \rangle_{H^{-1/2}(\Gamma)^3, H^{1/2}(\Gamma)^3}.$$

Known results on traces

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Integration by parts formula: for all $u \in H_0(\mathrm{rot})$ and $\varphi \in H(\mathrm{rot})$

$$\int_\Omega u \cdot \mathrm{rot}\,\varphi \,\mathrm{d}x = \int_\Omega \varphi \cdot \mathrm{rot}\, u \,\mathrm{d}x.$$

Well-posedness on a Lipschitz domain Ω

Consider $X = L^2(\Omega)^6$ with its closed subspace

$$X_0 = \Big\{(\mathbf{E},\mathbf{H}) \in X : \operatorname{div}(\varepsilon\mathbf{E}) = \operatorname{div}(\mu\mathbf{H}) = 0,\ (\mu\mathbf{H})\cdot\nu = 0 \text{ on } \Gamma\Big\}$$

equipped with weighted scalar product

$$\big((\mathbf{E},\mathbf{H})|(u,v)\big)_X = (\mathbf{E}|u)_\varepsilon + (\mathbf{H}|v)_\mu = \int_\Omega \mathbf{E}\cdot u\,\varepsilon\,\mathrm{d}x + \int_\Omega \mathbf{H}\cdot v\,\mu\,\mathrm{d}x.$$

Maxwell operator:

$$M = \begin{pmatrix} 0 & \frac{1}{\varepsilon}\operatorname{rot} \\ -\frac{1}{\mu}\operatorname{rot} & 0 \end{pmatrix}, \qquad D(M) = H_0(\operatorname{rot}) \times H(\operatorname{rot})$$

Restriction of M to X_0:

$$M_0 = M\big|_{X_0}, \qquad D(M_0) = D(M) \cap X_0$$

Well-posedness on a Lipschitz domain Ω

Theorem. M and M_0 are skew-adjoint on X and X_0, respectively.

Hence,

- M and M_0 generate unitary C_0 groups $T(t) = e^{tM}$ on X and $T_0(t) = e^{tM_0}$ on X_0 (Stone's theorem)
- for every $(\mathbf{E}^0, \mathbf{H}^0) \in D(M_0)$, Maxwell's equations have a unique solution
$$(\mathbf{E}(t), \mathbf{H}(t)) \in C^1(\mathbb{R}; X_0) \cap C\big(\mathbb{R}; D(M_0)\big)$$
with constant norm (energy)
- $Mw \in X_0$ for all $w \in D(M)$ (because $\operatorname{div}\operatorname{rot} = 0$)
- $D(M_0^j) = D(M^j) \cap X_0$, $j \in \mathbb{N}$

Proof of skew-symmetry

Let $w = (\mathbf{E}, \mathbf{H})$, $\widetilde{w} = (\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}}) \in D(M) = H_0(\mathrm{rot}) \times H(\mathrm{rot})$.

$$M \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \frac{1}{\varepsilon} \operatorname{rot} \mathbf{H} \\ -\frac{1}{\mu} \operatorname{rot} \mathbf{E} \end{pmatrix}$$

Skew-symmetry of M:

$$\begin{aligned} (Mw|\widetilde{w})_X &= (\tfrac{1}{\varepsilon} \operatorname{rot} \mathbf{H} | \widetilde{\mathbf{E}})_\varepsilon - (\tfrac{1}{\mu} \operatorname{rot} \mathbf{E} | \widetilde{\mathbf{H}})_\mu \\ &= \int_\Omega \operatorname{rot} \mathbf{H} \cdot \widetilde{\mathbf{E}} \, \mathrm{d}x - \int_\Omega \operatorname{rot} \mathbf{E} \cdot \widetilde{\mathbf{H}} \, \mathrm{d}x \\ &= \int_\Omega \mathbf{H} \cdot \operatorname{rot} \widetilde{\mathbf{E}} \, \mathrm{d}x - \int_\Omega \mathbf{E} \cdot \operatorname{rot} \widetilde{\mathbf{H}} \, \mathrm{d}x \\ &= -(\mathbf{H} | -\tfrac{1}{\mu} \operatorname{rot} \widetilde{\mathbf{E}})_\mu - (\mathbf{E} | \tfrac{1}{\varepsilon} \operatorname{rot} \widetilde{\mathbf{H}})_\varepsilon = -(w|M\widetilde{w})_X, \end{aligned}$$

Analogously for M_0.

Regularity I

- $\Omega = (a_1^-, a_1^+) \times (a_2^-, a_2^+) \times (a_3^-, a_3^+)$ cuboid
- $\Gamma_j^\pm$ open face of Ω given by $x_j = a_j^\pm$, $j = 1, 2, 3$

Lemma

Let $0 < \delta \leq \varepsilon, \mu \in W^{1,\infty}(\Omega)$ and $\partial_{ij}\varepsilon, \partial_{ij}\mu \in L^3(\Omega)$, $\forall\, i, j$.

Then $D(M_0^2) \hookrightarrow H^2(\Omega)^6$ and $(\mathbf{E}, \mathbf{H}) \in D(M_0^2)$ has traces

$$\begin{aligned} \text{on } \Gamma_1^\pm : \quad & H_1 = E_2 = E_3 = 0, \\ & \partial_2 E_2 = \partial_3 E_2 = \partial_2 E_3 = \partial_3 E_3 = \partial_2 H_1 = \partial_3 H_1 = 0, \\ \text{on } \Gamma_{2,3}^\pm : \quad & \ldots \end{aligned}$$

(Costabel, Dauge, 2000)

The NZCZ method

$$A = \begin{pmatrix} 0 & \frac{1}{\varepsilon} C_1 \\ \frac{1}{\mu} C_2 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -\frac{1}{\varepsilon} C_2 \\ -\frac{1}{\mu} C_1 & 0 \end{pmatrix}$$

A and B act on $X = L^2(\Omega)^6$ with domains

$$\begin{aligned} D(A) &= \{(u,v) \in X \,|\, (C_1 v, C_2 u) \in X, \\ &\qquad\qquad u_1 = 0 \text{ on } \Gamma_2^{\pm},\ u_2 = 0 \text{ on } \Gamma_3^{\pm},\ u_3 = 0 \text{ on } \Gamma_1^{\pm}\}, \\ D(B) &= \dots \end{aligned}$$

Then: $Aw + Bw = Mw$ for $w \in D(A) \cap D(B) \subset D(M)$.

Lemma: A and B are skew-adjoint in X.

Consequence: $(I - \tau A)^{-1}$ and $(I + \tau A)(I - \tau A)^{-1}$ are contractions
$(I - \tau B)^{-1}$ and $(I + \tau B)(I - \tau B)^{-1}$ are contractions

Unconditional stability

Maxwell's equations on a cuboid Ω

$$\partial_t u = Mu = (A + B)u, \qquad u = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \qquad u(0) = w$$

Peaceman–Rachford method (alternating direction implicit method)

$$u(n\tau) \approx u^n = S_\tau^n w, \qquad S_\tau = (I - \tfrac{\tau}{2}B)^{-1}(I + \tfrac{\tau}{2}A)(I - \tfrac{\tau}{2}A)^{-1}(I + \tfrac{\tau}{2}B)$$

A and B are skew-adjoint in X, hence

$$\left\| (I + \tfrac{\tau}{2}A)(I - \tfrac{\tau}{2}A)^{-1} \right\| = \left\| (I + \tfrac{\tau}{2}B)(I - \tfrac{\tau}{2}B)^{-1} \right\| = 1$$

and thus

$$\|u^n\| \le \underbrace{\left\| (I - \tfrac{\tau}{2}B)^{-1} \right\|}_{\le 1} \cdot \underbrace{\| \ \dots \ \|}_{=1} \cdot \left\| (I + \tfrac{\tau}{2}B)w \right\|$$

Local and global error

$$u^n = S_\tau^n w, \qquad u(t_n) = T_0(t_n) w, \qquad u(0) = w, \qquad t_n = n\tau$$

Global error

$$S_\tau^n w - T_0(t_n) w = \sum_{j=0}^{n-1} S_\tau^{n-j-1} \underbrace{\big(S_\tau - T_0(\tau)\big) T_0(t_j) w}_{\text{local error}}$$

Goal: error bound for local error:

$$\|\big(S_\tau - T_0(\tau)\big) v\| \leq c\tau^3 (\|v\| + \|M_0^3 v\|)$$

this implies an error bound for global error:

$$\|S_\tau^n w - T_0(t_n) w\| \leq c\, t_{\text{end}} \tau^2 (\|w\| + \|M_0^3 w\|), \qquad t_n \leq t_{\text{end}}$$

Error bound for the local error

Preparation: For $j \in \mathbb{N}$, $\tau > 0$ and $v \in X_0$ define

$$\Lambda_j(\tau)v = \int_0^1 \frac{(1-\theta)^{j-1}}{(j-1)!} T_0(\theta\tau)v \, \mathrm{d}\theta, \qquad \|\Lambda_j(\tau)\| \le 1$$

- for $v \in D(M^3) \cap X_0$, local error has the form

$$\begin{aligned}
&(S_\tau - T_0(\tau))v \\
&= \tau^3 (I - \tfrac{\tau}{2}B)^{-1}(I - \tfrac{\tau}{2}A)^{-1}[\tfrac{1}{2}\Lambda_2(\tau) - \Lambda_3(\tau)]M_0^3 v \\
&\quad - \frac{\tau^3}{4}(I - \tfrac{\tau}{2}B)^{-1}(I - \tfrac{\tau}{2}A)^{-1} AB(I - M_0)^{-2}\Lambda_1(\tau)(I - M_0)^2 M_0 v
\end{aligned}$$

  [Hansen, Ostermann, 2008]
- it remains to bound

$$\|AB(I - M_0)^{-2}(\quad \ldots \quad)v\|.$$

Regularity II

Lemma

If $0 < \delta \leq \varepsilon, \mu \in W^{1,\infty}(\Omega)$ and $\partial_{ij}\varepsilon, \partial_{ij}\mu \in L^3(\Omega)$, $\forall\, i, j$, then

$$D(M_0^2) \hookrightarrow H^2(\Omega)^6 \cap D(AB) \cap D(A)$$

cf. [Costabel & Dauge 2000]

Consequence: $AB(I - M_0)^{-2} : X_0 \to X$ bounded:

$$\begin{aligned} \|AB(I - M_0)^{-2} v\| &\leq \|(I - M_0)^{-2} v\|_{H^2} \\ &\leq c\|(I - M_0)^{-2} v\|_{D(M_0^2)} \\ &\leq C\|v\| \end{aligned}$$

Main result: Error bound for the NZCZ method

Theorem

Assumptions:

- $\varepsilon, \mu \in W^{1,\infty}(\Omega)$ with $\varepsilon, \mu \geq \delta > 0$
- $\partial_i \partial_j \varepsilon,\ \partial_i \partial_j \mu \in L^3(\Omega)$ for all $i, j \in \{1, 2, 3\}$
- initial data satisfies $w = (\mathbf{E}, \mathbf{H}) \in D(M^3) \cap X_0 = D(M_0^3)$

Then, the global error of the NZCZ method is bounded by

$$\|S_\tau^n w - T_0(t_n) w\| \leq c\, t_{\text{end}} \tau^2 \left(\|w\| + \|M_0^3 w\| \right)$$

for all $n \in \mathbb{N}$, $\tau > 0$ with $t_n = n\tau \in [0, t_{\text{end}}]$.

c is independent of n, τ.

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Numerical example

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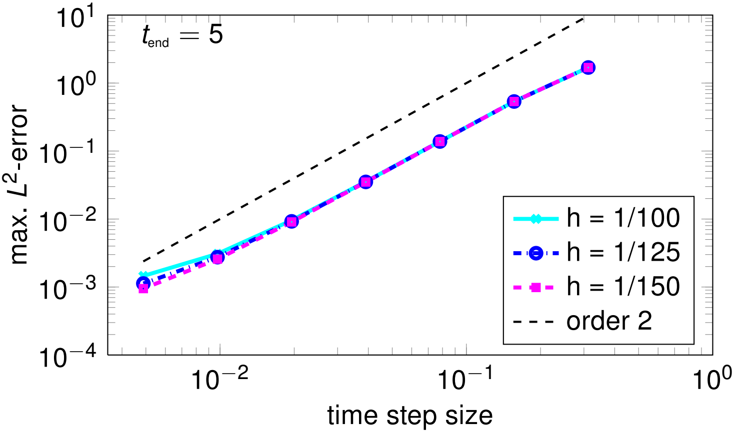
  $+$ b.c. $+$ i.c. $+$ divergence conditions
- special exact solution for $\varepsilon \equiv \mu \equiv 1$, $(\kappa, \lambda) \in \mathbb{Z}^2 \setminus \{0,0\}$

$$u^1_{\kappa\lambda}(t,x) = \begin{pmatrix} \sin(\kappa\pi x_2)\sin(\lambda\pi x_3)\cos(\sqrt{\kappa^2+\lambda^2}\pi t) \\ 0 \\ 0 \\ 0 \\ -\frac{\lambda}{\sqrt{\kappa^2+\lambda^2}}\sin(\kappa\pi x_2)\cos(\lambda\pi x_3)\sin(\sqrt{\kappa^2+\lambda^2}\pi t) \\ \frac{\kappa}{\sqrt{\kappa^2+\lambda^2}}\cos(\kappa\pi x_2)\sin(\lambda\pi x_3)\sin(\sqrt{\kappa^2+\lambda^2}\pi t) \end{pmatrix}$$

  $u^{2,3}_{\kappa\lambda}$ analogously (zeros at other positions, x_i permuted)
- superposition: $a^\ell_{\kappa\lambda} \in \mathbb{R}$, $a^\ell_{00} = 0$, $\ell = 1,2,3$

$$u(t,x) = \sum_{\kappa=0}^{\kappa_{\max}} \sum_{\lambda=0}^{\lambda_{\max}} \Big(a^1_{\kappa\lambda} u^1_{\kappa\lambda}(t,x) + a^2_{\kappa\lambda} u^2_{\kappa\lambda}(t,x) + a^3_{\kappa\lambda} u^3_{\kappa\lambda}(t,x) \Big)$$

Smooth data: $a^j_{11} \neq 0$



$\|w\|_X = 1$, $\|M_0^3 w\|_X = \mathcal{O}(1)$: order two

Nonsmooth data: a^j_{11}, a^1_{54}, a^2_{35}, $a^3_{55} \neq 0$

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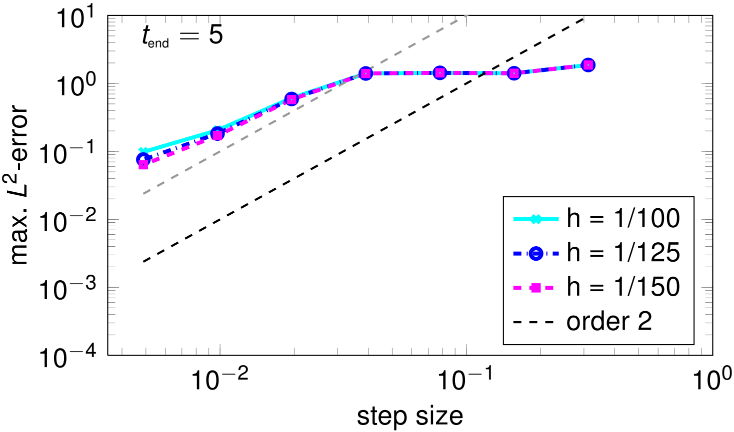

$\|w\|_X = 1$, $\|M_0^3 w\|_X \gg 1$: order reduction

Summary and outlook

- error analysis for Namiki; Zheng, Chen, Zhang method for Maxwell's equations on a cuboid
- proved rigorous bounds for abstract problem
- numerical results clearly indicate order reduction for nonsmooth data

M. Hochbruck, T. Jahnke, R. Schnaubelt,
Convergence of an ADI splitting for Maxwell's equations,
Preprint 2013, available online.

future work

- error analysis for full discretization
- nonlinear material laws