Numerische Mathematik manuscript No.
(will be inserted by the editor)

Error analysis of implicit Euler methods for quasilinear hyperbolic evolution equations

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January 27, 2015

Abstract In this paper we study the convergence of the semi-implicit and the implicit Euler methods for the time integration of abstract, quasilinear hyperbolic evolution equations. The analytical framework considered here includes certain quasilinear Maxwell's and wave equations as special cases. Our analysis shows that the Euler approximations are well-posed and convergent of order one. The techniques will be the basis for the future investigation of higher order time integration methods and full discretizations of certain quasilinear hyperbolic problems.

Keywords hyperbolic evolution equations, quasilinear Maxwell's equations, quasilinear wave equation, implicit Euler method, semi-implicit Euler method, well-posedness, error analysis, time integration

Mathematics Subject Classification (2000) Primary: 65M12, 65J15 Secondary: 35Q61, 35L90

1 Introduction

We consider the time discretization of a quasilinear hyperbolic evolution equation of the form

$$\Lambda(u(t))\partial_t u(t) = Au(t) + Q(u(t))u(t), \qquad u(0) = u_0, \tag{1a}$$

on a Hilbert space X by two variants of the implicit Euler method. Here, A is a linear, skew-adjoint operator, Λ is a symmetric positive definite operator on some neighborhood of zero and Q is a "nice" operator. The motivation to consider (1) is that Maxwell's equations with certain quasilinear constitution

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laws (such as Kerr nonlinearities) and quasilinear wave equations fit into this framework.

The aim of the present paper is to prove well-posedness and convergence of the semi-implicit and the implicit Euler methods applied to (1). In the following, we write (1a) in the equivalent short form as

$$\partial_t u(t) = A_{u(t)} u(t), \qquad u(0) = u_0, \tag{1b}$$

where

$$A_\varphi := \Lambda(\varphi)^{-1}(A + Q(\varphi)). \tag{1c}$$

Then, the approximation of the semi-implicit Euler method is given by

$$u_{n+1} = u_n + \tau A_{u_n} u_{n+1}, \qquad n = 0, \ldots, N-1, \tag{2}$$

while the implicit Euler method yields

$$u_{n+1} = u_n + \tau A_{u_{n+1}} u_{n+1}, \qquad n = 0, \ldots, N-1. \tag{3}$$

Here $u_n \approx u(t_n)$ is an approximation of the exact solution at time $t_n = n\tau$, where $\tau > 0$ denotes a fixed stepsize.

Well-posedness of (1) was proved by Kato in [12; 10; 11]. He linearized the problem and used a contraction mapping argument to prove existence and uniqueness of the solution under the assumption that the initial data is smooth enough. Using similar techniques, Müller [15] refined Kato's result by proving the result for somewhat relaxed assumptions on the initial data. Since these refined results are essential for our convergence analysis, we use the analytical framework provided in [15].

Surprisingly, there are only very few convergence results for time integration methods for quasilinear hyperbolic problems. The implicit Euler method for nonlinear evolution equations of the form $\partial_t u(t) = N(u)$ has been considered in [9; 13; 18] for various types of nonlinear operators N: dissipative operators in [18], ω-quasi dissipative operators in [13], and directed L-dissipative operators in [9]. These papers show convergence of order $1/2$ under the assumption that the numerical solution exists. In [4], Crandall and Souganidis showed that the approximations of the semi-implicit Euler method for (1) are well-posed and converge with order $1/2$. This enabled them to prove a well-posedness result being equivalent to that of Kato [10].

For nonlinear parabolic problems, the situation is different. Convergence of implicit time integration methods (Runge–Kutta and multistep methods) for quasilinear problems are given in [2; 14; 16] and for fully nonlinear parabolic problems in [1; 7; 17], for instance. However, since the analytic framework does not fit to hyperbolic problems of the form (1), they cannot be applied here.

The aim of this paper is to prove convergence of order one for the semi-implicit and the implicit Euler method for the abstract evolution equation (1) in the framework of Kato [10] and Müller [15]. We believe that the analysis presented here constitutes an essential tool to prove well-posedness and convergence for a class of higher-order implicit Runge–Kutta methods and also to study full discretizations based on finite elements or discontinuous Galerkin discretizations in space. For linear Maxwell's equations, such an analysis was carried out in [8], where we proved error bounds for implicit Runge–Kutta methods in time and discontinuous Galerkin method in space.

The paper is organized as follows. In Section 2 we review the analytical framework [10; 15], i.e., we state the precise assumptions on the operators Λ, A, and Q and give the local well-posedness result. Moreover, we show that quasilinear Maxwell and wave equations fit into this framework.

Section 3 contains additional analytical results that are required for the error analysis of the numerical methods. Here, stability estimates for the resolvents are of great importance. For the sake of readability, some technical details are postponed to the appendix.

The semi-implicit Euler method is analyzed in Section 4. Under the same regularity assumptions as in the continuous case, the well-posedness and the stability results are proven. We derive the error recursion and prove convergence of order one in the L^2-norm. Under additional regularity conditions of the solution, we also prove convergence in a stronger norm.

Section 5 deals with the analysis of the implicit Euler method. While for the semi-implicit Euler method, well-posedness and stability follow easily by using the stability estimates for the resolvents, for the implicit Euler method this is much harder. Our proof is based on linearization and a fixed point argument.

Notation. For two normed spaces X, Y we denote the space of bounded linear operators from X to Y by $\mathcal{L}(X,Y)$ and for $A \in \mathcal{L}(X,Y)$ we have $\|A\|_{Y \leftarrow X} := \max_{x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X}$. Given $R > 0$, we denote by $\overline{\mathcal{B}}_X(R)$, $\overline{\mathcal{B}}_Y(R)$ the closed balls of radius R around 0 in X and Y, respectively. Finally $a \lesssim b$ means that there exists a constant $c > 0$ such that $a \leq cb$.

2 Analytical framework and applications

In this section we provide the analytical framework, state the known well-posedness result for (1), and present two applications.

2.1 Assumptions and well-posedness

The refined analytical framework given in [15] uses three Hilbert spaces $(X, (\cdot,\cdot)_X)$, $(Y, (\cdot,\cdot)_Y)$, and $(Z, (\cdot,\cdot)_Z)$ with continuous and dense embeddings $Z \hookrightarrow Y \hookrightarrow X$. In addition, Y is an exact interpolation space between Z and X.

We start with the assumptions on the operator A.

Assumption 2.1 (Operator A)**.** *Let $A \in \mathcal{L}(Z,Y)$ be a skew-adjoint operator in X with $Y \subseteq D(A) \subseteq X$ and let $\alpha = \|A\|_{Y \leftarrow Z}$, i.e.,*

$$\|Az\|_Y \leq \alpha \|z\|_Z \quad \textit{for all } z \in Z. \tag{4}$$

For Λ we impose the following assumption.

Assumption 2.2 (Operator Λ)**.** *There exist a radius $R > 0$ and a family of linear operators $\{\Lambda(y) : y \in \overline{\mathcal{B}}_Y(R)\}$ on X such that for all $y, \widetilde{y} \in \overline{\mathcal{B}}_Y(R)$ the following holds:*

(a) *$\Lambda(y) \in \mathcal{L}(X)$ is self-adjoint and there is a constant $\nu > 0$ such that*

$$\Lambda(y) \geq \nu^{-1} I, \quad \textit{i.e.,} \quad (x, \Lambda(y)x)_X \geq \nu^{-1} \|x\|_X^2 \quad \textit{for all } x \in X. \tag{5a}$$

Hence, $\Lambda(y)$ is invertible in X with

$$\left\| \Lambda(y)^{-1} \right\|_{X \leftarrow X} \leq \nu. \tag{5b}$$

(b) *The range* $\operatorname{Ran}(I \mp \Lambda(y)^{-1} A)$ *is dense in X.*

(c) *There is a constant $\ell > 0$ such that*

$$\| \Lambda(y) - \Lambda(\widetilde{y}) \|_{X \leftarrow X} \leq \ell \, \| y - \widetilde{y} \|_Y . \tag{5c}$$

(d) *$\Lambda(y)^{-1} \in \mathcal{L}(Y)$ and there is a constant $\ell_Y > 0$ such that*

$$\left\| \Lambda(y)^{-1} - \Lambda(\widetilde{y})^{-1} \right\|_{Y \leftarrow Y} \leq \ell_Y \, \| y - \widetilde{y} \|_Y . \tag{5d}$$

In the following, R always refers to the radius from this assumption on Λ.

For the operator Q we require the following properties.

Assumption 2.3 (Operator Q)**.** *Let $r > 0$ be arbitrary. We assume that there is a constant $\mu_X = \mu_X(r)$ such that*

$$\| Q(y) \|_{X \leftarrow X} \leq \mu_X \quad \textit{for all} \quad y \in \overline{\mathcal{B}}_Y(R) \cap \overline{\mathcal{B}}_Z(r). \tag{6a}$$

Moreover, $Q(y) \in \mathcal{L}(Z, Y)$ for all $y \in \overline{\mathcal{B}}_Y(R)$ and there is a constant $m_Y > 0$ such that

$$\| Q(y) - Q(\widetilde{y}) \|_{Y \leftarrow Z} \leq m_Y \, \| y - \widetilde{y} \|_Y \quad \textit{for all} \quad y, \widetilde{y} \in \overline{\mathcal{B}}_Y(R). \tag{6b}$$

To obtain error bounds in stronger norms, we also need the following assumption on Λ.

Assumption 2.4. *Let $r > 0$ be arbitrary. We assume that there exists a continuous isomorphism $S : Z \to X$ such that for all $z \in \overline{\mathcal{B}}_Y(R) \cap \overline{\mathcal{B}}_Z(r)$*

$$A_z^S := S A_z S^{-1} = A_z + B(z) \tag{7a}$$

with linear operators $B(z) \in \mathcal{L}(X)$ being uniformly bounded, i.e.,

$$\| B(z) \|_{X \leftarrow X} \leq \beta \tag{7b}$$

for some constant $\beta > 0$.

Remark 2.5. *Let* $\lambda_0 := \|\Lambda(0)\|_{X\leftarrow X}$. *Then the operators* $\Lambda(y)$, $\Lambda(y)^{-1}$, *and* $Q(y)$ *are uniformly bounded in the corresponding norms, i.e., for all* $y \in \overline{\mathcal{B}}_Y(R)$ *we have*

$$\|\Lambda(y)\|_{X\leftarrow X} \le \lambda_X := \lambda_0 + \ell R \tag{8a}$$
$$\left\|\Lambda(y)^{-1}\right\|_{Y\leftarrow Y} \le \nu_Y := \left\|\Lambda(0)^{-1}\right\|_{Y\leftarrow Y} + \ell_Y R \tag{8b}$$
$$\|Q(y)\|_{Y\leftarrow Z} \le \mu_Y := \|Q(0)\|_{Y\leftarrow Z} + m_Y R \tag{8c}$$

The first bound follows from (5c), *the second from* (5d), *and the third from* (6b).

In the following, $\gamma > 0$ denotes a given parameter, which will be determined later. The constants

$$k_0 = (\nu\lambda_X)^{1/2} \ge 1 \qquad k_1 = k_1(\gamma) = \tfrac{1}{2}\,\nu\ell\gamma, \tag{9a}$$
$$c_0 = \|S\|_{X\leftarrow Z}\left\|S^{-1}\right\|_{Z\leftarrow X} k_0 \ge 1, \qquad c_1 = c_0\nu_Y(\alpha+\mu_Y) \tag{9b}$$
$$\omega = \nu\mu_X, \qquad \widetilde{\omega} = \omega + k_0\beta, \tag{9c}$$
$$L_X = \ell_X(\alpha+\mu_Y) + \nu m_X, \qquad L_Y = \ell_Y(\alpha+\mu_Y) + \nu_Y m_Y \tag{9d}$$

will be used throughout the paper.

The following well-posedness result was given in [15, Theorem 3.41].

Theorem 2.6. *Let Assumptions* 2.1–2.4 *be fulfilled and let* $\kappa \in (0,1)$ *and* $r > 0$ *be arbitrary. By*

$$R_0 = R_0(\kappa) = \kappa\frac{R}{c_0}, \qquad r_0 = r_0(\kappa, r) = \kappa\frac{r}{c_0} \tag{10}$$

we define two radii satisfying $R_0 < R$ *and* $r_0 < r$. *Then the following assertions hold:*

(a) For each $u_0 \in \overline{\mathcal{B}}_Y(R_0) \cap \overline{\mathcal{B}}_Z(r_0)$ *there exists a time*

$$T = T(\kappa, r, R) \ge \min\left\{\frac{-\ln\kappa}{k_1(\gamma)+\widetilde{\omega}}, \frac{\kappa}{rc_0L_Y}\right\} > 0, \qquad \gamma = \gamma(r) = r\frac{c_1}{c_0},$$

and a solution

$$u(\cdot, u_0) = u \in C([0,T], Z) \cap C^1([0,T], Y)$$

of (1) *with*

$$\|u(t)\|_Y \le R, \quad \textit{and} \quad \|u(t)\|_Z \le r \quad (0 \le t \le T). \tag{11}$$

(b) If $v \in C([0,T'], Z) \cap C^1([0,T'], Y)$ *is another solution of* (1) *with* $\|v(t)\|_Y \le R$ *for all* $t \in [0,T']$ *then* v *coincides with* u *on the interval* $[0, \min\{T, T'\}]$.

2.2 Quasilinear Maxwell's equations

Let $\Omega \subseteq \mathbb{R}^3$ be either a bounded domain with boundary $\partial\Omega$ or the full space $\mathbb{R}^3$. We consider the Maxwell's equations

$$
\begin{aligned}
\partial_t \mathbf{D}(t,x) &= \nabla\times \mathbf{H}(t,x), && t\in[0,T],\ x\in\Omega, && \text{(12a)}\\
\partial_t \mathbf{B}(t,x) &= -\nabla\times \mathbf{E}(t,x), && t\in[0,T],\ x\in\Omega, && \text{(12b)}\\
\nabla\cdot \mathbf{D}(t,x) &= 0, && t\in[0,T],\ x\in\Omega, && \text{(12c)}\\
\nabla\cdot \mathbf{B}(t,x) &= 0, && t\in[0,T],\ x\in\Omega, && \text{(12d)}
\end{aligned}
$$

with the constitution relations of the form

$$\mathbf{D}(t,x) = \mathbf{E}(t,x) + P(\mathbf{E}(t,x)), \quad \mathbf{B}(t,x) = \mathbf{H}(t,x) + M(\mathbf{H}(t,x)). \tag{12e}$$

Here, $P, M \in C^n(\mathbb{R}^3,\mathbb{R}^3)$ $(n\in\mathbb{N})$ are vector fields with positive definite matrices $I+P'(0)$ and $I+M'(0)$, respectively.

For Maxwell's equations on $\mathbb{R}^3$ let $n=\lfloor s\rfloor+1$ for some $s>3/2$. Then for

$$X := L^2(\mathbb{R}^3)^6, \qquad Y := H^s(\mathbb{R}^3)^6, \qquad Z := H^{s+1}(\mathbb{R}^3)^6,$$

the problem satisfies the assumptions of Theorem 2.6, cf. [15, Theorem 4.9], and is therefore well-posed.

On a bounded domain the well-posedness is proven for the case of Dirichlet boundary condition for the **E**-field, see [15, Theorem 4.6]. Suppose that the boundary $\partial\Omega$ is C^4 and that $n=5$. Then with

$$
\begin{aligned}
&X := L^2(\Omega)^6, \qquad Y := H^2(\Omega)^6 \cap H_0^1(\Omega)^6,\\
&Z := \{u\in H^4(\Omega)^3\cap H_0^1(\Omega)^3 : \Delta u \in H_0^1(\Omega)^3\}^2
\end{aligned}
$$

the assumptions of Theorem 2.6 are satisfied.

Remark 2.7. *In this paper we consider only real-valued vector fields, but the results also hold in the complex-valued case with minor modifications of the proofs.*

Example. In applications arising in physics, for instance the propagation of light through optical materials in photonic crystals, the so called Kerr nonlinearity, where the polarization and magnetization are given by

$$P(\mathbf{E}) = \chi\,|\mathbf{E}|^2\,\mathbf{E} \qquad (\chi\in\mathbb{R}), \qquad M=0$$

is of interest, cf. [3]. For $\chi>0$, the operator Λ is globally invertible and we can replace $\overline{\mathcal{B}}_Y(R)$ by Y in Assumption 2.2. Therefore, (1) is well-posed for all initial data in Y, i.e., R can be chosen arbitrarily large. ⋄

2.3 Quasilinear wave equation

Let $\Omega \subset \mathbb{R}^d$ ($d \le 3$) be a bounded domain with boundary $\partial\Omega$. We are interested in solving a quasilinear wave equation with homogeneous Dirichlet boundary conditions of the form

$$\begin{aligned} \partial_{tt} w(t,x) + \partial_{tt}(K \circ w)(t,x) &= \Delta w(t,x), \quad && t \in [0,T],\ x \in \Omega, \\ w(t,x) &= 0, && t \in [0,T],\ x \in \partial\Omega. \end{aligned} \tag{13}$$

For the nonlinearity we assume

$$K \in C^4(\mathbb{R}) \qquad \text{and} \qquad 1 + K'(0) > 0.$$

Local well-posedness of a strong solution was shown in [5]. It can also be obtained from Theorem 2.6 by reformulating the problem as a first order system of the form (1) where, for $u = (u_1, u_2) = (w, \partial_t w)$,

$$\Lambda(u) = \begin{pmatrix} 1 & 0 \\ 0 & 1 + K'(u_1) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}, \quad Q(u) = \begin{pmatrix} 0 & 0 \\ 0 & -K''(u_1)u_2 \end{pmatrix},$$

cf. [15, Theorem 3.45 and Theorem 4.12] with

$$\begin{aligned} &X := H_0^1(\Omega) \times L^2(\Omega), \qquad Y := (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega), \\ &Z := \{u \in H^3(\Omega) \cap H_0^1(\Omega) : \Delta u \in H_0^1(\Omega)\} \times (H^2(\Omega) \cap H_0^1(\Omega)). \end{aligned}$$

3 Additional properties of the operators

For the error analysis of the time integration methods additional properties of the operators are required. These are collected in this section.

Lemma 3.1. *Let Assumption 2.2 be fulfilled. For all $y, \widetilde{y} \in \overline{\mathcal{B}}_Y(R)$ we have*

(a) $\left\| \Lambda(y)^{1/2} \right\|_{X \leftarrow X} \le \lambda_X^{1/2}$.
(b) $(x, \Lambda(y)^{1/2} x)_X \ge \nu^{-1/2} \|x\|_X^2$ *for all* $x \in X$.
(c) There is a positive constant ℓ' such that

$$\left\| \Lambda(y)^{1/2} - \Lambda(\widetilde{y})^{1/2} \right\|_{X \leftarrow X} \le \ell' \|y - \widetilde{y}\|_Y .$$

Proof. (a) follows from (8a) and (b) follows from (5a).

(c) Since the spectrum $\sigma\big(\Lambda(y)\big) \subset J = [\nu^{-1}, \lambda_X]$, for $y \in \overline{\mathcal{B}}_Y(R)$, there exists an ellipse Γ in the right complex half-plan which encloses $\sigma\big(\Lambda(y)\big)$. By using the Cauchy integral formula for operators and the resolvent identity

$$(\lambda - A)^{-1} - (\lambda - B)^{-1} = (\lambda - A)^{-1}(A - B)(\lambda - B)^{-1} \tag{14}$$

we obtain

$$\Lambda(y)^{1/2} - \Lambda(\widetilde{y})^{1/2}$$

$$= \frac{1}{2\pi i} \int_\Gamma s^{1/2} \big(s - \Lambda(y)\big)^{-1} \big(\Lambda(y) - \Lambda(\widetilde{y})\big) \big(s - \Lambda(\widetilde{y})\big)^{-1} ds.$$

Since $\Lambda(y)$ and $\Lambda(\widetilde{y})$ are self-adjoint, we have

$$\left\| \Lambda(y)^{1/2} - \Lambda(\widetilde{y})^{1/2} \right\|_{X \leftarrow X} \leq \frac{1}{2\pi \, \mathrm{d}(\Gamma, J)^2} \left\| \Lambda(y) - \Lambda(\widetilde{y}) \right\|_{X \leftarrow X} \int_\Gamma |s|^{1/2} \, |ds| \,,$$

where $d(\Gamma, J)$ denotes the minimal distance between Γ and J. The claim then follows from (5c) for $\ell' = C(\Gamma, \nu, \lambda_X)\ell$. □

To show the convergence of numerical methods in the X-norm we need the following assumption.

Assumption 3.2. *Let $r > 0$ be arbitrary. Then there exist constants $\ell_X > 0$ and $m_X = m_X(r) > 0$ such that*

$$\left\| \Lambda(y)^{-1} - \Lambda(\widetilde{y})^{-1} \right\|_{X \leftarrow Y} \leq \ell_X \left\| y - \widetilde{y} \right\|_X \qquad \text{for all } y, \widetilde{y} \in \overline{\mathcal{B}}_Y(R), \tag{15a}$$

$$\left\| Q(z) - Q(\widetilde{z}) \right\|_{X \leftarrow Z} \leq m_X \left\| z - \widetilde{z} \right\|_X \qquad \text{for all } z, \widetilde{z} \in \overline{\mathcal{B}}_Z(r). \tag{15b}$$

Remark 3.3. *The previous assumption can be easily verified for both quasilinear Maxwell's equations (12) (on both $\mathbb{R}^3$ and on a bounded domain) and quasilinear wave equations (13).*

For Maxwell's equations we have $Q \equiv 0$, so (15b) is obviously true. We thus only sketch how (15a) can be shown. First we observe that

$$\Lambda(y) = \begin{pmatrix} I + P'(y_1) & 0 \\ 0 & I + M'(y_2) \end{pmatrix} \qquad \text{for} \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Next we use the resolvent identity (14) for $A = -P'(y_1)$ and $B = -P'(\widetilde{y}_1)$ and also for the M field. The boundedness of the resolvents and the Lipschitz-continuity of P' and M' imply

$$\left\| \Lambda(y)^{-1} - \Lambda(\widetilde{y})^{-1} \right\|_{L^2(\Omega)^{6\times 6}} \lesssim \left\| y - \widetilde{y} \right\|_X .$$

The claim then follows from

$$\begin{aligned} \left\| \left(\Lambda(y)^{-1} - \Lambda(\widetilde{y})^{-1} \right) u \right\|_X &\leq \left\| u \right\|_{L^\infty(\Omega)^6} \left\| \Lambda(y)^{-1} - \Lambda(\widetilde{y})^{-1} \right\|_{L^2(\Omega)^{6\times 6}} \\ &\lesssim \left\| u \right\|_Y \left\| y - \widetilde{y} \right\|_X . \end{aligned}$$

The following assumption is needed for the proof of the Z-norm convergence.

Assumption 3.4. *Let $r > 0$ and z, $\widetilde{z} \in \overline{\mathcal{B}}_Z(r)$ be arbitrary. Then there exist positive constants $\ell_Z = \ell_Z(r)$, $m_Z = m_Z(r)$, and $\mu_Z = \mu_Z(r)$ such that*

$$\left\| \Lambda(z)^{-1} - \Lambda(\widetilde{z})^{-1} \right\|_{Z \leftarrow Z} \leq \ell_Z \left\| z - \widetilde{z} \right\|_Z , \tag{16a}$$

$$\left\| Q(z) - Q(\widetilde{z}) \right\|_{Z \leftarrow Z} \leq m_Z \left\| z - \widetilde{z} \right\|_Z , \tag{16b}$$

$$\left\| Q(z) \right\|_{Z \leftarrow Z} \leq \mu_Z. \tag{16c}$$

Remark 3.5. *All properties can be verified for both quasilinear Maxwell and quasilinear wave equation by straightforward (but rather lengthy) computations, i.e., by differentiating the expressions and bounding all the terms.* (16a) *additionally requires* $P^{(5)}$ *and* $M^{(5)}$ *to be Lipschitz continuous for quasilinear Maxwell's equations with Dirichlet boundary conditions and* (16b) *additionally requires* $K^{(4)}$ *to be Lipschitz continuous for quasilinear wave equations.*

Under the last assumption we also have

$$\left\|\Lambda(z)^{-1}\right\|_{Z\leftarrow Z} \leq \ell_Z\, r + \left\|\Lambda(0)^{-1}\right\|_{Z\leftarrow Z} =: \nu_Z \qquad \text{for all} \qquad z \in \overline{\mathcal{B}}_Z(r) \quad (17)$$

Lemma 3.6. *Let Assumptions 2.1–2.3 be fulfilled and let* $r > 0$ *be arbitrary. Moreover, let* $\varphi, \psi \in \overline{\mathcal{B}}_Y(R) \cap \overline{\mathcal{B}}_Z(r)$. *Then the operator* A_φ *defined in* (1c) *satisfies the following estimates.*

(a) For ν_Y *and* μ_Y *defined in* (8) *we have*

$$\|A_\varphi\|_{Y\leftarrow Z} \leq \nu_Y(\alpha + \mu_Y). \qquad (18a)$$

(b) For L_Y *defined in* (9d) *we have*

$$\|A_\varphi - A_\psi\|_{Y\leftarrow Z} \leq L_Y \|\varphi - \psi\|_Y. \qquad (18b)$$

(c) If in addition Assumption 3.2 is fulfilled, for L_X *defined in* (9d) *we have*

$$\|A_\varphi - A_\psi\|_{X\leftarrow Z} \leq L_X \|\varphi - \psi\|_X. \qquad (18c)$$

(d) If in addition Assumption 3.4 holds and if u *satisfies*

$$\|u\|_Z + \|Au\|_Z \leq r_A \qquad (18d)$$

for some $r_A > 0$, *we have*

$$\|(A_\varphi - A_\psi)u\|_Z \leq L_Z r_A \|\varphi - \psi\|_Z. \qquad (18e)$$

with $L_Z = \ell_Z(1 + \mu_Z) + \nu_Z m_Z$.

Proof. (a) The inequality follows from (8b), (4), and (8c).

(b) Here we use

$$\|A_\varphi - A_\psi\|_{Y\leftarrow Z} \leq \left\|(\Lambda(\varphi)^{-1} - \Lambda(\psi)^{-1})A\right\|_{Y\leftarrow Z} + \left\|(\Lambda(\varphi)^{-1} - \Lambda(\psi)^{-1})Q(\psi)\right\|_{Y\leftarrow Z} + \left\|\Lambda(\varphi)^{-1}(Q(\varphi) - Q(\psi))\right\|_{Y\leftarrow Z}.$$

The claim follows by using (5d) and (4) for the first term, (5d) and (8c) for the second term, and (8b) and (6b) for the last term.

(c) We use (15a) instead of (5d), (5b) instead of (8b) and (15b) instead of (6b).

(d) Analogously to (c), but we use (16) and (17) instead of (15) and (5b). We have

$$\|(A_\varphi - A_\psi)u\|_Z \leq (\ell_Z(\|Au\|_Z + \mu_Z \|u\|_Z) + \nu_Z m_Z \|u\|_Z) \|\varphi - \psi\|_Z.$$

The statement then follows from $\|u\|_Z + \|Au\|_Z \leq r_A$.

□

3.1 Stability estimates

For $N \in \mathbb{N}$ and $r, \zeta > 0$ we define the function space $E := E(N, r, \zeta)$ by

$$
\begin{aligned}
E(N, r, \zeta) := \{\varphi = (\varphi_1, \dots, \varphi_N) \in Z^N :& \\
&\|\varphi_k\|_Y \le R,\ \|\varphi_k\|_Z \le r \text{ for } k = 1, \dots, N, \\
&\|\varphi_{k+1} - \varphi_k\|_Y \le \zeta \text{ for } k = 1, \dots, N-1\}.
\end{aligned} \tag{19}
$$

The following stability result is similar to the proofs of Lemma 3.21, Lemma 3.29 and Theorem 3.41 in [15]. Because of its importance for the well-posedness results for the numerical solution, the sketch of the proof can be found in the appendix.

Lemma 3.7. *Let $\gamma > 0$ and $\varphi = (\varphi_1, \dots, \varphi_N) \in E(N, r, \tau\gamma)$ be given. Then*

$$
\begin{aligned}
\big\|(I - \tau A_{\varphi_k})^{-1} \cdots (I - \tau A_{\varphi_j})^{-1}\big\|_{X \leftarrow X} &\le k_0 (1 - \tau\omega)^{-(k-j+1)} e^{k_1 (k-j)\tau}, \\
\big\|(I - \tau A_{\varphi_k})^{-1} \cdots (I - \tau A_{\varphi_j})^{-1}\big\|_{Y \leftarrow Y} &\le c_0 (1 - \tau\widetilde{\omega})^{-(k-j+1)} e^{k_1 (k-j)\tau}, \\
\big\|(I - \tau A_{\varphi_k})^{-1} \cdots (I - \tau A_{\varphi_j})^{-1}\big\|_{Z \leftarrow Z} &\le c_0 (1 - \tau\widetilde{\omega})^{-(k-j+1)} e^{k_1 (k-j)\tau},
\end{aligned}
$$

for all $\tau\omega < 1$ in the first inequality, $\tau\widetilde{\omega} < 1$ in the second and third inequality, and all $1 \le j < k \le N$. The constants used here are given in (9).

4 Semi-implicit Euler method

In this section we consider the semi-implicit Euler method (2), which can be written in the equivalent form

$$
u_{n+1} = (I - \tau A_{u_n})^{-1} u_n = (I - \tau A_{u_n})^{-1} \dots (I - \tau A_{u_0})^{-1} u_0. \tag{20}
$$

4.1 Well-posedness and stability

We first prove that the approximations defined in (20) are well-posed and uniformly bounded. Recall that the constants have been defined in (9).

Theorem 4.1. *Let Assumptions 2.1–2.4 be satisfied and let $\kappa \in (0,1)$ and $r > 0$ be arbitrary. For each $u_0 \in \overline{\mathcal{B}}_Y(R_0) \cap \overline{\mathcal{B}}_Z(r_0)$, where the radii R_0, r_0 are defined in (10), there exists a time*

$$
T \ge -\frac{\ln \kappa}{k_1(\gamma) + 2\widetilde{\omega}} > 0, \qquad \text{where} \qquad \gamma = \gamma(r) = r\frac{c_1}{c_0},
$$

such that for all $\tau \le \tau_0 = 3/(4\widetilde{\omega})$ and $(N+1)\tau \le T$, there is a unique finite sequence $(u_n)_{n=0}^N$ of semi-implicit Euler approximations (20) satisfying

$$
\|u_n\|_Y \le R, \quad \text{and} \quad \|u_n\|_Z \le r, \quad n = 1, \dots, N. \tag{21}
$$

Proof. We use induction on n to prove (21) and that the bound

$$\|u_{k+1} - u_k\|_Y \le \gamma\tau \quad \text{for all} \quad k = 0, \dots, n-1 \tag{22}$$

holds. For $n = 0$ the bounds (21) hold by assumption and the condition (22) is empty.

Suppose that (21) and (22) hold for some $n \in \{0, \dots, N-1\}$. Then, $(u_0, \dots, u_n) \in E(n+1, r, \gamma\tau)$ and we can apply Lemma 3.7 to (20) which gives

$$\|u_{n+1}\|_Y \le c_0\, e^{(k_1+2\widetilde{\omega})T} \|u_0\|_Y\,, \qquad \|u_{n+1}\|_Z \le c_0\, e^{(k_1+2\widetilde{\omega})T} \|u_0\|_Z\,. \tag{23}$$

Here we used the bound

$$(1-\xi)^{-1} \le e^{2\xi} \quad \text{for} \quad \xi \le \frac{3}{4} \tag{24}$$

for $\xi = \tau\widetilde{\omega}$. This proves (21).

Finally, by using a resolvent identity we obtain

$$u_{n+1} - u_n = \big((I - \tau A_{u_n})^{-1} - I\big)\, u_n = \tau A_{u_n}(I - \tau A_{u_n})^{-1} u_n.$$

By taking the Y-norm and using (18a) and (23) we can bound

$$\|u_{n+1} - u_n\|_Y \le \tau\nu_Y(\alpha + \mu_Y)\, \|u_{n+1}\|_Z \le \tau c_1 e^{(k_1+2\widetilde{\omega})T} \|u_0\|_Z\,,$$

where c_1 was defined in (9b). Inserting γ and T completes the proof. □

4.2 Error recursion

The Taylor expansion of the exact solution of (1) yields

$$u(t_n) = u(t_{n+1}) - \tau\partial_t u(t_{n+1}) - \delta_{n+1} \tag{25a}$$

with defect

$$\delta_{n+1} = \int_{t_n}^{t_{n+1}} \partial_t^2 u(t)(t_n - t)dt. \tag{25b}$$

Hence we have

$$u(t_{n+1}) = u(t_n) + \tau A_{u(t_{n+1})} u(t_{n+1}) + \delta_{n+1}. \tag{26}$$

We define the error as

$$e_n = u_n - \widehat{u}_n, \qquad \widehat{u}_n = u(t_n).$$

To simplify the following presentation we write

$$\begin{aligned} \Lambda_n &= \Lambda(u_n), & \widehat{\Lambda}_n &= \Lambda\big(u(t_n)\big), \\ A_n &= A_{u_n}, & \widehat{A}_n &= A_{u(t_n)}. \end{aligned}$$

By subtracting (26) from (2) we obtain the error equation

$$\begin{aligned} e_{n+1} &= e_n + \tau\big(A_n u_{n+1} - \widehat{A}_{n+1}\widehat{u}_{n+1}\big) - \delta_{n+1} \\ &= e_n + \tau A_n e_{n+1} + \tau(A_n - \widehat{A}_n)\widehat{u}_{n+1} + \tau(\widehat{A}_n - \widehat{A}_{n+1})\widehat{u}_{n+1} - \delta_{n+1}. \end{aligned} \tag{27}$$

4.3 Convergence in the X-norm

Motivated by the energy techniques presented in [14] for parabolic problems and [8] for linear Maxwell's equations, we now prove the convergence of the semi-implicit Euler method in the X-norm.

Lemma 4.2. *Let Assumptions 2.1–2.4 and Assumption 3.2 be fulfilled. Let $u \in C([0,T],Z) \cap C^1([0,T],Y)$ be the exact solution of (1). Then for τ sufficiently small, the error $e_n = u_n - u(t_n)$ of the semi-implicit Euler approximation (2) satisfies*

$$\|e_N\|_X^2 \le \tau M \sum_{n=0}^{N-1} \|e_{n+1}\|_X^2 + \tau\nu\lambda_X \left(\sum_{n=0}^{N-1} \left\| \frac{\delta_{n+1}}{\tau} \right\|_X^2 + L_X r \sum_{n=0}^{N-1} \|\rho_n\|_X^2 \right),$$

for $0 \le N\tau \le T$, where

$$\rho_n := \widehat{u}_{n+1} - \widehat{u}_n = \int_{t_n}^{t_{n+1}} u'(t)dt \tag{28}$$

and

$$M = 2\nu\big(\mu_X + \tfrac{3}{2} L_X \lambda_X r + \ell' \lambda_X^{1/2} \nu_Y (\alpha + \mu_Y) r + \tfrac{1}{2}\lambda_X\big). \tag{29}$$

Proof. Let $\Lambda_{-1} = 0$. By taking the X-inner product of (27) with $\Lambda_n e_{n+1}$ we obtain

$$\begin{aligned}
(\Lambda_n^{1/2} e_{n+1} - \Lambda_{n-1}^{1/2} e_n, \Lambda_n^{1/2} e_{n+1})_X =& \tau(A_n e_{n+1}, \Lambda_n e_{n+1})_X \\
&+ \tau((A_n - \widehat{A}_n)\widehat{u}_{n+1}, \Lambda_n e_{n+1})_X \\
&+ \tau((\widehat{A}_n - \widehat{A}_{n+1})\widehat{u}_{n+1}, \Lambda_n e_{n+1})_X \\
&+ ((\Lambda_n^{1/2} - \Lambda_{n-1}^{1/2}) e_n, \Lambda_n^{1/2} e_{n+1})_X \\
&- (\delta_{n+1}, \Lambda_n e_{n+1})_X.
\end{aligned} \tag{30}$$

We bound each of the terms on the right-hand side separately. For the first term we use that Λ is self-adjoint, A is skew-adjoint, and (6a) to get

$$\tau(A_n e_{n+1}, \Lambda_n e_{n+1})_X = \tau((A + Q(u_n))e_{n+1}, e_{n+1})_X \le \tau\mu_X \|e_{n+1}\|_X^2 . \tag{31}$$

By applying the Cauchy-Schwarz inequality, (18c), (8a), and (11), for the second and the third term we obtain

$$\begin{aligned}
\tau((A_n - \widehat{A}_n)\widehat{u}_{n+1}, \Lambda_n e_{n+1})_X &\le \tau\, L_X \lambda_X \|\widehat{u}_{n+1}\|_Z \|e_n\|_X \|e_{n+1}\|_X \\
&\le \frac{\tau}{2} L_X \lambda_X r \left(\|e_n\|_X^2 + \|e_{n+1}\|_X^2 \right)
\end{aligned} \tag{32}$$

and, with ρ_n defined in (28),

$$\begin{aligned}
\tau((\widehat{A}_n - \widehat{A}_{n+1})\widehat{u}_{n+1}, \Lambda_n e_{n+1})_X &\le \tau L_X \lambda_X \|\widehat{u}_{n+1}\|_Z \|\widehat{u}_{n+1} - \widehat{u}_n\|_X \|e_{n+1}\|_X \\
&\le \frac{\tau}{2} L_X \lambda_X r \left(\|\rho_n\|_X^2 + \|e_{n+1}\|_X^2 \right).
\end{aligned} \tag{33}$$

To bound the fourth term we use Lemma 3.1 to get

$$
\begin{aligned}
((\Lambda_n^{1/2} - \Lambda_{n-1}^{1/2})e_n, \Lambda_n^{1/2} e_{n+1})_X &\leq \ell' \lambda_X^{1/2} \|u_n - u_{n-1}\|_Y \|e_n\|_X \|e_{n+1}\|_X \\
&\leq \frac{\tau}{2} \ell' \lambda_X^{1/2} \nu_Y (\alpha + \mu_Y) r \left(\|e_n\|_X^2 + \|e_{n+1}\|_X^2 \right), \qquad (34)
\end{aligned}
$$

where we used (21) and

$$
\|u_n - u_{n-1}\|_Y = \tau \|A_{n-1} u_n\|_Y \leq \tau \nu_Y (\alpha + \mu_Y) \|u_n\|_Z
$$

for the second inequality. The latter bound follows from (2) and (18a).

For the fifth term it holds

$$
(\delta_{n+1}, \Lambda_n e_{n+1})_X \leq \frac{\tau}{2} \lambda_X \left(\left\| \frac{\delta_{n+1}}{\tau} \right\|_X^2 + \|e_{n+1}\|_X^2 \right). \qquad (35)
$$

The claim now follows by summing (30) for $n = 0, \ldots, N-1$ and using the estimates (31)–(35) to bound the right-hand side. For the left-hand side we use $e_0 = 0$ to show

$$
\begin{aligned}
\sum_{n=0}^{N-1} (\Lambda_n^{1/2} e_{n+1} - \Lambda_{n-1}^{1/2} e_n, \Lambda_n^{1/2} e_{n+1})_X &\geq \frac{1}{2} \left\| \Lambda_{N-1}^{1/2} e_N \right\|_X^2 \\
&\geq \frac{1}{2} \nu^{-1} \|e_N\|_X^2, \qquad (36)
\end{aligned}
$$

where the first inequality follows from

$$
\begin{aligned}
\sum_{n=0}^{N-1} (v_n - v_{n-1}, v_n) = \tfrac{1}{2} \|v_{N-1}\|^2 &+ \tfrac{1}{2} \|v_{N-1}\|^2 - (v_{N-2}, v_{N-1}) + \tfrac{1}{2} \|v_{N-2}\|^2 \\
&+ \tfrac{1}{2} \|v_{N-2}\|^2 - (v_{N-3}, v_{N-2}) + \tfrac{1}{2} \|v_{N-3}\|^2 \\
&+ \ \ldots \\
&+ \tfrac{1}{2} \|v_0\|^2 - (v_{-1}, v_0) + \tfrac{1}{2} \|v_{-1}\|^2 \\
&- \tfrac{1}{2} \|v_{-1}\|^2 \\
\geq \tfrac{1}{2} \big(\|v_{N-1}\|^2 - \|v_{-1}\|^2 \big)&,
\end{aligned}
$$

for $v_n = \Lambda_n^{1/2} e_{n+1}$ (where $v_{-1} = 0$). The last inequality in (36) is a consequence of (5a). □

Theorem 4.3. *Let the assumptions of Lemma 4.2 be fulfilled. We additionally assume $u'' \in L^2(0,T;X)$. Then for τ sufficiently small, the error $e_n = u_n - u(t_n)$ of the semi-implicit Euler method is bounded by*

$$
\|e_N\|_X^2 \leq e^{2MT} \nu \lambda_X \tau^2 \left(\int_0^T \|u''(t)\|_X^2 \, dt + L_X r \int_0^T \|u'(t)\|_X^2 \, dt \right)
$$

with $M = M(r)$ given in (29), i.e., the method is convergent of order one.

Proof. By a discrete Gronwall inequality and the previous lemma, it follows that

$$\|e_N\|_X^2 \le e^{2MT}\nu\lambda_X\tau\left(\sum_{n=0}^{N-1}\left\|\frac{\delta_{n+1}}{\tau}\right\|_X^2 + L_X r\sum_{n=0}^{N-1}\|\rho_n\|_X^2\right)$$

for $\tau \le \frac{3}{4}M^{-1}$ The bounds

$$\sum_{n=0}^{N-1}\left\|\frac{\delta_{n+1}}{\tau}\right\|_X^2 \le \tau\int_0^T \|u''(t)\|_X^2\,dt, \qquad \sum_{n=0}^{N-1}\|\rho_n\|_X^2 \le \tau\int_0^T \|u'(t)\|_X^2\,dt,$$

for the defects δ_{n+1} and ρ_n defined in (25) and (28), respectively, imply the stated result. □

4.4 Convergence in the Z-norm

Lemma 4.4. *Let Assumptions 2.1–2.4 and Assumption 3.4 be fulfilled. Let $u \in C([0,T],Z)\cap C^1([0,T],Y)$ be the exact solution of (1) and assume that it satisfies (18d) uniformly in t. Then for τ sufficiently small, the error $e_n = u_n - u(t_n)$ of the semi-implicit Euler approximation (2) satisfies*

$$\begin{aligned}\|e_N\|_Z^2 \le{}& \tau M_Z\sum_{n=0}^{N-1}\|e_{n+1}\|_Z^2\\ &+\tau\nu\kappa(S)^2\lambda_X\left(\sum_{n=0}^{N-1}\left\|\frac{\delta_{n+1}}{\tau}\right\|_Z^2 + L_Z r_A\sum_{n=0}^{N-1}\|\rho_n\|_Z^2\right),\end{aligned}$$

where $M_Z = M_Z(r)$ is given in

$$M_Z = 2\nu\kappa(S)^2\Big(\mu_X + \lambda_X\beta + \frac{3}{2}L_Z\lambda_X r_A + \ell'\lambda_X^{1/2}\nu_Y(\alpha+\mu_Y)r + \frac{1}{2}\lambda_X\Big) \tag{37}$$

and where $\kappa(S) := \|S\|_{X\leftarrow Z}\left\|S^{-1}\right\|_{Z\leftarrow X}$ denotes the condition number of the operator S.

Proof. We multiply (27) by S and use (7a) to obtain

$$\begin{aligned}Se_{n+1} - Se_n ={}& \tau(A_n + B(u_n))Se_{n+1} + \tau S(A_n - \widehat{A}_n)\widehat{u}_{n+1}\\ &+\tau S(\widehat{A}_n - \widehat{A}_{n+1})\widehat{u}_{n+1} - S\delta_{n+1}.\end{aligned}$$

By taking the X-inner product with $\Lambda_n Se_{n+1}$ we have

$$\begin{aligned}(\Lambda_n^{1/2}Se_{n+1} - \Lambda_{n-1}^{1/2}Se_n, \Lambda_n^{1/2}Se_{n+1})_X ={}& \tau((A_n + B(u_n))Se_{n+1}, \Lambda_n Se_{n+1})_X\\ &+\tau(S(A_n - \widehat{A}_n)\widehat{u}_{n+1}, \Lambda_n Se_{n+1})_X\\ &+\tau(S(\widehat{A}_n - \widehat{A}_{n+1})\widehat{u}_{n+1}, \Lambda_n Se_{n+1})_X\\ &+((\Lambda_n^{1/2} - \Lambda_{n-1}^{1/2})Se_n, \Lambda_n^{1/2}Se_{n+1})_X\end{aligned}$$

$$- (S\delta_{n+1}, \Lambda_n S e_{n+1})_X.$$

We bound each of the terms on the right-hand side separately. For the first term, we again use that A is skew-adjoint, Λ is self-adjoint, (6a), (8a), and (7b) to show

$$\tau((A_n + B(u_n))Se_{n+1}, \Lambda_n S e_{n+1})_X \leq \tau(\mu_X + \lambda_X \beta) \|S\|^2_{X \leftarrow Z} \|e_{n+1}\|^2_Z.$$

We apply the Cauchy-Schwarz inequality and (18e) to bound the second term by

$$\begin{aligned}
&\tau(S(A_n - \widehat{A}_n)\widehat{u}_{n+1}, \Lambda_n S e_{n+1})_X \\
&\quad \leq \tau \|S\|_{X \leftarrow Z} \left\| (A_n - \widehat{A}_n)\widehat{u}_{n+1} \right\|_Z \|\Lambda_n S e_{n+1}\|_X \\
&\quad \leq \frac{\tau}{2} \|S\|^2_{X \leftarrow Z} \lambda_X L_Z r_A (\|e_n\|^2_Z + \|e_{n+1}\|^2_Z).
\end{aligned}$$

Similarly, as in (33), for the third term it holds

$$\begin{aligned}
&\tau(S(\widehat{A}_n - \widehat{A}_{n+1})\widehat{u}_{n+1}, \Lambda_n S e_{n+1})_X \\
&\quad \leq \frac{\tau}{2} \|S\|^2_{X \leftarrow Z} \lambda_X L_Z r_A \left(\|\rho_n\|^2_Z + \|e_{n+1}\|^2_Z \right),
\end{aligned}$$

where ρ_n was defined in (28). We bound the fourth and the fifth term analogously as in the proof of Lemma 4.2, which gives

$$\begin{aligned}
&((\Lambda_n^{1/2} - \Lambda_{n-1}^{1/2})Se_n, \Lambda_n^{1/2} S e_{n+1})_X \\
&\quad \leq \frac{\tau}{2} \|S\|^2_{X \leftarrow Z} \ell' \lambda_X^{1/2} \nu_Y (\alpha + \mu_Y) r \left(\|e_n\|^2_Z + \|e_{n+1}\|^2_Z \right)
\end{aligned}$$

and

$$(S\delta_{n+1}, \Lambda_n S e_{n+1})_X \leq \frac{\tau}{2} \|S\|^2_{X \leftarrow Z} \lambda_X \left(\left\| \frac{\delta_{n+1}}{\tau} \right\|^2_Z + \|e_{n+1}\|^2_Z \right).$$

Summing these bounds from 0 to $N-1$ and using

$$\sum_{n=0}^{N-1} (\Lambda_n^{1/2} S e_{n+1} - \Lambda_{n-1}^{1/2} S e_n, \Lambda_n^{1/2} S e_{n+1})_X \geq \frac{1}{2} \left\| S^{-1} \right\|^{-2}_{Z \leftarrow X} \nu^{-1} \|e_N\|^2_Z$$

for the left-hand side proves the lemma. □

Theorem 4.5. *Let the assumptions of Lemma 4.4 be fulfilled and in addition assume $u' \in L^2(0,T;Z)$ and $u'' \in L^2(0,T;Z)$. Then for τ sufficiently small, the error of the semi-implicit Euler method is bounded by*

$$\|e_N\|^2_Z \leq e^{2M_Z T} \nu \kappa(S)^2 \lambda_X \tau^2 \left(\int_0^T \|u''(t)\|^2_Z \, dt + L_Z r_A \int_0^T \|u'(t)\|^2_Z \, dt \right).$$

with $M_Z = M_Z(r)$ given in (37), i.e., the method is convergent of order one.

Proof. The proof is analogous to the proof of Theorem 4.3. □

5 Implicit Euler method

Next we consider the (fully) implicit Euler method (3) which we write as

$$u_{n+1} = (I - \tau A_{u_{n+1}})^{-1} u_n. \tag{38}$$

In contrast to the semi-implicit Euler method where the approximations are defined via a linear problem, here a nonlinear problem has to be solved to compute u_{n+1} from u_n. This makes the analysis more involved.

5.1 Well-posedness

We start with proving that the approximations are well-posed by following ideas from [15, Theorem 3.41] which are based on Banach's fixed point theorem.

Theorem 5.1. *Let Assumptions 2.1–2.4 be satisfied and let $\kappa \in (0,1)$ and $r > 0$ be arbitrary. For each $u_0 \in \overline{\mathcal{B}}_Y(R_0) \cap \overline{\mathcal{B}}_Z(r_0)$, where the radii R_0, r_0 are defined in (10), there exists a time*

$$T = T(\kappa, r, R) \geq \min\left\{ \frac{-\ln \kappa}{k_1(\gamma) + 2\widetilde{\omega}}, \frac{\kappa}{r c_0 L_Y} \right\} > 0, \qquad \gamma = \gamma(r) = r\frac{c_1}{c_0},$$

such that for all $\tau \leq \tau_0 = 3/(4\widetilde{\omega})$ and $(N+1)\tau \leq T$, there is a unique finite sequence $(u_n)_{n=0}^N$ of implicit Euler approximations (38) which satisfy (21).

Proof. The space $E = E(N, r, \tau\gamma)$ defined in (19) equipped with the metric

$$d(\varphi, \psi) := \max_{k=1,\dots N} \|\varphi_k - \psi_k\|_Y, \qquad \varphi, \psi \in E$$

is a complete metric space. For $\varphi = (\varphi_1, \dots, \varphi_N) \in E$ we define a function $\Phi_{u_0} : E \to X^N$ by

$$\Phi_{u_0}(\varphi)_k := (I - \tau A_{\varphi_k})^{-1} \cdots (I - \tau A_{\varphi_1})^{-1} u_0, \qquad k = 1, \dots, N.$$

It is easy to see that a fixed point of Φ_{u_0} is a solution of (38), i.e., $\Phi_{u_0}(\varphi) = \varphi$ is equivalent to

$$\varphi_k = (I - \tau A_{\varphi_k})^{-1} \cdots (I - \tau A_{\varphi_1})^{-1} u_0, \qquad k = 1, \dots, N.$$

For the existence of a unique fixed point we have to prove that Φ_{u_0} is a contraction on (E, d). We first prove that Φ_{u_0} leaves E invariant. Analogously to the proof of Theorem 4.1 we obtain for T given in the theorem

$$\begin{aligned} \|\Phi_{u_0}(\varphi)_k\|_Y &\leq c_0\, e^{(k_1+2\widetilde{\omega})T} \|u_0\|_Y \leq R, \\ \|\Phi_{u_0}(\varphi)_k\|_Z &\leq c_0\, e^{(k_1+2\widetilde{\omega})T} \|u_0\|_Z \leq r, \\ \|\Phi_{u_0}(\varphi)_{k+1} - \Phi_{u_0}(\varphi)_k\|_Y &\leq \tau c_1 e^{(k_1+2\widetilde{\omega})T} \|u_0\|_Z \leq \gamma\tau, \end{aligned} \tag{39}$$

where c_0 and c_1 are defined in (9b). Thus $\Phi_{u_0}(\varphi) \in E$.

It remains to prove that Φ_{u_0} is a contraction on E, i.e.,

$$d(\Phi_{u_0}(\varphi), \Phi_{u_0}(\psi)) \le \kappa\, d(\varphi, \psi), \qquad \varphi, \psi \in E. \tag{40}$$

Writing $G^{\psi}_{k,k} = I$ and

$$G^{\psi}_{k,j} := (I - \tau A_{\psi_k})^{-1} \cdots (I - \tau A_{\psi_{j+1}})^{-1}, \qquad 0 \le j < k \le N,$$

we have

$$\begin{aligned}\Phi_{u_0}(\varphi)_k &- \Phi_{u_0}(\psi)_k \\ &= \sum_{i=0}^{k-1} G^{\psi}_{k,k-i} \left((I - \tau A_{\varphi_{k-i}})^{-1} - (I - \tau A_{\psi_{k-i}})^{-1} \right) G^{\varphi}_{k-i-1,0} u_0.\end{aligned}$$

For the factor in the middle we use the resolvent identity (14) to obtain

$$\Phi_{u_0}(\varphi)_k - \Phi_{u_0}(\psi)_k = \tau \sum_{i=0}^{k-1} G^{\psi}_{k,k-i-1} (A_{\varphi_{k-i}} - A_{\psi_{k-i}}) G^{\varphi}_{k-i,0} u_0.$$

Lemma 3.7 and (18b) yield

$$\begin{aligned}&\|\Phi_{u_0}(\varphi)_k - \Phi_{u_0}(\psi)_k\|_Y \\ &\quad \le \tau \sum_{i=0}^{k-1} \left\| G^{\psi}_{k,k-i-1} \right\|_{Y \leftarrow Y} \left\| (A_{\varphi_{k-i}} - A_{\psi_{k-i}}) \right\|_{Y \leftarrow Z} \left\| G^{\varphi}_{k-i,0} u_0 \right\|_Z \\ &\quad \le \tau c_0^2 (1 - \tau \widetilde{\omega})^{-(k+1)} e^{k_1 (k-1) \tau} L_Y \left\| u_0 \right\|_Z \sum_{i=0}^{k-1} \left\| \varphi_{k-i} - \psi_{k-i} \right\|_Y .\end{aligned}$$

By taking the maximum over all k we finally obtain the bound

$$d\big(\Phi_{u_0}(\varphi), \Phi_{u_0}(\psi)\big) \le T c_0^2 e^{(k_1 + 2\widetilde{\omega})T} L_Y \left\| u_0 \right\|_Z d(\varphi, \psi).$$

Therefore, Φ_{u_0} is a contraction for the T given in the theorem. □

Remark 5.2. *The interval of existence for the implicit Euler method differs from the one in the continuous case (Theorem 2.6) only by a factor of two in front of $\widetilde{\omega}$. This factor comes from the factor two in the estimate (24) which we have used in the well-posedness proof of the semi-implicit Euler method and also in the proof for the implicit Euler method. Note that we can choose a factor arbitrarily close to one if ξ is sufficiently small. Thus, the time interval on which the implicit Euler method is well-posed can be extended to the existence interval of the continuous problem in the limit $\tau \to 0$.*

5.2 Error recursion

Subtracting (26) from (3) and using the same notation as in Section 4 we have

$$e_{n+1} = e_n + \tau(A_{n+1}u_{n+1} - \widehat{A}_{n+1}\widehat{u}_{n+1}) - \delta_{n+1},$$

or, equivalently,

$$e_{n+1} = e_n + \tau A_{n+1}e_{n+1} + \tau(A_{n+1} - \widehat{A}_{n+1})\widehat{u}_{n+1} - \delta_{n+1}. \tag{41}$$

5.3 Convergence in the X-norm

We proceed by proving the convergence of the method in the X-norm.

Lemma 5.3. *Let Assumptions 2.1–2.4 and Assumption 3.2 be fulfilled. Let $u \in C([0,T],Z) \cap C^1([0,T],Y)$ be the exact solution of (1). Then for τ sufficiently small, the error $e_n = u_n - u(t_n)$ of the implicit Euler approximation (3) satisfies*

$$\|e_N\|_X^2 \le \tau \widehat{M} \sum_{n=0}^{N-1} \|e_{n+1}\|_X^2 + \tau\nu\lambda_X \sum_{n=0}^{N-1} \left\| \frac{\delta_{n+1}}{\tau} \right\|_X^2,$$

where

$$\widehat{M} = \widehat{M}(r) = 2\nu\Big(\mu_X + L_X\lambda_X r + \ell'\lambda_X^{1/2}\nu_Y(\alpha + \mu_Y)r + \frac{1}{2}\lambda_X\Big). \tag{42}$$

Proof. By taking the X-inner product of (41) with $\Lambda_{n+1}e_{n+1}$ we obtain

$$\begin{aligned}
(\Lambda_{n+1}^{1/2}e_{n+1} - \Lambda_n^{1/2}e_n, \Lambda_{n+1}^{1/2}e_{n+1})_X =& \tau(A_{n+1}e_{n+1}, \Lambda_{n+1}e_{n+1})_X \\
&+ \tau((A_{n+1} - \widehat{A}_{n+1})\widehat{u}_{n+1}, \Lambda_{n+1}e_{n+1})_X \\
&+ ((\Lambda_{n+1}^{1/2} - \Lambda_n^{1/2})e_n, \Lambda_{n+1}^{1/2}e_{n+1})_X \\
&- (\delta_{n+1}, \Lambda_{n+1}e_{n+1})_X.
\end{aligned}$$

We bound all terms on the right-hand side as in the proof of Lemma 4.2 and obtain

$$\begin{aligned}
&(\Lambda_{n+1}^{1/2}e_{n+1} - \Lambda_n^{1/2}e_n, \Lambda_{n+1}^{1/2}e_{n+1})_X \\
&\quad\le \tau\mu_X \|e_{n+1}\|_X^2 + \tau L_X\lambda_X \|\widehat{u}_{n+1}\|_Z \|e_{n+1}\|_X^2 \\
&\quad+ \frac{\tau}{2}\ell'\lambda_X^{1/2}\nu_Y(\alpha + \mu_Y) \|u_{n+1}\|_Z \left(\|e_n\|_X^2 + \|e_{n+1}\|_X^2\right) \\
&\quad+ \frac{\tau}{2}\lambda_X \left(\left\| \frac{\delta_{n+1}}{\tau} \right\|_X^2 + \|e_{n+1}\|_X^2 \right).
\end{aligned}$$

The claim now follows by summing the last inequality from 0 to $N-1$ and using the same estimate for the left-hand side as in the proof of Lemma 4.2. □

Theorem 5.4. *Let the assumptions of Lemma 5.3 be fulfilled and in addition assume $u'' \in L^2(0,T;X)$. Then for τ sufficiently small, the error $e_n = u_n - u(t_n)$ of the implicit Euler method is bounded by*

$$\|e_N\|_X^2 \le e^{2\widehat{M}T} \nu \lambda_X \tau^2 \int_0^T \|u''(t)\|_X^2 \, dt,$$

where $\widehat{M} = \widehat{M}(r)$ is given in (42), i.e., the method is convergent of order one.

Proof. Analogously to the proof of Theorem 4.3. □

5.4 Convergence in the Z-norm

Lemma 5.5. *Let Assumptions 2.1–2.4 and Assumption 3.4 be fulfilled. Let $u \in C([0,T],Z) \cap C^1([0,T],Y)$ be the exact solution of (1) and assume that it satisfies (18d) uniformly in t. Then for τ sufficiently small, the error $e_n = u_n - u(t_n)$ of the implicit Euler approximation (3) satisfies*

$$\|e_N\|_Z^2 \le \tau \widehat{M}_Z \sum_{n=0}^{N-1} \|e_{n+1}\|_Z^2 + \tau \nu \kappa(S)^2 \lambda_X \sum_{n=0}^{N-1} \left\| \frac{\delta_{n+1}}{\tau} \right\|_Z^2,$$

where $\widehat{M}_Z = \widehat{M}_Z(r)$ is defined as

$$\widehat{M}_Z := 2\nu\kappa(S)^2 \Big(\mu_X + \lambda_X \beta + L_Z \lambda_X r_A + \ell' \lambda_X^{1/2} \nu_Y (\alpha + \mu_Y) r + \frac{1}{2}\lambda_X \Big). \quad (43)$$

The condition number $\kappa(S)$ was defined in Lemma 4.4.

Proof. We multiply (41) with S and use (7a) to obtain

$$Se_{n+1} - Se_n = \tau(A_n + B(u_n))Se_{n+1} + \tau S(A_{n+1} - \widehat{A}_{n+1})\widehat{u}_{n+1} - S\delta_{n+1}.$$

By taking the X-inner product with $\Lambda_{n+1} S e_{n+1}$ we have

$$\begin{aligned}
(\Lambda_{n+1}^{1/2} S e_{n+1} - \Lambda_n^{1/2} S e_n, \Lambda_{n+1}^{1/2} S e_{n+1})_X& \\
&= \tau((A_{n+1} + B(u_{n+1}))Se_{n+1}, \Lambda_{n+1} S e_{n+1})_X \\
&+ \tau(S(A_{n+1} - \widehat{A}_{n+1})\widehat{u}_{n+1}, \Lambda_{n+1} S e_{n+1})_X \\
&+ ((\Lambda_{n+1}^{1/2} - \Lambda_n^{1/2}) S e_n, \Lambda_{n+1}^{1/2} S e_{n+1})_X \\
&- (S\delta_{n+1}, \Lambda_{n+1} S e_{n+1})_X.
\end{aligned} \quad (44)$$

The claim now follows by using the same estimates as in the proof of Lemma 4.4.

□

Theorem 5.6. *Let the assumptions of Lemma 5.5 be fulfilled and assume $u'' \in L^2(0,T;Z)$. Then for τ sufficiently small, the error $e_n = u_n - u(t_n)$ of the implicit Euler method is bounded by*

$$\|e_N\|_Z^2 \le e^{2\widehat{M}_Z T} \nu \kappa(S)^2 \lambda_X \tau^2 \int_0^T \|u''(t)\|_Z^2 \, dt$$

with $\widehat{M}_Z = \widehat{M}_Z(r)$ given in (43), i.e., the method is convergent of order one.

Proof. The proof is analogous to the proof of Theorem 4.3. □

A Stability estimates

In this appendix we sketch the proof of Lemma 3.7.

For $\varphi \in \bar{\mathcal{B}}_Y(R)$ we define inner product

$$(x, y)_\varphi = (\Lambda(\varphi)x, y)_X.$$

With X_φ we denote the space X endowed with this inner product. From (5a) and (8a) follows that the associated norm is uniformly equivalent to the X-norm, i.e.,

$$\lambda_X^{-1} \|x\|_\varphi^2 \le \|x\|_X^2 \le \nu \|x\|_\varphi^2, \qquad x \in X. \tag{45}$$

By using (5c) and (45), for $\varphi, \psi \in \bar{\mathcal{B}}_Y(R)$, we have

$$\begin{aligned}\|x\|_\varphi^2 &= (\Lambda(\varphi)x, x)_X = (\Lambda(\psi)x, x)_X + ((\Lambda(\varphi) - \Lambda(\psi))x, x)_X \\ &\le \|x\|_\psi^2 + \ell \|\varphi - \psi\|_Y \|x\|_X^2 \le (1 + \ell\nu \|\varphi - \psi\|_Y) \|x\|_\psi^2 .\end{aligned}$$

It follows that

$$\|x\|_\varphi \le e^{k_1 \tau} \|x\|_\psi \qquad \text{for} \qquad \|\varphi - \psi\|_Y \le \gamma\tau, \tag{46}$$

where $k_1 := k_1(\gamma)$ is defined in (9a).

For a Banach space V and real numbers $C \ge 1$ and $a > 0$ we denote by $G(V, C, a)$ the set of all infinitesimal generators of C_0-semigroups of type (C, a) on V. We show that for $\varphi \in \bar{\mathcal{B}}_Y(R)$ there holds

$$A_\varphi \in G(X_\varphi, 1, \omega), \tag{47a}$$

where ω is defined in (9c), which then implies the following bound for the resolvent

$$\left\|(I - \tau A_\varphi)^{-1}\right\|_{X_\varphi \leftarrow X_\varphi} \le (1 - \tau\omega)^{-1} \quad \text{for} \quad \tau\omega < 1. \tag{47b}$$

From $(\Lambda(\varphi)^{-1}Ax, x)_\varphi = 0$, Assumption 2.2(b), the fact that A is a closed operator in X (since it is skew-adjoint) and the norm equivalence (45) we can conclude by using the Lumer-Phillips theorem, cf. [6, Theorem II.3.15], that $\Lambda(\varphi)^{-1}A$ generates a contraction semigroup on X_φ. Further on, $\Lambda(\varphi)^{-1}Q(\varphi) - \omega I$ is a bounded operator on X_φ and

$$((\Lambda(\varphi)^{-1}Q(\varphi) - \omega I)x, x)_\varphi \le \mu_X \|x\|_X^2 - \omega \|x\|_\varphi^2 \le (\mu_X \nu - \omega) \|x\|_\varphi^2 = 0,$$

i.e., $\Lambda(\varphi)^{-1}Q(\varphi) - \omega I$ is dissipative in $(\cdot,\cdot)_\varphi$. Therefore, by the perturbation result [6, Theorem III.2.7], we have that $A_\varphi - \omega I$ generates a contraction semigroup on X_φ, i.e. $A_\varphi - \omega I \in G(X_\varphi, 1, 0)$. (47) now follows by the bounded perturbation theorem (cf. [6, Theorem III.1.3]).

We proceed as follows by using (45), (46) and (47) to obtain the X-norm estimate. For $\tau\omega < 1$ there holds

$$\left\|(I - \tau A_{\varphi_k})^{-1} \cdots (I - \tau A_{\varphi_j})^{-1} u\right\|_X$$

$$
\begin{aligned}
&\leq \nu^{1/2} \left\|(I-\tau A_{\varphi_k})^{-1}\cdots(I-\tau A_{\varphi_j})^{-1}u\right\|_{\varphi_k} \\
&\leq \nu^{1/2}(1-\tau\omega)^{-1} \left\|(I-\tau A_{\varphi_{k-1}})^{-1}\cdots(I-\tau A_{\varphi_j})^{-1}u\right\|_{\varphi_k} \\
&\leq \nu^{1/2}(1-\tau\omega)^{-1}e^{k_1\tau} \left\|(I-\tau A_{\varphi_{k-1}})^{-1}\cdots(I-\tau A_{\varphi_j})^{-1}u\right\|_{\varphi_{k-1}} \\
&\leq \ldots \leq \nu^{1/2}(1-\tau\omega)^{-(k-j+1)}e^{k_1(k-j)\tau} \|u\|_{\varphi_j} \\
&\leq k_0(1-\tau\omega)^{-(k-j+1)}e^{k_1(k-j)\tau} \|u\|_X \,.
\end{aligned}
$$

To get the Z-norm estimate we use the operator $A_\varphi^S = A_\varphi + B(\varphi)$ defined in (7a). For $\varphi \in \overline{\mathcal{B}}_Y(R) \cap \overline{\mathcal{B}}_Z(r)$, by (7b) and (45), we obtain

$$
\|B(\varphi)x\|_\varphi \leq \lambda_X^{1/2} \|B(\varphi)x\|_X \leq \lambda_X^{1/2}\beta \|x\|_X \leq k_0\beta \|x\|_\varphi \,,
$$

i.e., $\|B(\varphi)\|_{X_\varphi \leftarrow X_\varphi} \leq k_0\beta$. Applying the bounded perturbation theorem again gives that for $\varphi \in \overline{\mathcal{B}}_Y(R) \cap \overline{\mathcal{B}}_Z(r)$ it holds

$$
A_\varphi^S \in G(X_\varphi, 1, \widetilde{\omega})
$$

where $\widetilde{\omega}$ is defined in (9c). For $\tau\widetilde{\omega} < 1$ we can write

$$
\begin{aligned}
\left\|(I-\tau A_{\varphi_k})^{-1}\cdots(I-\tau A_{\varphi_j})^{-1}u\right\|_Z &= \left\|S^{-1}(I-\tau A_{\varphi_k}^S)^{-1}\cdots(I-\tau A_{\varphi_j}^S)^{-1}Su\right\|_Z \\
&\leq \left\|S^{-1}\right\|_{Z\leftarrow X} \left\|(I-\tau A_{\varphi_k}^S)^{-1}\cdots(I-\tau A_{\varphi_j}^S)^{-1}Su\right\|_X
\end{aligned}
$$

and proceed as above. This yields the bound in the Z-norm. Since Y is an exact interpolation space between Z and X, the second inequality follows immediately.

Acknowledgements

The authors thank Roland Schnaubelt and Dominik Müller for helpful discussions on the well-posedness of the Euler approximations.

This work was supported by the Deutsche Forschungsgemeinschaft (DFG) via RTG 1294.

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