A convergence analysis of the exponential Euler iteration for nonlinear ill-posed problems

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Abstract. Asymptotic regularization is a well established tool for treating nonlinear ill-posed problems. For its numerical realization, an appropriate numerical method for solving differential equations is required. In this paper, we analyze an exponential Euler method for this purpose. The scheme requires a certain approximation to the Jacobian of the Showalter differential equation and gives the exact solution for linear problems. Under a suitable discrepancy principle, the method is shown to be convergent under the same assumptions that are needed for the continuous analysis. Our convergence analysis admits variable step sizes and moreover yields optimal convergence rates.

Keywords: nonlinear ill-posed problems, asymptotic regularization, exponential integrators, variable step sizes, convergence, optimal convergence rates

1. Introduction

In this paper we consider the problem

$$F(x) = y, \tag{1}$$

where $F : \mathcal{D}(F) \subset X \to Y$ is a nonlinear differentiable operator between the Hilbert spaces X and Y, whose Fréchet derivative $F'(u)$ is locally uniformly bounded. In the following, we always assume that (1) has a solution $x_* \in \mathcal{D}(F)$ but we do not assume that this solution is unique. We are interested in ill-posed problems, where the solution does not depend continuously on the data y. In order to solve the perturbed problem

$$F(u) = y^\delta, \tag{2}$$

with perturbed data $y^\delta \approx y$ satisfying

$$\left\| y^\delta - y \right\| \leq \delta, \tag{3}$$

regularization is indispensable. Throughout the paper, the norm in both Hilbert spaces X and Y is denoted by $\|\cdot\|$, the corresponding inner product by $\langle \cdot , \cdot \rangle$.

Among several regularization techniques, one possible option is asymptotic regularization based on the Showalter differential equation [14]

$$u'(t) = F'\big(u(t)\big)^* \Big(y^\delta - F\big(u(t)\big)\Big), \qquad t \geq 0, \qquad u(0) = x_0. \qquad (4)$$

It was shown in [15] that the exact solution of (4) yields an optimal order regularization scheme under suitable assumptions. The explicit Euler method applied to (4) leads to the well-known Landweber iteration analyzed in [5]. In [7] we proposed to use the exponential Euler scheme for solving (4) numerically and we presented numerical examples which showed that this scheme is competitive for certain applications.

In this paper, we will analyze the convergence properties of the exponential Euler regularization method. In particular, we will prove that, under the same assumptions as for the continuous analysis of Tautenhahn [15], the scheme converges in the limit $\delta \to 0$ with the optimal convergence rate. Our convergence analysis is performed for variable step sizes satisfying a suitable discrepancy principle.

The paper is organized as follows. In Section 2 we briefly recall the exponential Euler regularization scheme and introduce some notation to improve the readability of this paper. Section 3 is devoted to the convergence analysis. The analysis in this section is motivated by previous work by Hanke [3]. The optimal convergence rates are shown in Section 4. The sections about the convergence both start with a review of the corresponding continuous results from Tautenhahn [15], so that the reader can easily compare the assumptions and the results. Finally, we comment on generalizations of our analysis to other regularization methods.

2. Exponential Euler regularization

The exponential Euler method is the simplest variant of the class of exponential Rosenbrock integrators [8]. It is based on the variation-of-constants formula which allows to integrate the linear part of semilinear differential equations exactly. In [7] we proposed to use the following modification of the original exponential Euler scheme

$$u_{n+1} = u_n + h_n \varphi(-h_n J(u_n)) F'(u_n)^* \big(y^\delta - F(u_n)\big), \quad u_0 = x_0 \qquad (5)$$

for solving (4). Here, we denote

$$J(u) = F'(u)^* F'(u). \qquad (6)$$

Hence, $-J(u_n)$ is an approximation to the Jacobian of the right-hand side of (4). Moreover, φ denotes the entire function

$$\varphi(z) = \frac{e^z - 1}{z}. \qquad (7)$$

If $u_n \approx u(t_n)$, then this scheme gives an approximation $u_{n+1} \approx u(t_{n+1})$, at time

$$t_{n+1} = t_n + h_n, \qquad t_0 = 0.$$

For the convergence of the scheme it will be essential to use appropriate time steps h_n. Details will be discussed in Section 3.3. The iteration is stopped as soon as the standard discrepancy principle

$$\left\| \Delta F_{n_*}^\delta \right\| \leq \tau \delta < \left\| \Delta F_n^\delta \right\| \qquad \text{for all } n < n_*, \qquad (8)$$

is satisfied. Here $\tau > 1$ is a parameter and

$$\Delta F_n^\delta = y^\delta - F(u_n)$$

denotes the residual of the perturbed problem. The stopping criterion (8) is the discrete analogue of the discrepancy principle used by Tautenhahn [15], where the regularization parameter t_* is chosen such that

$$\left\| y^\delta - F(u(t_*)) \right\| \leq \tau\delta < \left\| y^\delta - F(u(t)) \right\|, \qquad 0 \leq t < t_*. \tag{9}$$

Note that the exponential Euler method gives the exact solution for the Showalter differential equation for linear inverse problems. For nonlinear problems this scheme can also be interpreted as Newton's method with asymptotic regularization for solving the linearized problem, see [7] for details. This interpretation allows to apply the convergence results of Rieder [12, 13], which hold for a whole class of Newton methods. However, the convergence rates shown there are not yet optimal.

3. Convergence analysis

The aim of this section is to prove the convergence of the exponential Euler regularization for exact and perturbed data. Before we start, we present the analytic framework for our analysis and we review the convergence analysis for the continuous solution of the (Showalter) differential equation (4) from [15].

3.1. Assumptions

For the convergence analysis we impose the so-called tangential cone condition, which is also used in [4, 15].

Assumption 1. *$F : \mathcal{D}(F) \subset X \to Y$ is Fréchet differentiable and satisfies*

$$\|F(\widetilde{x}) - F(x) - F'(x)(\widetilde{x} - x)\| \leq \eta \|F(x) - F(\widetilde{x})\| \tag{10}$$

for $\eta < 0.7$ and for all $x, \widetilde{x} \in B_r(x_+) \subset \mathcal{D}(F)$, where $B_r(x_+)$ denotes the ball of radius $r > 0$ around the center x_+.

This assumption guarantees the following bounds for all $x, \widetilde{x} \in B_r(x_+)$

$$\frac{1}{1+\eta} \|F'(x)(x - \widetilde{x})\| \leq \|F(x) - F(\widetilde{x})\| \leq \frac{1}{1-\eta} \|F'(x)(x - \widetilde{x})\|. \tag{11}$$

To simplify the presentation we further assume without loss of generality that the problem is appropriately scaled, i.e.,

$$\|F'(x)\| \leq 1, \qquad x \in B_r(x_+). \tag{12}$$

Remark. Instead of the tangential cone condition, some papers impose that

$$\|F(\widetilde{x}) - F(x) - F'(x)(\widetilde{x} - x)\| \leq C \|x - \widetilde{x}\| \, \|F(x) - F(\widetilde{x})\|,$$

for $x, \widetilde{x} \in B_r(x_+)$, cf. [4]. Our analysis is valid in this case as well. A discussion about the different conditions can be found in [5, 11].

Recall that we always assume that (1) has a solution $x_* \in \mathcal{D}(F)$. In this case, Assumption 1 ensures that a unique solution x_+ of minimal distance to x_0 exists, cf. [11, Prop. 2.1]. This solution satisfies $x_+ - x_0 \in \mathcal{N}\big(F'(x_+)\big)^\perp$.

3.2. Continuous case

We now review the main results on the convergence of the solution of (4). Theorem 3 in [15] states that the method converges when applied to the unperturbed problem (1).

Theorem 3.1. *Let Assumption* 1 *be satisfied and let* (1) *be solvable in* $B_r(x_+)$. *If* $u(t)$ *is a solution of* (4), *then* $u(t)$ *converges to a solution of* (1) *in* $B_r(x_+)$ *for* $t \to \infty$. *If in addition* $\mathcal{N}\big(F'(x_+)\big) \subset \mathcal{N}\big(F'(x)\big)$ *for all* $x \in B_r(x_+)$, *then* $u(t)$ *converges to* x_+.

Theorem 4 of [15] states the corresponding result for perturbed data.

Theorem 3.2. *Let Assumption* 1, (3), *and* $\big\|y^\delta - F(x_0)\big\| > \tau\delta > 0$ *be satisfied and let* (1) *be solvable in* $B_r(x_+)$. *If* $u(t)$ *is a solution of* (4), *and* t_* *is chosen from* (9) *with*

$$\tau > \frac{1+\eta}{1-\eta},$$

then $u(t_*)$ *converges to a solution* $x_* \in B_r(x_+)$ *of* (1) *for* $\delta \to 0$. *If in addition* $\mathcal{N}\big(F'(x_+)\big) \subset \mathcal{N}\big(F'(x)\big)$ *for all* $x \in B_r(x_+)$, *then* $u(t_*)$ *converges to* x_+ *for* $\delta \to 0$.

It is shown in [15] that the stopping-time t_* is uniquely defined via (9) and finite. An important auxiliary result is the monotonicity of the error function $\|u(t) - x_+\|$. The analogous discrete results will be verified in the analysis of the exponential Euler method below.

3.3. Discrete case

Next we analyze the convergence of the exponential Euler method (5). To simplify the presentation of our paper, we introduce the following notation. The operators will be denoted by

$$\begin{aligned}
A_+ &= F'(x_+), & A_n &= F'(u_n), \\
J_+ &= A_+^* A_+, & J_n &= A_n^* A_n, \\
K_+ &= A_+ A_+^*, & K_n &= A_n A_n^*,
\end{aligned}$$

and the corresponding operator functions by

$$\begin{aligned}
\Phi_{n,+} &= \varphi(-h_n J_+), & \Phi_n &= \varphi(-h_n J_n), \\
\widetilde{\Phi}_{n,+} &= \varphi(-h_n K_+), & \widetilde{\Phi}_n &= \varphi(-h_n K_n).
\end{aligned}$$

To ensure monotonicity of the errors (cf. Lemma 3.4 below), the step sizes h_n have to be chosen appropriately. For this we propose to use the following discrepancy principle

$$p_n^\delta(h_n) := \big\|\psi(-h_n K_n)\Delta F_n^\delta - A_n \Delta u_n\big\| = \mu \big\|\Delta F_n^\delta\big\|, \quad \mu < 1, \quad (13)$$

where the updates Δu_n are written as

$$\Delta u_n = u_{n+1} - u_n = h_n A_n^* \widetilde{\Phi}_n \Delta F_n^\delta$$

and the function ψ is defined by

$$\psi(z) = \varphi(z)(1-z). \tag{14}$$

Due to (14), we have

$$p_n^\delta(h_n) = \big\|\widetilde{\Phi}_n \Delta F_n^\delta\big\|. \tag{15}$$

We will verify later in Theorems 3.5 and 3.6 that if h_n is chosen according to (13), then there exists a parameter $\nu > 1$ such that the nth residual satisfies

$$\left\| \psi(-h_n K_n)\Delta F_n^\delta + A_n e_n \right\| \leq \frac{\mu}{\nu}\left\| \Delta F_n^\delta \right\|, \tag{16}$$

where

$$e_n = u_n - x_+ \tag{17}$$

denotes the error of the nth iterate u_n. This will be used to prove existence of a unique step size h_n satisfying (13).

Remark. For the Levenberg-Marquardt method the situation simplifies due to $\psi \equiv 1$. In this case, the step size selection (13) and assumption (16) are the same as in [3].

Lemma 3.3. *Assume that (16) holds, then there is a unique solution $h_n > 0$ of (13).*

Proof. Obviously, the function $p_n^\delta(h)$ is continuous and strictly decreasing if $\Delta F_n^\delta \neq 0$. By (15) and due to $\varphi(0) = 1$ and $\varphi(-z) \to 0$ for $z \to \infty$ we have

$$\lim_{h \to 0} p_n^\delta(h) = \left\| \Delta F_n^\delta \right\|, \qquad \lim_{h \to \infty} p_n^\delta(h) = \left\| \Pi_n \Delta F_n^\delta \right\|,$$

where Π_n denotes the orthogonal projection onto $\mathcal{R}(A_n)^\perp$. By definition of ψ we have $\psi(0) = 1$, and thus

$$\Pi_n \Delta F_n^\delta = \Pi_n\left(\psi(-hK_n)\Delta F_n^\delta + A_n(u_n - u) \right)$$

for all $u \in \mathcal{D}(F)$. This yields

$$\left\| \Pi_n \Delta F_n^\delta \right\| \leq \left\| \psi(-hK_n)\Delta F_n^\delta + A_n(u_n - u) \right\|$$

for all $h > 0$ and for all $u \in \mathcal{D}(F)$. The existence of h_n satisfying (13) then follows from (16). $\qquad\square$

The next lemma states that the sequence of errors is monotonically decreasing.

Lemma 3.4. *Let $0 < \mu < 1 < \nu$ and assume that (16) is fulfilled so that h_n can be defined according to (13). Then for $\Delta u_n \neq 0$ we have*

$$\|e_n\|^2 - \|e_{n+1}\|^2 > \frac{2h_n(\nu - 1)\mu}{\nu}\left\| \Delta F_n^\delta \right\| \cdot \left\| \widetilde{\Phi}_n \Delta F_n^\delta \right\|, \tag{18}$$

$$\|e_n\|^2 - \|e_{n+1}\|^2 > -\frac{2\varphi^{-1}(\mu)(\nu - 1)\mu^2}{\nu \|A_n\|^2}\left\| \Delta F_n^\delta \right\|^2. \tag{19}$$

Note that $\varphi^{-1}(\mu) < 0$ for $\mu < 1$.

Proof. Our proof is inspired by the proof of Proposition 2.1 in [3]. The errors (17) satisfy

$$
\begin{aligned}
\|e_{n+1}\|^2 &= \|\Delta u_n + e_n\|^2 \\
&= \|\Delta u_n\|^2 + 2\langle \Delta u_n, e_n \rangle + \|e_n\|^2 \\
&= h_n^2 \left\langle \Phi_n A_n^* \Delta F_n^\delta, \Phi_n A_n^* \Delta F_n^\delta \right\rangle \\
&\quad + 2h_n \left\langle \Phi_n A_n^* \Delta F_n^\delta, e_n \right\rangle + \|e_n\|^2
\end{aligned}
$$

$$= h_n^2 \left\langle \Delta F_n^\delta, K_n \widetilde{\Phi}_n^2 \Delta F_n^\delta \right\rangle$$
$$- 2h_n \left\langle \widetilde{\Phi}_n \Delta F_n^\delta, \psi(-h_n K_n) \Delta F_n^\delta \right\rangle$$
$$+ 2h_n \left\langle \widetilde{\Phi}_n \Delta F_n^\delta, \psi(-h_n K_n) \Delta F_n^\delta + A_n e_n \right\rangle + \|e_n\|^2$$
$$= -h_n^2 \left\langle \Delta F_n^\delta, K_n \widetilde{\Phi}_n^2 \Delta F_n^\delta \right\rangle$$
$$+ 2h_n \left\langle \widetilde{\Phi}_n \Delta F_n^\delta, h_n K_n \widetilde{\Phi}_n \Delta F_n^\delta - \psi(-h_n K_n) \Delta F_n^\delta \right\rangle$$
$$+ 2h_n \left\langle \widetilde{\Phi}_n \Delta F_n^\delta, \psi(-h_n K_n) \Delta F_n^\delta + A_n e_n \right\rangle + \|e_n\|^2.$$

By (14) we have $z\varphi(-z) - \psi(-z) = -\varphi(-z)$, so that the expression above can be written as

$$\|e_{n+1}\|^2 - \|e_n\|^2 = -h_n^2 \left\langle \Delta F_n^\delta, K_n \widetilde{\Phi}_n^2 \Delta F_n^\delta \right\rangle - 2h_n \left\langle \widetilde{\Phi}_n \Delta F_n^\delta, \widetilde{\Phi}_n \Delta F_n^\delta \right\rangle$$
$$+ 2h_n \left\langle \widetilde{\Phi}_n \Delta F_n^\delta, \psi(-h_n K_n) \Delta F_n^\delta + A_n e_n \right\rangle$$
$$= -\|\Delta u_n\|^2 - 2h_n \left\| \widetilde{\Phi}_n \Delta F_n^\delta \right\|^2$$
$$+ 2h_n \left\langle \widetilde{\Phi}_n \Delta F_n^\delta, \psi(-h_n K_n) \Delta F_n^\delta + A_n e_n \right\rangle$$
$$< -2h_n \left\| \widetilde{\Phi}_n \Delta F_n^\delta \right\|^2 + 2h_n \left\langle \widetilde{\Phi}_n \Delta F_n^\delta, \psi(-h_n K_n) \Delta F_n^\delta + A_n e_n \right\rangle$$
$$\leq 2h_n \left\| \widetilde{\Phi}_n \Delta F_n^\delta \right\| \left(\left\| \psi(-h_n K_n) \Delta F_n^\delta + A_n e_n \right\| - \left\| \widetilde{\Phi}_n \Delta F_n^\delta \right\| \right)$$
$$\leq 2h_n \mu \frac{1-\nu}{\nu} \left\| \widetilde{\Phi}_n \Delta F_n^\delta \right\| \left\| \Delta F_n^\delta \right\|.$$

For the last inequality we have used (13), (15) and (16).

For the second estimate (19) we note that the relation

$$\mu \left\| \Delta F_n^\delta \right\| = \left\| \widetilde{\Phi}_n \Delta F_n^\delta \right\| \geq \varphi(-h_n \|K_n\|) \left\| \Delta F_n^\delta \right\|$$

yields

$$h_n \geq -\frac{\varphi^{-1}(\mu)}{\|A_n\|^2}.$$

This proves the desired result. $\qquad\square$

Using these lemmas, we will prove that the method converges if it is applied to the unperturbed problem (1). We will write the unperturbed problem formally as a perturbed problem with $\delta = 0$, so that $y^\delta = y$ in our scheme.

Theorem 3.5. *Let $0.3 + \eta < \mu < 1$ and let in addition Assumption 1 and (12) be satisfied. If the step sizes h_n are chosen according to (13) and if $y^\delta = y = F(x_+)$, then u_n converges to a solution of (1) as $n \to \infty$. If in addition $\mathcal{N}\big(F'(x_+)\big) \subset \mathcal{N}\big(F'(x)\big)$ for all $x \in B_r(x_+)$, then u_n converges to x_+ as $n \to \infty$.*

Proof. The proof is done analogously to the proof of Theorem 2.2 in [3]. First we prove that the error decreases monotonically.

Let $n = 0$ and denote by

$$\Delta F_n = y - F(u_n),$$

the residual of the unperturbed problem. Then, by definition of ψ and (10), we have

$$
\begin{aligned}
\|\psi(-h_n K_n)\Delta F_n + A_n e_n\| \\
&= \left\|\left(\widetilde{\Phi}_n - \exp(-h_n K_n)\right)\Delta F_n + \Delta F_n + A_n e_n\right\| \\
&\leq \left\|\widetilde{\Phi}_n - \exp(-h_n K_n)\right\| \|\Delta F_n\| + \|\Delta F_n + A_n e_n\| \\
&\leq \sup_{z \in [-\infty, 0]} |\varphi(z) - \mathrm{e}^z| \|\Delta F_n\| + \eta \|\Delta F_n\| \\
&\leq (0.3 + \eta) \|\Delta F_n\|.
\end{aligned}
$$

Note that the function $|\varphi(z) - \mathrm{e}^z|$ has a unique maximum for $z \leq 0$. The stated bound follows by straightforward numerical calculations. For $\mu > 0.3 + \eta$ the above reasoning yields (16) with $\nu = \dfrac{\mu}{0.3 + \eta} > 1$. Moreover, Lemma 3.4 gives

$$\|e_{n+1}\| < \|e_n\|.$$

Induction shows that the error decreases monotonically for all $n \in \mathbb{N}$.

As in the proof of Theorem 4.2 of [4], we proceed by proving that $\|e_n\|$ is a Cauchy sequence. Let $m, n \in \mathbb{N}$ be given indices satisfying $m > n$ and let l be an index chosen such that $n \leq l \leq m$ and

$$\|\Delta F_l\| \leq \|\Delta F_i\|, \qquad i = n, \ldots, m.$$

From (10) we have

$$
\begin{aligned}
\|A_i e_l\| &= \|A_i e_i - A_i(e_i - e_l)\| \\
&\leq \|\Delta F_i + A_i e_i\| + \|F(u_i) - F(u_l) - A_i(e_i - e_l)\| + \|\Delta F_l\| \\
&\leq (3\eta + 1)\|\Delta F_i\|.
\end{aligned}
$$

Recall that the error can be written as

$$e_{n+1} = e_n + h_n A_n^* \widetilde{\Phi}_n \Delta F_n.$$

Using (18) we obtain

$$
\begin{aligned}
|\langle e_l - e_n, e_l \rangle| &= \left|\sum_{i=n}^{l-1} \left\langle h_i A_i^* \widetilde{\Phi}_i \Delta F_i, e_l \right\rangle\right| \\
&= \left|\sum_{i=n}^{l-1} \left\langle h_i \widetilde{\Phi}_i \Delta F_i, A_i e_l \right\rangle\right| \\
&\leq \frac{(3\eta + 1)\nu}{2(\nu - 1)\mu} \left(\|e_n\|^2 - \|e_l\|^2\right).
\end{aligned}
$$

The identity

$$\|e_l - e_n\|^2 = 2\langle e_l - e_n, e_l \rangle + \|e_n\|^2 - \|e_l\|^2$$

thus yields

$$\|e_l - e_n\|^2 \leq \left(\frac{(3\eta + 1)\nu}{(\nu - 1)\mu} + 1\right)\left(\|e_n\|^2 - \|e_l\|^2\right)$$

and analogously

$$\|e_m - e_l\|^2 \leq \left(\frac{(3\eta + 1)\nu}{(\nu - 1)\mu} + 1 \right) \left(\|e_l\|^2 - \|e_m\|^2 \right),$$

so that

$$
\begin{aligned}
\|u_m - u_n\|^2 &= \|e_m - e_n\|^2 \\
&\leq 2 \|e_m - e_l\|^2 + 2 \|e_l - e_n\|^2 \\
&\leq 2 \left(\frac{(3\eta + 1)\nu}{(\nu - 1)\mu} + 1 \right) \left(\|e_n\|^2 - \|e_m\|^2 \right).
\end{aligned}
$$

From the monotonicity of the errors it follows that the sequence $\{\|e_n\|\}$ converges. Consequently, $\{u_n\}$ is a Cauchy sequence which converges to a limit u_∞. By (19) and (12), the series

$$\sum_{n=0}^{\infty} \|\Delta F_n\|^2$$

converges and therefore u_∞ is a solution of (1).

The second part of the theorem, the convergence of u_n to x_+, follows as in Theorem 2.3 in [5]. ◻

Our main result of this section is that the exponential Euler iteration applied to the perturbed problem (2) converges to a solution of the unperturbed problem (1) if the perturbation tends to zero.

Theorem 3.6. *Let the assumptions of Theorem 3.5 and (3) be satisfied and let $\mu > (1 + \eta + \tau(0.3 + \eta))/\tau$. Then the exponential Euler iteration stopped by the discrepancy principle (8) terminates after $n_* = n_*(\delta) < \infty$ iterations and the corresponding approximations u_{n_*} converge to a solution of (1) for $\delta \to 0$. If in addition $\mathcal{N}(F'(x_+)) \subset \mathcal{N}(F'(x))$ for all $x \in B_r(x_+)$, then u_{n_*} converges to x_+.*

Proof. We follow the proof of [3, Theorem 2.3] and show the monotonic decrease of the error as long as the discrepancy principle (8) is not fulfilled. As in the proof of Theorem 3.5 we have for $n \in \mathbb{N}$

$$
\begin{aligned}
\Big\| \psi&(-h_n K_n)\Delta F_n^\delta + A_n e_n \Big\| \\
&\leq \left\| \widetilde{\Phi}_n - \exp(-h_n K_n) \right\| \left\| \Delta F_n^\delta \right\| + \left\| \Delta F_n^\delta + A_n e_n \right\| \\
&\leq \sup_{z \in [-\infty, 0]} |\varphi(z) - e^z| \left\| \Delta F_n^\delta \right\| + \delta + \eta \|\Delta F_n\| \\
&\leq (0.3 + \eta) \left\| \Delta F_n^\delta \right\| + \delta(1 + \eta).
\end{aligned}
$$

If $n_* = 0$, there is nothing to prove. For $n_* > 0$ and $n < n_*$ we have $\delta < \left\| \Delta F_n^\delta \right\|/\tau$ and hence

$$\left\| \psi(-h_n K_n)\Delta F_n^\delta + A_n e_n \right\| \leq \frac{1 + \eta + \tau(0.3 + \eta)}{\tau} \left\| \Delta F_n^\delta \right\|,$$

which shows that (16) holds with $\nu = \mu\tau/(1 + \eta + \tau(0.3 + \eta))$ for all $n < n_*$. Applying Lemma 3.4 finally proves the monotonicity of $\{\|e_n\|\}$.

From the discrepancy principle (8), (12), and (19) we conclude

$$
\begin{aligned}
n_* \tau^2 \delta^2 &\leq \sum_{n=0}^{n_*-1} \left\| \Delta F_n^\delta \right\|^2 \\
&\leq \frac{\nu}{2\varphi^{-1}(\mu)(\nu-1)\mu^2} \left(\|e_{n_*}\|^2 - \|e_0\|^2 \right) \\
&< \infty
\end{aligned}
$$

and therefore the stopping index n_* is finite. The remaining part is proved as in [3, Theorem 2.3]. $\qquad\square$

Remark. The convergence results can be generalized by replacing the parameter μ in (13) by variable parameters μ_n satisfying

$$
\mu_{\min} \leq \mu_n \leq \mu_{\max} < 1.
$$

Theorems 3.5 and 3.6 remain true if the assumptions are fulfilled for $\mu_{\min}$ instead of μ.

4. Convergence rates

The aim of this section is to show that the exponential Euler regularization in fact converges with optimal rates under suitable assumptions.

4.1. Assumptions

The first additional assumption is a so-called source condition. In many applications, in particular in parameter identification problems for partial differential equations, this condition reflects the smoothness of the initial error.

Assumption 2. *There exists $w \in X$ and constants $\gamma \in (0, 1/2]$ and $\rho \geq 0$ such that*

$$
e_0 = x_0 - x_+ = J(x_+)^\gamma w, \qquad \|w\| \leq \rho
$$

Moreover, we have to assume relations between the Fréchet derivatives F' evaluated at certain points in $B_r(x_+)$.

Assumption 3. *For all $x, \widetilde{x} \in B_r(x_+)$ there exist linear bounded operators $R(x, \widetilde{x}) : Y \to Y$ and a constant $C_R \geq 0$ such that*

(i) $F'(x) = R(x, \widetilde{x}) F'(\widetilde{x})$

(ii) $\|R(x, \widetilde{x}) - I\| \leq C_R \|x - \widetilde{x}\|.$

Both assumption are standard assumptions arising in the literature, see, e.g. [10, 11].

4.2. Continuous case

As in Section 3, we first present the main result for the continuous solution of (4) from [15, Theorem 6].

Theorem 4.1. *If in addition to the assumptions of Theorem 3.2, Assumptions 2 and 3, and (12) are satisfied and if*

$$
\tau > \frac{2 - \eta}{1 - \eta}, \tag{20}
$$

then there exists a constant $c_* = c_*(\tau, \eta, C_R, \gamma)$ such that

$$\|u(t_*) - x_+\| \le c_* \rho^{1/(2\gamma+1)} \delta^{2\gamma/(2\gamma+1)}$$

for ρ sufficiently small.

This theorem proves that asymptotic regularization converges with optimal rates.

4.3. Discrete case

Next we will derive the corresponding results for the discrete exponential Euler regularization scheme. These results are valid under weak restrictions on the step sizes, namely we assume that there exist constants c_0 and c_h such that

$$h_0 \le c_0, \qquad h_j \le c_h t_j, \qquad j \ge 1, \tag{21}$$

where, as before, $t_{j+1} = t_j + h_j$. Note that this step size restriction allows to choose $(h_j)_{j \ge 0}$ as a geometric sequence, i.e. $h_j = h_0 \sigma^j$ for some $\sigma > 1$.

Our main result is the following theorem.

Theorem 4.2. *Let Assumptions* 1 *to* 3 *hold and assume that the step sizes* h_j *satisfy* (21) *for all* $j \le n_*$. *Here, the stopping index* n_* *is defined by* (8), *with* τ *satisfying* (20). *Then for* ρ *sufficiently small, there exists a constant* $C = C(\tau, \eta, C_R, c_0, c_h, \gamma, r) > 0$ *such that*

$$\|e_{n_*}\| \le C \rho^{1/(2\gamma+1)} \delta^{2\gamma/(2\gamma+1)}.$$

The proof of this theorem is rather involved and therefore split into several pieces. The auxiliary results required will be presented in a series of lemmas. First we give two immediate consequences of our assumptions.

Lemma 4.3. *Let Assumption* 1 *and* (3) *hold and let the stopping index* n_* *be defined by* (8). *Then we have*

$$\left\| \Delta F_n^\delta \right\| \le \frac{\tau}{(\tau-1)(1-\eta)} \|A_+ e_n\|, \qquad n < n_*.$$

Proof. By (11) and (3) we have

$$\begin{aligned}
\left\| \Delta F_n^\delta \right\| &\le \left\| y^\delta - y \right\| + \|F(x_+) - F(u_n)\| \\
&\le \delta + \frac{1}{1-\eta} \|A_+ e_n\|.
\end{aligned}$$

The estimate thus follows from the stopping criterion (8). $\qquad \square$

Lemma 4.4. *Let Assumption* 3 *hold. Then, for* $x \in B_r(x_+)$, *we have*

$$\|F(x) - F(x_+) - F'(x_+)(x - x_+)\| \le \frac{1}{2} C_R \|x - x_+\| \, \|F'(x_+)(x - x_+)\|.$$

Proof. Proposition 4 in [15]. $\qquad \square$

One of the main ingredients of the theory presented by Tautenhahn [15] is the variation-of-constants formula for the error and the error premultiplied by A_+. Our analysis uses the discrete variation-of-constants formula (Theorem 4.8), which is derived from the following suitably written error recursion.

Lemma 4.5. *Let Assumption 3 hold. Then the error $e_n = u_n - x_+$ of the exponential Euler recursion* (5) *satisfies*

$$e_{n+1} = \exp(-h_n J_+)e_n + h_n \Phi_{n,+} A_+^* \big(r_n^{(1)} + y^\delta - y\big) - h_n r_n^{(2)}. \quad (22)$$

where

$$r_n^{(1)} = F(x_+) - F(u_n) + A_+ e_n + (R^*(u_n, x_+) - I)\Delta F_n^\delta,$$
$$r_n^{(2)} = \big(\Phi_{n,+} - \Phi_n\big) A_n^* \Delta F_n^\delta.$$

Proof. By (5), the following error recursion holds

$$\begin{aligned}
e_{n+1} &= e_n + h_n \Phi_n A_n^* \Delta F_n^\delta \\
&= e_n + h_n \Phi_{n,+} A_+^* (F(x_+) - F(u_n)) \\
&\quad - h_n \big(\Phi_{n,+} A_+^* - \Phi_n A_n^*\big) \Delta F_n^\delta \\
&\quad + h_n \Phi_{n,+} A_+^* (y^\delta - y).
\end{aligned}$$

By writing e_n in the form

$$e_n = \exp(-h_n J_+)e_n + h_n \Phi_{n,+} A_+^* A_+ e_n$$

we get

$$\begin{aligned}
e_{n+1} &= \exp(-h_n J_+)e_n \\
&\quad + h_n A_+^* \widetilde{\Phi}_{n,+}(F(x_+) - F(u_n) + A_+ e_n) \\
&\quad - h_n \big(\Phi_{n,+} A_+^* - \Phi_n A_n^*\big) \Delta F_n^\delta \\
&\quad + h_n A_+^* \widetilde{\Phi}_{n,+}(y^\delta - y).
\end{aligned}$$

The identity

$$\Phi_{n,+} A_+^* - \Phi_n A_n^* = \Phi_{n,+} A_+^* \big(I - R^*(u_n, x_+)\big) + (\Phi_{n,+} - \Phi_n))A_n^*$$

finally proves the desired relation. □

Lemma 4.6. *Let Assumptions 1 and 3 hold. Then there are constants $C_i = C_i(\tau, \eta, C_R, r)$, $i = 1, 2, 3$ such that for $j < n \leq n_*$ we have*

$$\left\| r_j^{(1)} \right\| \leq C_1 \|e_j\| \|A_+ e_j\|,$$

$$\left\| \exp\big(-(t_n - t_{j+1})J_+\big) r_j^{(2)} \right\| \leq C_2 \frac{1}{\sqrt{1 + t_n - t_j}} \|e_j\| \|A_+ e_j\|,$$

$$\left\| A_+ \exp\big(-(t_n - t_{j+1})J_+\big) r_j^{(2)} \right\| \leq C_3 \frac{1}{1 + t_n - t_j} \|e_j\| \|A_+ e_j\|.$$

Proof. From Lemmas 4.3 and 4.4 we obtain the first bound with

$$C_1 = C_R \left(\frac{1}{2} + \frac{\tau}{(\tau - 1)(1 - \eta)} \right).$$

The terms involving $r_j^{(2)}$ can be written with the help of the Cauchy integral formula

$$\Phi_{j,+} - \Phi_j = \frac{1}{2\pi i} \int_\Gamma \varphi(\lambda) \Big((\lambda I + h_j J_+)^{-1} - (\lambda I + h_j J_j)^{-1}\Big) \, d\lambda, \quad (23)$$

see [1] and [2]. The contour Γ is parameterized by

$$s \mapsto 1 - |s|\, e^{i\theta \operatorname{sign} s}, \quad s \in \mathbb{R} \tag{24}$$

for some $0 < \theta < \pi/2$. Using the resolvent identity

$$
\begin{aligned}
(\lambda + h_j J_+)^{-1} &- (\lambda + h_j J_j)^{-1} \\
&= h_j(\lambda + h_j J_+)^{-1}(J_j - J_+)(\lambda + h_j J_j)^{-1} \\
&= h_j(\lambda + h_j J_+)^{-1}\big((A_j^* - A_+^*)A_j - A_+^*(A_+ - A_j)\big)(\lambda + h_j J_j)^{-1} \\
&= h_j(\lambda + h_j J_+)^{-1} A_+^* \big(R^*(u_j, x_+) - R(x_+, u_j)\big) A_j (\lambda + h_j J_j)^{-1}
\end{aligned}
$$

we obtain

$$(\Phi_{j,+} - \Phi_j)A_j^* = \frac{1}{2\pi i} \int_\Gamma \varphi(\lambda)(\lambda + h_j J_+)^{-1} A_+^* \Delta R_j h_j K_j (\lambda + h_j K_j)^{-1}\, d\lambda \tag{25}$$

with $\Delta R_j = R^*(u_j, x_+) - R(x_+, u_j)$. From

$$\sup_{\sigma \geq 0} \left| \frac{\sigma}{\lambda + \sigma} \right| \leq 1 + \left| \frac{\lambda}{\operatorname{Im}\lambda} \right| = 1 + \left| \frac{\lambda}{\lambda - 1} \right| \frac{1}{\sin\theta}, \qquad \lambda \in \Gamma$$

we have the bound

$$\left\| \exp(-tJ_+) r_j^{(2)} \right\| \leq C_4 \int_\Gamma |\varphi(\lambda)| \left\| \exp(-tJ_+)(\lambda + h_j J_+)^{-1} A_+^* \right\| \|e_j\|\, d\,|\lambda| \left\| \Delta F_n^\delta \right\|$$

with $C_4 = C_4(C_R, \theta)$. By Lemma 4.7, we thus obtain the second bound of Lemma 4.6.

In order to prove the third estimate, we start with

$$\Phi_{j,+} - \Phi_j = \boldsymbol{\chi}_{j,+} - \boldsymbol{\chi}_j + (I + h_j J_+)^{-1} - (I + h_j J_j)^{-1}$$

with $\boldsymbol{\chi}_{j,+} = \chi(-h_j J_+)$ and $\boldsymbol{\chi}_j = \chi(-h_j J_j)$, where

$$\chi(\lambda) = \varphi(\lambda) - \frac{1}{1 - \lambda}.$$

Note that $\chi(\lambda) = \mathcal{O}(\lambda^{-2})$ for $\operatorname{Re}\lambda \to -\infty$. Using the Cauchy integral formula again yields

$$
\begin{aligned}
A_+ \exp(-tJ_+)(\boldsymbol{\chi}_{j,+} &- \boldsymbol{\chi}_j)A_j^* \\
&= \frac{1}{2\pi i} \int_\Gamma \chi(\lambda) \exp(-tK_+)(\lambda + h_j K_+)^{-1} K_+ \Delta R_j h_j K_j (\lambda + h_j K_j)^{-1}\, d\lambda.
\end{aligned}
$$

By Lemma 4.7, we have

$$\left\| \exp(-tK_+)(\lambda + h_j K_+)^{-1} K_+ \right\| \leq \frac{C_5}{1 + t + h_j}$$

which shows that the above integral possesses the desired bound. Using the above resolvent identity with $\lambda = 1$ finally shows

$$
\begin{aligned}
A_+ \exp(-tJ_+)\big((I + h_j J_+)^{-1} &- (I + h_j J_j)^{-1}\big)A_j^* \\
&= \exp(-tK_+)(I + h_j K_+)^{-1} K_+ \Delta R_j h_j K_j (\lambda + h_j K_j)^{-1}.
\end{aligned}
$$

This can be bounded with the help of Lemma 4.7. $\qquad \square$

Lemma 4.7. *For $0 \leq \alpha \leq 1$ and $\lambda \in \Gamma$, where the curve Γ is defined by (24), it holds*

$$\sup_{0 \leq \sigma \leq 1} \left| \sigma^{\alpha} \mathrm{e}^{-\sigma t} \varphi(-\sigma h) \right| \leq \min \left\{ \frac{1}{(t+h)^{\alpha}}, \frac{2}{(1+t+h)^{\alpha}} \right\},$$

$$\sup_{0 \leq \sigma \leq 1} \left| \sigma^{\alpha} \mathrm{e}^{-\sigma t} \frac{1}{\lambda + \sigma h} \right| \leq \frac{C_5}{(1+t+h)^{\alpha} |\lambda|^{1-\alpha}},$$

with a constant $C_5 = C_5(\theta)$.

Proof. We consider the function $f(x,y) = (1 + x + y)^{\alpha} \mathrm{e}^{-x} \varphi(-y)$ on the cone $\Omega = \{x, y \geq 0\}$. It is easily verified that f does not have stationary points in the interior of Ω. Since f is bounded in Ω, local extrema have to appear on the boundary $\partial\Omega$. Obviously, $|f(x,y)| \leq 2$ for $(x,y) \in \partial\Omega$. The other bounds follows in the same way. $\square$

We are now ready to prove that the errors $\|e_n\|$ and $\|A_+ e_n\|$ decay with the same rate proportional to $(1 + t_n)^{\gamma}$ and $(1 + t_n)^{\gamma + 1/2}$, respectively, as their continuous counterparts in [15].

Theorem 4.8. *Let the assumptions of Theorem 4.2 hold. Then for ρ sufficiently small there is a constant $C_* = C_*(\tau, \eta, C_R, c_0, c_h, \gamma, r)$ such that for $n < n_*$*

$$\|e_n\| \quad \leq C_* \frac{\rho}{(1 + t_n)^{\gamma}},$$

$$\|A_+ e_n\| \leq C_* \frac{\rho}{(1 + t_n)^{\gamma + 1/2}}.$$

Proof. For an arbitrary $n \in \mathbb{N}$ the error recursion (22) leads to the following discrete variation-of-constants formulas

$$e_n = \exp(-t_n J_+) e_0 \tag{26}$$

$$+ \sum_{j=0}^{n-1} h_j \exp\left(-(t_n - t_{j+1}) J_+\right) \left(\Phi_{j,+} A_+^* \left(r_j^{(1)} + y^{\delta} - y \right) - r_j^{(2)} \right)$$

and

$$A_+ e_n = \exp(-t_n K_+) A_+ e_0 \tag{27}$$

$$+ \sum_{j=0}^{n-1} h_j \exp\left(-(t_n - t_{j+1}) K_+\right) \widetilde{\Phi}_{j,+} K_+ \left(r_j^{(1)} + y^{\delta} - y \right)$$

$$- \sum_{j=0}^{n-1} h_j A_+ \exp\left(-(t_n - t_{j+1}) J_+\right) r_j^{(2)}.$$

By Lemma 4.7, the sum multiplying $y^{\delta} - y$ in (26) can be bounded by

$$\left\| \sum_{j=0}^{n-1} h_j \exp(-(t_n - t_{j+1}) J_+) A_+^* \widetilde{\Phi}_{j,+} \right\| \leq \sum_{j=0}^{n-1} \frac{h_j}{\sqrt{t_n - t_j}}$$

$$\leq \int_0^{t_n} \frac{1}{\sqrt{t_n - s}} \, \mathrm{d}s = 2\sqrt{t_n} \tag{28}$$

while the corresponding sum in (27) can be bounded by one by using the (telescopic) identity

$$\sum_{j=0}^{n-1} h_j \exp(-(t_n - t_{j+1})K_+)K_+\widetilde{\Phi}_{j,+} = I - \exp(-t_n K_+). \tag{29}$$

Thus, by Assumption 2, Lemmas 4.6 and 4.7 we have

$$\|e_n\| \leq \frac{\rho}{(1 + t_n)^\gamma} + 2\sqrt{t_n}\,\delta$$

$$+ (2C_1 + C_2)\sum_{j=0}^{n-1} h_j \frac{1}{\sqrt{1 + t_n - t_j}}\|e_j\|\,\|A_+ e_j\|$$

and

$$\|A_+ e_n\| \leq \frac{\rho}{(1 + t_n)^{\gamma+1/2}} + \delta$$

$$+ (2C_1 + C_3)\sum_{j=0}^{n-1} h_j \frac{1}{1 + t_n - t_j}\|e_j\|\,\|A_+ e_j\|.$$

The proof now proceeds by induction for $n = 0, 1, \ldots, n_* - 1$. By Assumption 2, the statement is true for $n = 0$ if $C_* \geq 1$. Assuming that the bounds hold for all indices up to $n - 1$, we obtain

$$\|e_n\| \leq \frac{\rho}{(1 + t_n)^\gamma} + 2\sqrt{t_n}\,\delta + C_*^2 \rho^2 (2C_1 + C_2)\, S_n\left(\tfrac{1}{2}, 2\gamma + \tfrac{1}{2}\right)$$

and

$$\|A_+ e_n\| \leq \frac{\rho}{(1 + t_n)^{\gamma+1/2}} + \delta + C_*^2 \rho^2 (2C_1 + C_3)\, S_n\left(1, 2\gamma + \tfrac{1}{2}\right)$$

where

$$S_n(\alpha, \beta) = \sum_{j=0}^{n-1} \frac{h_j}{(1 + t_n - t_j)^\alpha (1 + t_j)^\beta}. \tag{30}$$

The inequalities in Lemma 4.11 below lead to

$$\|e_n\| \quad \leq \frac{\rho}{(1 + t_n)^\gamma}\left(1 + C_*^2 \rho C_6\right) + 2\sqrt{t_n}\,\delta, \tag{31}$$

$$\|A_+ e_n\| \leq \frac{\rho}{(1 + t_n)^{\gamma+1/2}}\left(1 + C_*^2 \rho C_7\right) + \delta \tag{32}$$

with

$$C_6 = (2C_1 + C_2)C_9,$$
$$C_7 = (2C_1 + C_3)C_9.$$

Applying (8) and Lemma 4.3 we get

$$\delta \leq \frac{1}{(\tau - 1)(1 - \eta)}\|A_+ e_n\|$$

$$\leq \frac{1}{(\tau - 1)(1 - \eta)}\left(\frac{\rho}{(1 + t_n)^{\gamma+1/2}}\left(1 + C_*^2 \rho C_7\right) + \delta\right).$$

By (20), this yields for $n < n_*$

$$\delta \leq C_8 \frac{\rho}{(1 + t_n)^{\gamma + 1/2}} \tag{33}$$

with

$$C_8 = \frac{1}{(\tau - 1)(1 - \eta) - 1} \left(1 + C_*^2 \rho C_7\right).$$

Inserting this relation into (31) shows

$$\|e_n\| \leq \frac{\rho}{(1 + t_n)^\gamma} \left(1 + C_*^2 \rho C_6 + 2C_8\right).$$

This yields the desired result, as long as

$$1 + C_*^2 \rho C_6 + 2C_8 \leq C_*,$$
$$1 + C_*^2 \rho C_7 + C_8 \leq C_*$$

hold. This can be achieved for ρ sufficiently small. $\qquad\square$

Remark. If the maximum possible step sizes $h_j = c_h t_j$, $j = 1, \ldots, n_* - 1$ are chosen, then Lemma 4.3, Theorem 4.8, and (8) show that there is a constant c such that the stopping index satisfies $n_* \leq c |\log \delta|$.

Theorem 4.9. *Let the assumptions of Theorem 4.2 be satisfied. If in addition the step size sequence is bounded away from 0, then the stopping index n_* defined by (8) is finite and the iterates u_n satisfy $u_n \in B_{C_* \rho}(x_+)$ for $n = 0, 1, \ldots, n_*$, and C_* as in Theorem 4.8. Moreover, the approximation u_{n_*} converges to a solution of (1) for $\delta \to 0$.*

Proof. The assumption that the step sizes are bounded away from 0 ensures that $t_n \to \infty$ for $n \to \infty$. By Lemma 4.3 and Theorem 4.8 we have that $\left\|\Delta F_n^\delta\right\| \to 0$ for $n \to \infty$. Hence the discrepancy principle (8) is satisfied for $n_* < \infty$. Moreover, by Theorem 4.8 we have $\|e_n\| \leq C_* \rho$ for $n = 0, 1, \ldots$, which shows that the method is well-defined with $u_n \in B_{C_* \rho}(x_+)$. $\qquad\square$

We close this section with two auxiliary lemmas which are used to prove our main theorem.

Lemma 4.10. *Let $f_T : [0, T] \to [0, \infty)$, $T > 0$, be a family of convex integrable functions and let $I : [0, \infty) \to [0, \infty)$ be a function such that*

$$f_T(0) \leq C_f I(T) \quad and \quad \int_0^T f_T(t) \, \mathrm{d}t \leq I(T).$$

Then, for any step size sequence $(h_j)_{0 \leq j \leq n-1}$ satisfying (21), we have

$$\sum_{j=0}^{n-1} h_j f_{t_n}(t_j) \leq C_I I(t_n), \qquad C_I = 1 + c_0 C_f + c_h. \tag{34}$$

We stress the fact that the constant C_I is independent of the particular choice of the step size sequence.
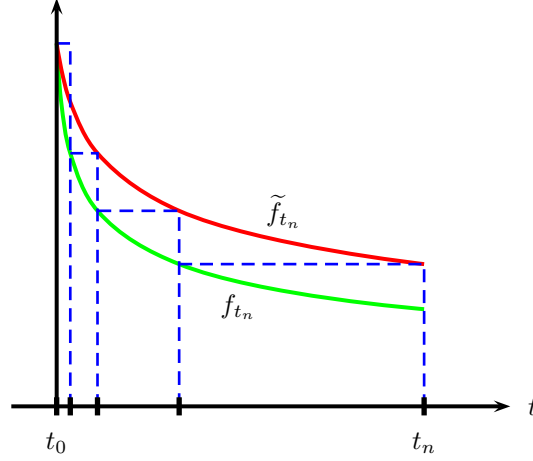
Figure 1. Illustration of the construction in the proof of Lemma 4.10.

Proof. Let T^* denote the point in $[0, t_n]$ at which f_{t_n} achieves its minimum. Then f_{t_n} is monotonically decreasing on $[0, T^*]$. For $0 \le t \le (1 + c_h)T^*$ we define the auxiliary function

$$\widetilde{f}_{t_n}(t) = f_{t_n}\left(\tfrac{1}{1+c_h}t\right).$$

This function is constructed such that on the interval $[0, T^*]$, any Riemann sum of f_{t_n} is a lower sum if the step size sequence satisfies (21), see Fig. 1 for an illustration. Let $1 \le j^* \le n$ be such that $t_{j^*-1} < T^* \le t_{j^*}$. By construction we have

$$\int_0^{T^*} f_{t_n}(t)\, \mathrm{d}t \le \sum_{j=0}^{j^*-1} h_j f_{t_n}(t_j)$$

$$\le h_0 f_{t_n}(0) + \int_0^{(1+c_h)T^*} \widetilde{f}_{t_n}(t)\, \mathrm{d}t$$

$$= h_0 f_{t_n}(0) + (1+c_h)\int_0^{T^*} f_{t_n}(t)\, \mathrm{d}t.$$

For $t \ge T^*$, the Riemann sum is in fact a lower sum for the integral so that

$$\sum_{j=j^*}^{n-1} h_j f_{t_n}(t_j) \;\le\; \int_{T^*}^{t_n} f_{t_n}(t)\, \mathrm{d}t.$$

Due to the assumptions this finally yields (34). ☐

Lemma 4.11. *Let $0 < \gamma \le 1/2$ and let S_n be given by (30) and assume that (21) holds. Then, for $\alpha = \frac{1}{2}, 1, \frac{1}{2} - \gamma$, there is a constant $C_9 = C_9(\alpha, \gamma, c_0, c_h)$ such that*

$$S_n\left(\alpha, 2\gamma + \tfrac{1}{2}\right) \le C_9 \frac{1}{(1 + t_n)^{\alpha + \gamma - 1/2}}. \tag{35}$$

Proof. The bound follows from Lemma 4.10 for the function

$$f_{t_n} = (1 + t_n - t)^{-\alpha}(1 + t)^{-\beta}, \qquad \beta = 2\gamma + \frac{1}{2}.$$

This function satisfies all the assumptions of Lemma 4.10 with

$$I(t) = I\left(t, \alpha, 2\gamma + \frac{1}{2}\right) = c_{\alpha,\gamma} \frac{1}{(1+t)^{\alpha+\gamma-1/2}}, \qquad \alpha = \frac{1}{2}, \ 1, \ \frac{1}{2} - \gamma,$$

for some constants $c_{\alpha,\gamma}$ and $C_f = 1/c_{\alpha,\gamma}$, see [15, Proposition 6]. $\square$

It remains to prove our main theorem.

Proof of Theorem 4.2. We write (26) for $n = n_*$ in the form

$$e_{n_*} = J_+^\gamma v_* + \sum_{j=0}^{n_*-1} h_j \exp(-(t_{n_*} - t_{j+1})J_+)\Phi_{j,+}A_+^*(y^\delta - y)$$

where

$$v_* = \exp(-t_{n_*}J_+)w + \sum_{j=0}^{n_*-1} h_j \exp(-(t_{n_*} - t_{j+1})J_+)J_+^{-\gamma}\left(\Phi_{j,+}A_+^* r_j^{(1)} - r_j^{(2)}\right).$$

Note that v_* is well defined as $r_j^{(2)} \in \mathcal{R}(A_+^*)$, see (25), and since

$$J_+^{-\gamma}A_+^* : \mathcal{N}(A_+^*)^\perp \to X$$

is a bounded operator for $\gamma \le \frac{1}{2}$.

Using (25) and Lemmas 4.7 and 4.11, we bound v_* by

$$\begin{aligned}
\|v_*\| &\le \rho + C_*^2\rho^2(2C_1 + C_5)S_{n_*}\left(\frac{1}{2} - \gamma, 2\gamma + \frac{1}{2}\right) \\
&\le C_{10}\rho
\end{aligned}$$

with

$$C_{10} = 1 + C_*^2\rho(2C_1 + C_5)C_9.$$

Next using the telescopic identity (29)

$$\begin{aligned}
A_+ J_+^\gamma v_* &= A_+ e_{n_*} - \sum_{j=0}^{n_*-1} h_j \exp(-(t_{n_*} - t_{j+1})K_+)K_+\widetilde{\Phi}_{j,+}(y^\delta - y) \\
&= A_+ e_{n_*} + (\exp(-t_{n_*}K_+) - I)(y^\delta - y)
\end{aligned}$$

implies

$$\begin{aligned}
\left\|J_+^{\gamma+1/2}v_*\right\| = \left\|A_+ J_+^\gamma v_*\right\| &\le \|A_+ e_{n_*}\| + \delta \\
&\le (1 + \eta)\|F(u_{n_*}) - F(x_+)\| + \delta \\
&\le (1 + \eta)\left(\left\|\Delta F_{n_*}^\delta\right\| + \delta\right) + \delta \\
&\le C_{11}\delta
\end{aligned}$$

with

$$C_{11} = (1 + \eta)(1 + \tau) + 1$$

by (11) and the discrepancy principle (8).

With the help of the moment inequality, we obtain

$$\left\| J_+^\gamma v_* \right\| \leq \left\| J_+^{\gamma+1/2} v_* \right\|^{2\gamma/(2\gamma+1)} \|v_*\|^{1/(2\gamma+1)}$$

$$\leq (C_{11}\delta)^{2\gamma/(2\gamma+1)} (C_{10}\rho)^{1/(2\gamma+1)}.$$

Using (28), we obtain

$$\|e_{n_*}\| \leq \left\| J_+^\gamma v_* \right\| + 2\sqrt{1+t_{n_*}}\,\delta. \tag{36}$$

By (33) for $n = n_* - 1$ we further have

$$\delta \leq C_8 \, (1+c_h)^{\gamma+1/2} \frac{\rho}{(1+t_{n_*})^{\gamma+1/2}}$$

so that

$$\sqrt{1+t_{n_*}}\,\delta^{1/(2\gamma+1)} \leq C_8^{1/(2\gamma+1)} (1+c_h)^{1/2}\, \rho^{1/(2\gamma+1)}.$$

Inserting this into (36) and defining

$$C = C_{11}^{2\gamma/(2\gamma+1)} C_{10}^{1/(2\gamma+1)} + 2C_8^{1/(2\gamma+1)}(1+c_h)^{1/2}$$

gives the desired result. $\qquad\square$

5. Concluding Remarks

In this paper we proved that the exponential Euler regularization method converges with optimal rates under suitable assumptions. If the step sizes are chosen according to the discrepancy principle (13), then the method converges without requiring a source condition. If the source condition (cf. Assumption 2) is satisfied, then Theorem 4.2 shows that the rate of convergence is optimal, if the step sizes chosen by (13) do not grow faster than (21). Note that (21) is satisfied if $h_{j+1}/h_j \leq$ const, $j = 0, 1, \ldots$, so that this result appears to be relevant for practical applications. However, if (21) fails to be true, then the results of Theorems 4.2 and 4.9 guarantee that one can switch to any step size sequence satisfying $h_{\min} \leq h_j \leq c_h t_j$ and still gets optimal convergence rates.

It is possible to generalize the results of this paper to other methods, for instance to the closely related Levenberg–Marquardt scheme, where the function φ in (5) is taken as

$$\varphi(z) = (1-z)^{-1}.$$

As noted before, Hanke [3] proved that the convergence results of Section 3 hold without any changes for this method as well. A proof of optimal convergence rates is presented in [6]. The convergence of more general methods will be studied elsewhere.

Further, we would like to mention the recent paper by Jin and Tautenhahn [10], where the authors analyze Newton type methods of the form

$$u_{n+1} = x_0 + h_n\varphi(-h_n J_n)A_n^*\big(\Delta F_n^\delta + A_n(u_n - x_0)\big).$$

In that paper, optimal convergence rates have been proved for a whole class of φ functions, among those the function (7) of the exponential Euler scheme. The techniques used in that paper, however, are quite different from ours and cannot be used to analyze our method (5).

During the revision of the present paper, the new preprint [9] by Jin appeared where optimal convergence rates for the Levenberg–Marquardt scheme for an a priori given step size sequence $h_j = h_0\sigma^j$, $j \geq 0$ are shown.

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