Reliable Computing **10:** 423–435, 2004.

Enclosing Solutions of Linear Complementarity Problems for H-matrices

GÖTZ ALEFELD
Institut für Angewandte Mathematik, Universität Karlsruhe, 76128 Karlsruhe, Germany, e-mail: goetz.alefeld@math.uni-karlsruhe.de

ZHENGYU WANG*
Department of Mathematics, Nanjing University, Nanjing 210093, P.R.China, e-mail: wzhengyu@hotmail.com

and

ZUHE SHEN
Department of Mathematics, Nanjing University, Nanjing 210093, P.R.China, e-mail: shenzuhe@netra.edu.cn

(Received: 23 June 2003; accepted: 27 January 2004)

Abstract. The paper establishes a computational enclosure of the solution of the linear complementarity problem (q, M), where M is assumed to be an H-matrix with a positive main diagonal. A class of problems with interval data, which can arise in approximating the solutions of free boundary problems, is also treated successfully.

1. Introduction

Let $M = (m_{ij}) \in R^{n\times n}$ and $q = (q_i) \in R^n$. The linear complementarity problem, denoted by (q, M), is to compute a vector x such that

$$x \geq 0, \qquad Mx + q \geq 0, \qquad (Mx + q)^T x = 0, \tag{1.1}$$

or to show that no such solution exists. In this paper we consider the problem (q, M), where M is assumed to be an H-matrix with a positive main diagonal. Remember that M is an H-matrix if there is a vector $d = (d_i)$ with positive components d_i such that

$$\sum_{j\neq i} |m_{ij}|d_j < |m_{ii}|d_i, \qquad i = 1, 2, \dots, n. \tag{1.2}$$

Define the comparison matrix $\bar{M} = (\bar{m}_{ij})$, where

$$\bar{m}_{ij} = \begin{cases} |m_{ii}| & \text{if } i = j, \\ -|m_{ij}| & \text{if } i \neq j. \end{cases}$$

* The article was completed during the second author's stay in the Institut für Angewandte Mathematik, Universität Karlsruhe.

Then we can alternatively write the above condition (1.2) as

$$\bar{M}d > 0.$$

It is known that an H-matrix with a positive main diagonal is a P-matrix, which is defined as a matrix with positive principal minors; and M is an H-matrix if and only if $\bar{M}^{-1} \geq 0$. One can find the aforementioned results in Plemmons [11]. From the classic result [12] we know (q, M) has a unique solution for any vector q if M is an H-matrix with a positive main diagonal.

There are various numerical methods for solving linear complementarity problems [5], [9], but very few enclosure methods are studied. In [1], [3] the authors developed the Moore test [8] and applied Miranda's theorem [7], respectively, to the equation

$$\min\{x,\ Mx + q\} = 0, \tag{1.3}$$

where $\min(\cdot,\cdot)$ is the componentwise minimum of two vectors. (1.3) is an equivalent formulation of (1.1) given by Pang in [10]. The two papers both provide sufficient conditions for insuring the existence of solutions of the linear complementarity problem in a given interval, but neither points out how to compute the interval enclosing the solutions. This paper establishes an enclosure of the solution of (q, M), for which the main computational cost is to solve a system of linear equations. Furthermore, we extend the enclosure method to the problem with interval data in the vector q , which can arise in approximating the solutions of the free boundary problems [13]. Recall some necessary notations. Denote the one-dimensional real closed interval by $[x] = [\underline{x}, \overline{x}]$, where $\underline{x} \leq \overline{x}$ are real numbers. Denote the n-dimensional real closed interval by $[x] = ([x])_i$, where each of its components $([x])_i$ is a one-dimensional real closed interval. Also we can write an n-dimensional interval as $[x] = [\underline{x}, \overline{x}]$, where $\underline{x}, \overline{x} \in R^n$ and $\underline{x} \leq \overline{x}$ holds componentwise. We define the midpoint of an interval by $m([x]) = (\underline{x} + \overline{x}) / 2$ and the radius by $r([x]) = (\overline{x} - \underline{x}) / 2$. Refer to [2].

2. Existence Test

We begin with giving an existence test for the solution to the nonlinear complementarity problem NCP(f), i.e, the problem of finding a vector x such that

$$x \geq 0, \qquad f(x) \geq 0, \qquad x^T f(x) = 0,$$

where $f : R^n \to R^n$ is assumed to be continuously differentiable and has an interval extension $f'([x])$ over $[x]$. Define

$$p(x) = \max\{0, x - Df(x)\},$$

where D is a diagonal matrix with positive diagonal elements. We can call D a positive diagonal matrix. It is known that x solves NCP(f) if and only if x is a fixed point of the mapping $p(x)$, i.e., $x = p(x)$, which can also be written as the equation

$$\min\{x, Df(x)\} = 0.$$

We can see that Pang's formula (1.3) is a special case of it.

Introduce an interval operator

$$\max\{0, [x]\} = [\max\{0, \underline{x}\}, \max\{0, \overline{x}\}],$$

where $[x]$ is an n-dimensional interval, and $\max\{0, x\}$ is carried out componentwise. Notice that this interval operator is inclusion monotonic, i.e., the inclusion $[x] \subseteq [y]$ implies $\max\{0, [x]\} \subseteq \max\{0, [y]\}$, and the fact that $r(\max\{0, [x]\}) \leq r([x])$. The following is an interval test for the existence of solutions to the nonlinear complementarity problem NCP(f).

THEOREM 2.1. *Let $[x]$ be an n-dimensional interval, denote by $f'([x])$ an interval extension of f' over $[x]$. If*

$$\Gamma(x, [x], D) := \max\{0,\ x - Df(x) + \big(I - Df'([x])\big)([x] - x)\} \subseteq [x], \qquad (2.1)$$

where $x \in [x]$ is fixed and D is a positive diagonal matrix, then there is a solution x^* to NCP(f) *in $\Gamma(x, [x], D)$. Moreover, if a solution x^* of* NCP(f) *is contained in $[x]$, then $x^* \in \Gamma(x, [x], D)$.*

Proof. For any $y \in [x]$ we have

$$y - Df(y) \in x - Df(x) + \big(I - Df'([x])\big)([x] - x),$$

see [8], so

$$p(y) = \max\{0,\ y - Df(y)\} \in \max\{0,\ x - Df(x) + \big(I - Df'([x])\big)([x] - x)\},$$

i.e., $\Gamma(x, [x], D)$ is an interval extension of the mapping $p(\cdot)$ over $[x]$. Thus the condition (2.1) implies that $p(\cdot)$ maps $[x]$ into itself, from which, using the continuity of $p(\cdot)$, it follows that $p(\cdot)$ has a fixed point $x^* \in [x]$, where x^* is a solution to NCP(f). For any solution x^* of NCP(f) in $[x]$, we can conclude that

$$x^* = p(x^*) \in \max\{0,\ x - Df(x) + \big(I - Df'([x])\big)([x] - x)\},$$

which indicates $x^* \in \Gamma(x, [x], D)$. □

COROLLARY 2.1. *Let $\Gamma(x, [x], D)$ be defined as in (2.1). If $\Gamma(x, [x], D) \cap [x] = \emptyset$, then there is no solution to the problem* NCP(f) *in $[x]$.*

Theorem 2.1 indicates that if we can find an interval $[x]^0$, for which the condition (2.1) holds, then an inclusion monotonic sequence $\{[x]^k\}$ of n-dimensional intervals can be computed, where

$$[x]^{k+1} = \Gamma(x^k, [x]^k, D^k) \cap [x]^k, \qquad k = 0, 1, \ldots,$$

$x^k \in [x]^k$, D^k a positive diagonal matrix. Furthermore, we can guarantee that a solution x^* to NCP(f) is contained in each interval $[x]^k$. A real approximation of x^* can also be automatically given by $x^k \in [x]^k$ with the componentwise error less than $|[x]^k - [x^k, x^k]|$. A common choice of x^k is $x^k = m([x]^k)$. Then the componentwise error is less than or equal to the radius $r([x]^k)$. We apply the above results to linear complementarity problems.

COROLLARY 2.2. *Let $M \in R^{n\times n}$, $q \in R^n$, let $[x]$ be an n-dimensional interval, $x \in [x]$ be fixed and D a positive diagonal matrix. If*

$$\Gamma(x, [x], D) := \max\{0,\ x - D(Mx + q) + (I - DM)([x] - x)\} \subseteq [x], \qquad (2.2)$$

then there is a solution x^* to the linear complementarity problem (q, M) in $\Gamma(x, [x], D)$. Moreover, if a solution x^* of (q, M) is contained in $[x]$, then $x^* \in \Gamma(x, [x], D)$.

We give the following interval iterative algorithm for solving the linear complementarity problem (q, M), where $M = (m_{ij})$ is an H-matrix with a positive main diagonal.

ALGORITHM 2.1. Let $D = \mathrm{diag}(m_{11}^{-1}, m_{22}^{-1}, \ldots, m_{nn}^{-1})$, and [*enclosure*] be an interval in which the unique solution x^* of (q, M) is contained. Compute

$$\begin{cases} [x]^0 &:= [enclosure], \\ [x]^{k+1} &:= [x]^k \cap \max\{0,\ x^k - D(Mx^k + q) + (I - DM)([x]^k - x^k)\}, \end{cases}$$

where $x^k = m([x]^k)$.

Since the solution x^* of (q, M) is contained in [*enclosure*], Algorithm 2.1 will compute a nested sequence $\{[x]^k\}$. Furthermore, we can show that the sequence converges to the point interval $[x^*, x^*]$.

THEOREM 2.2. *Let M be an H-matrix with a positive main diagonal. If the unique solution x^* of (q, M) is contained in* [enclosure], *then Algorithm 2.1 will compute a nested sequence $\{[x]^k\}$, which converges to $[x^*, x^*]$.*

Proof. From Corollary 2.2 we know that Algorithm 2.1 will compute a nested sequence $\{[x]^k\}$ such that $x^* \in [x]^k$ for $k = 0, 1, \ldots$. Considering that

$$[x]^{k+1} \subseteq \max\{0,\ x^k - D(Mx^k + q) + (I - DM)([x]^k - x^k)\},$$

we have

$$\begin{aligned} r([x]^{k+1}) &\leq r\big(\max\{0,\ x^k - D(Mx^k + q) + (I - DM)([x]^k - x^k)\}\big) \\ &\leq r(x^k - D(Mx^k + q) + (I - DM)([x]^k - x^k)) \\ &= r\big((I - DM)([x]^k - x^k)\big). \end{aligned}$$

Since $x^k = m([x]^k)$, we have

$$\begin{aligned} (I - DM)([x]^k - x^k) &= (I - DM)[-r([x]^k), r([x]^k)] \\ &= [-(I - D\bar{M})r([x]^k), (I - D\bar{M})r([x]^k)], \end{aligned}$$

where $I - D\bar{M} \geq 0$. Hence

$$r([x]^{k+1}) \leq (I - D\bar{M})r([x]^k).$$

Since M is an H-matrix, $\rho(I - D\bar{M}) < 1$, see [11]. Consequently, $r([x]^k) \to 0$, and the conclusion holds. □

3. Enclosing Solution

We consider the problem of computing an [*enclosure*] needed in Algorithm 2.1 and in Theorem 2.2. Choose $x = 0$ in $\Gamma(x, [x], D)$ of (2.2), $D = \text{diag}(m_{ii}^{-1})$ and $[x] = [-d, d]$, where $d \geq 0$. Then

$$(I - DM)[-d, d] = [-(I - D\bar{M})d, (I - D\bar{M})d],$$

and we can write (2.2) as

$$\Gamma(x, [x], D) := \max\{0, -Dq + [-(I - D\bar{M})d, (I - D\bar{M})d]\} \subseteq [-d, d]. \quad (3.1)$$

In order to find a vector $d \geq 0$ such that the inclusion (3.1) holds, we need the following results.

THEOREM 3.1. *Let $a, b, c \in R^n$, $a \leq b$ and $c \geq 0$. Then*

$$\max\{0, [a, b]\} \subseteq [-c, c]$$

holds if and only if $b \leq c$.

Proof. Consider the i-th component of the inclusion. If

$$\max\{0, [a_i, b_i]\} \subseteq [-c_i, c_i],$$

then

$$\overline{\max\{0, [a_i, b_i]\}} = \max\{0, b_i\} \leq c_i,$$

and so $b_i \leq \max\{0, b_i\} \leq c_i$. Conversely, if $b_i \leq c_i$, then $\max\{0, b_i\} \leq c_i$ since $c_i \geq 0$. So $\max\{0, [a_i, b_i]\} \subseteq [-c_i, c_i]$ if and only if $b_i \leq c_i$, and the conclusion holds for the n-dimensional case. □

COROLLARY 3.1. *Let $M \in R^{n\times n}$ have a positive main diagonal (not necessarily an H-matrix), and let $d \geq 0$. Then the inclusion (3.1) holds for (q, M) if and only if*

$$\bar{M}d + q \geq 0.$$

Proof. From Theorem 3.1 it follows that the inclusion (3.1) holds if and only if

$$-Dq + (I - D\bar{M})d \leq d,$$

which is equivalent to $D(\bar{M}d + q) \geq 0$, and so the conclusion holds since D is a positive diagonal matrix. □

Assume that M is an H-matrix with a positive main diagonal. From Corollary 3.1, it follows that the unique solution x^* of (q, M) is contained in

$$\max\{0, -Dq + [-(I - \bar{M})d, (I - \bar{M})d]\}$$

or equivalently

$$\max\{0, -Dq + [-d + Du, d - Du]\}, \tag{3.2}$$

where $u = \bar{M}d$, $d \geq 0$ and $\bar{M}d + q \geq 0$. If $u \geq 0$ is given, then we can compute $d = \bar{M}^{-1}u \geq 0$; otherwise, we cannot guarantee that $\bar{M}d = u$ has the nonnegative solution. Hence, to get an enclosure, we have to compute a vector d satisfying the following system of linear inequalities

$$\begin{cases} d \geq 0, \\ \bar{M}d \geq 0, \\ \bar{M}d + q \geq 0. \end{cases} \tag{3.3}$$

We give the following algorithm to compute the enclosure of the solution of the problem (q, M).

ALGORITHM 3.1. For the linear complementarity problem (q, M), where M is assumed to be an H-matrix with a positive main diagonal, choose the positive diagonal matrix D as $D = \mathrm{diag}(m_{ii}^{-1})$, choose $u = (u_i)$, where

$$u_i = \begin{cases} 0 & \text{if } q_i \geq 0, \\ -q_i & \text{if } q_i < 0, \end{cases}$$

and compute the unique solution d of the system of linear equations $\bar{M}d = u$. Then the unique solution of (q, M) is contained in

$$[enclosure] = \max\{0, -Dq + [-d + Du,\ d - Du]\}. \tag{3.4}$$

Let $\check{d}$ satisfy (3.3) with $\bar{M}\check{d} = \check{u}$. Then we can get an enclosure of the type of (3.2)

$$\max\{0, -Dq + [-\check{d} + D\check{u},\ \check{d} - D\check{u}]\}.$$

Let d and u be defined as in Algorithm 3.1. Since $\check{u} \geq 0$ and $\check{u} + q \geq 0$, it is clear that $\check{u} \geq u$. Because $I - D\bar{M} \geq 0$, we have

$$\check{d} - D\check{u} = (I - D\bar{M})\check{d} \geq (I - D\bar{M})d = d - Du,$$

which indicates

$$\max\{0, -Dq + [-d + Du, d - Du]\} \subseteq \max\{0, -Dq + [-\check{d} + D\check{u},\ \check{d} - D\check{u}]\},$$

in other words, (3.4) is the sharpest enclosure of the type of (3.2) under the requirement of (3.3).

4. Numerical Results

In this section we apply Algorithm 3.1 for computing an enclosure to several typical test problems in the literature, and improve the enclosure via Algorithm 2.1 and the iterative algorithm presented in Alefeld et al. [1]. We terminate the iteration when the radius of the interval is not more than 1e−15 componentwise. The algorithms are coded in MATLAB 6.5. The numerical results include the enclosures computed by Algorithm 3.1, and we use the following abbreviations:

NUM1: the number of the iterations of Algorithm 2.1;

NUM2: the number of the iterations of the algorithm of [1];

RAD1: $\|r([x^*]^{(1)})\|_\infty$ for $[x^*]^{(1)}$ computed by the method of this paper;

RAD2: $\|r([x^*]^{(2)})\|_\infty$ for $[x^*]^{(2)}$ computed by the method from [1];

FUN1: $\|\min(x, Mx+q)\|_\infty$, where $x = m([x^*]^{(1)})$;

FUN2: $\|\min(x, Mx+q)\|_\infty$, where $x = m([x^*]^{(2)})$.

EXAMPLE 4.1 (Random test problems). We first apply Algorithm 3.1 to two random test problems studied in Alefeld et al.[1], which have the common matrix M and different column vectors q, where $M = (m_1, m_2, m_3, m_4)$,

$$m_1 = \begin{pmatrix} 1.388713122168711 \\ -4.699766249426920e{-1} \\ 7.370559770214220e{-2} \\ -4.110090461033111e{-1} \end{pmatrix},$$

$$m_2 = \begin{pmatrix} -4.699766249426920e{-1} \\ 1.453401598450949 \\ 3.334909523505895e{-2} \\ -5.175564143615730e{-1} \end{pmatrix},$$

$$m_3 = \begin{pmatrix} 7.370559770214220e{-2} \\ 3.334909523505895e{-2} \\ 6.604515405730874e{-1} \\ -1.651162344083680e{-1} \end{pmatrix},$$

$$m_4 = \begin{pmatrix} -4.110090461033111e{-1} \\ -5.175564143615730e{-1} \\ -1.651162344083680e{-1} \\ 1.477373564900058 \end{pmatrix}.$$

The matrix is diagonally dominant, and so it is an H-matrix. Furthermore, the diagonal elements are positive. For the vectors

$$q = \begin{pmatrix} 9.252128641303051 \\ 2.789538442487311 \\ 9.950524251712144 \\ -3.325681126317601 \end{pmatrix}$$

Table 1.

i	[*enclosure*]
1	[0.00000000000000, 0.00000000000000]
2	[0.00000000000000, 0.00000000000000]
3	[0.00000000000000, 0.00000000000000]
4	[1.00479765662298, 3.49735563125256]

Table 2.

i	[*enclosure*]
1	[0.00000000000000, 1.26839053831666]
2	[0.00000000000000, 0.09849992333873]
3	[1.15283989683645, 3.53837185135689]
4	[0.32032065803092, 3.53173054280243]

Table 3.

Example	NUM1	NUM2	RAD1	RAD2	FUN1	FUN2
1	1	1	0	0	0	0
2	21	14	4.4409e−16	4.8880e−17	0	4.8880e−17

and

$$q = \begin{pmatrix} 8.679035675427925\text{e}-1 \\ 2.692546385763099 \\ -1.549159013124430 \\ -2.845459307376360 \end{pmatrix}$$

respectively, Algorithm 3.1 gives the enclosures for the solutions to the corresponding linear complementarity problems. The results are presented in Table 1 and Table 2, respectively.

In Table 3 a comparison of the new algorithm is performed with the algorithm from [1].

For the second example, if we apply Algorithm 2.1 two times, and start the algorithm of [1] with the interval computed, then after one iteration, an enclosure is computed with the radius less than 1e−15 componentwise.

EXAMPLE 4.2 (Murty [9]).

$$M = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \qquad q = (-1, -1, \ldots, -1)^T.$$

M is an H-matrix with positive diagonal elements. The exact solution of the problem is $x^* = (0, \ldots, 0, 1)^T$. Applying Algorithm 3.1, we can compute the enclosure

$$[enclosure] = \begin{pmatrix} [0, 3^{n-1}] \\ [0, 3^{n-2}] \\ [0, 3^{n-2}] \\ \vdots \\ [0, 3] \\ [1, 1] \end{pmatrix},$$

which is very wide if the dimension of M is large. We do the numerical tests for $n = 5, 10, 20, 50, 100$, and the results show that it is needed just one iteration until the stopping criteria is fulfilled via the algorithm from [1] and also if Algorithm 2.1 is used.

EXAMPLE 4.3 (Journal bearing problem [4]). The following problem can arise in discretizing the free boundary problem for an infinite journal bearing by a finite difference method [6]: $M = (m_{ij})$ is a tridiagonal matrix, where

$$m_{ij} = \begin{cases} -h^3_{i+\frac{1}{2}}, & \text{if } j = i + 1, \\ h^3_{i-\frac{1}{2}} + h^3_{i+\frac{1}{2}}, & \text{if } j = i, \\ -h^3_{i-\frac{1}{2}}, & \text{if } j = i - 1, \\ 0, & \text{otherwise} \end{cases}$$

and $q = (q_i)$, where

$$q_i = \delta(h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}}), \qquad i = 1, 2, \ldots, n.$$

In a common model for the infinitely long cylindrical bearing,

$$h_{i-\frac{1}{2}} = \frac{1 + \varepsilon \cos(\pi(i - \frac{1}{2})\delta)}{\sqrt{\pi}}, \qquad i = 1, 2, \ldots, n + 1,$$

$\delta = \frac{T}{n+1}$, $T = 2$. Following Cryer [6], choose $\varepsilon = 0.8$. For $n = 10$ and $n = 100$ Algorithm 3.1 gives the enclosures presented in Tables 4 and 5.

For the case $n = 100$, the algorithm from [1] does not improve the enclosure. However, starting with the interval which the Algorithm 2.1 has computed after 1000 iterations, the algorithm of [1] computes an enclosure with RAD2 $\leq$ 4.4490e−16 and FUN2 $\leq$ 6.5161e−17 after just one iteration.

Table 4. ($n = 10$)

i	[*enclosure*]
1	[0.00000000000000, 0.15860695902414]
2	[0.00000000000000, 0.36679074313145]
3	[0.00000000000000, 0.73042625644525]
4	[0.00000000000000, 1.56250650372998]
5	[0.29659205265926, 3.08014830457683]
6	[0.00000000000000, 0.00000000000000]
7	[0.00000000000000, 0.00000000000000]
8	[0.00000000000000, 0.00000000000000]
9	[0.00000000000000, 0.00000000000000]
10	[0.00000000000000, 0.00352664824520]

Table 5. ($n = 100$)

i	[*enclosure*]
10	[0.00000000000000, 0.17608386065516]
20	[0.00000000000000, 0.42170736330022]
30	[0.00000000000000, 0.90669935946865]
40	[0.00000000000000, 2.10359042750378]
50	[0.00000000000000, 2.08064053243049]
60	[0.00000000000000, 0.24370748854934]
70	[0.00000000000000, 0.03790090545453]
80	[0.00000000000000, 0.01201508633063]
90	[0.00000000000000, 0.00452851819317]
100	[0.00000000000000, 0.00038814001364]

Table 6.

n	NUM1	NUM2	RAD1	RAD2	FUN1	FUN2
10	137	2	8.8818e−16	0	1.2837e−16	6.2450e−17
100	10836	–	9.9920e−16	1.3833	0	5.5635e−4

EXAMPLE 4.4 (Problems with interval data [13]). In [13] Schäfer develops the validation method of [1] to the linear complementarity problem (q, M), where

$$M = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & \cdots & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \cdots & 0 & -\frac{1}{2} & 1 \end{pmatrix},$$

and q is not known exactly, however is contained in

$$q \in [q] = \frac{1}{2}\begin{pmatrix} h^2 \cdot \left[\frac{1}{2}, 1\right] \cdot [F] + \frac{1}{2}h^3 \cdot [-D, D] - y_0 \\ h^2 \cdot \left[\frac{1}{2}, 1\right] \cdot [F] + \frac{1}{2}h^3 \cdot [-D, D] \\ \vdots \\ h^2 \cdot \left[\frac{1}{2}, 1\right] \cdot [F] + \frac{1}{2}h^3 \cdot [-D, D] \end{pmatrix}.$$

In [13] the details of computing $[q]$ are explained. The problem arises in discretizing a class of free boundary problems by taking account of the discretization error. In [12] an enclosure of the solution is also given

$$[x] = \begin{pmatrix} [0, y_0] \\ \vdots \\ [0, y_0] \\ [0, y_0] \end{pmatrix},$$

by some characterization of the free boundary problem, not via the validation method. Here we consider a modification of Algorithm 3.1 to the problem (q, M), where M is an H-matrix with a positive main diagonal, and $q \in [q]$ is unknown. However, the involved vector $[q]$ is given.

ALGORITHM 4.1. For the linear complementarity problem (q, M), where M is assumed to be an H-matrix with a positive main diagonal and $q \in [q]$ is unknown, choose the positive diagonal matrix $D = \mathrm{diag}(m_{ii}^{-1})$, choose $u = (u_i)$, where

$$u_i = \begin{cases} 0 & \text{if } \underline{q}_i \geq 0, \\ -\underline{q}_i & \text{otherwise.} \end{cases}$$

Solve $\bar{M}d = u$ for d, and set

$$[enclosure] = \max\{0, -D[q] + [-d + Du,\ d - Du]\}. \tag{4.1}$$

It is clear that the vector u chosen in Algorithm 4.1 satisfies $u \geq 0$ and $u \geq -q$ for any $q \in [q]$, so from Corollary 3.2 we know that the unique solution to (q, M) is contained in the enclosure (4.1) although $q \in [q]$ is unknown. We test Algorithm 4.1 for Examples 5.1 and 5.2 in [13].

Extensive numerical results show that the first $\frac{n}{2}$-th components of the enclosure (4.1) are a little wider than $[0, y_0]$, but the remaining components are all sharper than $[0, y_0]$, and the radius decreases rapidly along with the increase of the subscript. It seems very hard to improve the enclosures by the iteration presented in [13].

Our experience shows that the algorithm from [1] works well especially for an interval sufficiently sharp. The more large-scaled the problem is, the sharper

Table 7. ($n = 10$)

i	[*enclosure*]
1	[0.01783542862101, 0.17945394160203]
2	[0.00000000000000, 0.15971337039656]
3	[0.00000000000000, 0.14187794177556]
4	[0.00000000000000, 0.12404251315455]
5	[0.00000000000000, 0.10620708453354]
6	[0.00000000000000, 0.08837165591253]
7	[0.00000000000000, 0.07053622729153]
8	[0.00000000000000, 0.05270079867052]
9	[0.00000000000000, 0.03486537004951]
10	[0.00000000000000, 0.01702994142851]

Table 8. ($n = 300$)

i	[*enclosure*]
30	[0.00000000000000, 0.18005631584958]
60	[0.00000000000000, 0.16012366950035]
90	[0.00000000000000, 0.14019102315112]
120	[0.00000000000000, 0.12025837680189]
150	[0.00000000000000, 0.10032573045266]
180	[0.00000000000000, 0.08039308410343]
210	[0.00000000000000, 0.06046043775420]
240	[0.00000000000000, 0.04052779140496]
270	[0.00000000000000, 0.02059514505573]
300	[0.00000000000000, 0.00066249870650]

the starting interval is required. In practical computation, we prefer to sharpen the enclosure firstly via Algorithm 2.1, and then accelerate the convergence by the algorithm of [1].

References

1. Alefeld, G., Chen, X., and Potra, F.: Validation of Solution to Linear Complementarity Problems, *Numerische Mathematik* **83** (1999), pp. 1–23.
2. Alefeld, G. and Herzberger, J.: *Introduction to Interval Computations*, Academic Press, New York and London, 1983.
3. Alefeld, G. and Shen, Z.: *Miranda Theorem for Linear Complementarity Problems*, Tech. Rept., 2003.
4. Bierlein, J. C.: The Journal Bearing, *Scientific American* **233** (1975), pp. 50–64.
5. Cottle, R. W., Pang, J. S., and Stone, R. E.: *Linear Complementarity Problem*, Academic Press, Boston, 1992.
6. Cryer, C. W.: The Method of Christopherson for Solving Free Boundary Problems for Infinite Journal Bearings by Means of Finite Differences, *Math. of Computation* **25** (1971), pp. 435–443.
7. Miranda, C.: Un Osservazione su un teorema di Brouwer, *Belletino Unione Math. Ital. Ser.* **2** (1940), pp. 3–7.

8. Moore, R. E.: A Test of Existence of Solutions to Nonlinear Systems, *SIAM J. Numer. Anal.* **14** (1977), pp. 611–615.
9. Murty, K. G.: *Linear Complementarity, Linear and Nonlinear Programming*, Helderman, Berlin, 1988.
10. Pang, J. S.: Newton's Method for B-Differentiable Equations, *Mathematics of Operations Research* **15** (1990), pp. 311–341.
11. Plemmons, R. J.: M-matrix Characterizations I: Nonsingular M-matrices, *Linear Algebra and Its Applications* **18** (1977), pp. 175–188.
12. Samelson, H., Thrall, R. M., and Wesler, O.: A Partition Theorem for Euclidean n-Space, *Proceedings of the American Mathematical Society* **9** (1958), pp. 805–807.
13. Schäfer, U.: An Enclosure Method for Free Boundary Problems Based on a Linear Complementarity Problem with Interval Data, *Numer. Funct. Anal. Optim.* **22** (2001), pp. 991–1011.