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On Symmetric Solution Sets

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Abstract

Given an $n \times n$ interval matrix $[A]$ and an interval vector $[b]$ with n components we present an overview on existing results on the solution set S_{sym} of linear systems of equations $Ax = b$ with symmetric matrices $A \in [A]$ and vectors $b \in [b]$. Similarly we consider the set E_{sym} of eigenpairs associated with the symmetric matrices $A \in [A]$. We report on characterizations of S_{sym} by means of inequalities, by means of intersection of sets, and by an approach which is generalizable to more general dependencies of the entries. We also recall two methods for enclosing S_{sym} by means of interval vectors, and we mention a characterization of E_{sym}.

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1. Introduction

With this paper we intend to give an overview on existing results for the symmetric solution set

$$S_{sym} := \{x \in \mathbb{R}^n |\ Ax = b,\ A = A^T \in [A],\ b \in [b]\}, \tag{1}$$

where $[A]$ is a given $n \times n$ interval matrix with $[A] = [A]^T$, and $[b]$ is a given interval vector with n components. This set obviously is a subset of the general solution set

$$S := \{x \in \mathbb{R}^n |\ Ax = b,\ A \in [A],\ b \in [b]\}, \tag{2}$$

where the restriction $A = A^T$ on $A \in \mathbb{R}^{n\times n}$ is not required. Knowing S and S_{sym} is particularly interesting in the following situations:

(a) Assume that one has to solve a linear system $\check{A}x = \check{b}$ on a computer using floating point arithmetic. Due to rounding errors, the computed result $\tilde{x}$ normally will not fulfill $\check{A}\tilde{x} = \check{b}$. If $\Delta A \in \mathbb{R}^{n\times n}$, $\Delta b \in \mathbb{R}^n$are given nonnegative tolerances one may view $\tilde{x}$ as an acceptable solution whenever $\tilde{x} \in S$ with S formed as in (2) with respect to $[A] := \check{A} + [-\Delta A, \Delta A]$, $[b] := \check{b} + [-\Delta b, \Delta b]$; in this case $\tilde{x}$ can be interpreted as *exact* solution of a linear system $\tilde{A}x = \tilde{b}$ with some $\tilde{A} \in [A]$, $\tilde{b} \in [b]$.

(b) In contrast to (a), where the linear system is known we assume now that one has to solve a linear system $Ax = b$ where A, b are not given exactly, but they are known to differ from some $\check{A} \in \mathbb{R}^{n\times n}, \check{b} \in \mathbb{R}^n$ by at most $\Delta A \in \mathbb{R}^{n\times n}$ and $\Delta b \in \mathbb{R}^n$, respectively ($\Delta A$, Δb nonnegative). Then $A \in [A] := \check{A} + [-\Delta A, \Delta A]$, $b \in [b] := \check{b} + [-\Delta b, \Delta b]$. Compute a solution x^* of $\check{A}x = \check{b}$. Since $x^* \in S$ one can accept x^* as a good approximation for the unknown solution of $Ax = b$. This situation can occur due to

– conversion errors (from decimal to binary or vice versa),

– errors in measurements,

– errors in adjusting the technical devices.

As we shall see S and S_{sym} are not so easy to handle. Therefore, enclosures of S and S_{sym} are important. For S such enclosures can be computed by means of interval arithmetic. Since such methods are contained in textbooks like [1], [20], e.g., we will omit them here. They trivially deliver also enclosures for $S_{\text{sym}} \subseteq S$. But there are also methods to enclose S_{sym}without bounding S at the same time. We will study such methods later on. Although we shall concentrate on S_{sym} in this paper we will give a short glance at S in order to work out particularities of S_{sym}. So we start in Sect. 3 with several equivalent statements for $x \in S$ and list some properties of S. In Sect. 4, we characterize the boundary ∂S_{sym} of S_{sym} by means of parts of hyperplanes and quadrics. In Sect. 5, we introduce two methods for *enclosing* S_{sym} and in Sect. 6, we report on the eigenpair set

$$E := \left\{ (x^T, \lambda)^T \in \mathbb{R}^{n+1} \mid Ax = \lambda x,\ x \neq 0,\ A \in [A] \right\} \tag{3}$$

and the symmetric eigenpair set

$$E_{\text{sym}} := \left\{ (x^T, \lambda)^T \in \mathbb{R}^{n+1} \mid Ax = \lambda x,\ x \neq 0,\ A = A^T \in [A] = [A]^T \right\}. \tag{4}$$

It turns out that quadrics are needed in order to describe E and algebraic inequalities of order at most three in order to describe E_{sym}.

2. Notations

In the sequel we denote intervals in square brackets, i.e., $[a] = [\underline{a}, \overline{a}]$, and identify point intervals $[a, a]$ by their unique element omitting the brackets. We assume that the reader is familiar with the elementary rules and basic facts of interval arithmetic as introduced in the first chapters of [1] or [20], e.g. We will write $\mathbb{IR}$, $\mathbb{IR}^n$, $\mathbb{IR}^{m\times n}$ for the set of real compact intervals, interval vectors with n components and $m \times n$ interval matrices, respectively. We apply the notation $[A] = [\underline{A}, \overline{A}] = ([a]_{ij}) = ([\underline{a}_{ij}, \overline{a}_{ij}])$ simultaneously for interval matrices and have a similar notation for interval vectors, real vectors and real matrices. An interval matrix $[A]$ is called regular if each $A \in [A]$ has this property. By $\check{A}$ we denote the midpoint of $[A]$, i.e., $\check{A} := \frac{1}{2}(\underline{A} + \overline{A})$, and by $\text{rad}[A] := \frac{1}{2}(\overline{A} - \underline{A})$ its radius which we wrote as ΔA in Sect. 1. In connection with interval vectors and intervals this notation is used analogously. If T is any bounded subset of $\mathbb{R}^n$ the symbol $[]T$ denotes the interval hull of T, i.e., the smallest enclosure of T by an interval vector. For real matrices $A \in \mathbb{R}^{n\times n}$ we write $\rho(A)$ for the spectral radius of A, and define the absolute value $|A| = (|a_{ij}|) \in \mathbb{R}^{n\times n}$. In addition, we introduce the entrywise defined partial ordering '$\leq$', and proceed similarly with vectors. For intervals [a], [c] we define $|[a]| := \max\{|a| | a \in [a]\}$, $\langle [a] \rangle := \min\{|a| | a \in [a]\}$ and $q([a],[c]) := \max\{|\underline{a} - \underline{c}|, |\overline{a} - \overline{c}|\}$ (= Hausdorff distance). For interval matrices $[A], [B] \in \mathbb{R}^{n\times n}$ we introduce the distance $q([A],[B]) := (q([a]_{ij}, [b]_{ij})) \in \mathbb{R}^{n\times n}$ and the comparison matrix $\langle [A] \rangle = (c_{ij}) \in \mathbb{R}^{n\times n}$ which we define by

$$c_{ij} := \begin{cases} \langle [a]_{ij} \rangle, & \text{if } i = j, \\ |[a]_{ij}|, & \text{if } i \neq j. \end{cases}$$

We call a regular matrix $A \in \mathbb{R}^{n\times n}$ an M matrix if $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$. An $n \times n$ interval matrix $[A]$ is termed an interval M matrix if each element $A \in [A]$ is an M matrix. In particular, an interval M matrix is always regular, and it is an interval H matrix, i.e., it satisfies $\langle [A] \rangle^{-1} \geq 0$.

From interval analysis (cf. [1], e.g.) we use the following result:

Theorem 1. *Let $f(x_1, \ldots, x_n)$ be a rational expression in $x_1, \ldots, x_n$ in which each variable x_i occurs at most once. If $[x]_1, \ldots, [x]_n$ are given intervals then*

$$f([x]_1, \ldots, [x]_n) = \{f(x_1, \ldots, x_n) |\, x_1 \in [x]_1, \ldots, x_n \in [x]_n\},$$

i.e., replacing $x_1, \ldots, x_n$ in f by $[x]_1, \ldots, [x]_n$ and evaluating all appearing arithmetic operations according to their interval arithmetic definition yields to the range of f restricted to $[x]_1 \times [x]_2 \times \ldots \times [x]_n$. □

By I we denote the unit matrix, by O an orthant of $\mathbb{R}^n$ and by O_1 the first orthant, i.e., $O_1 = \{x \in \mathbb{R}^n |\, x_i \geq 0 \text{ for } i = 1, \ldots, n\}$.

3. The Solution Set S

In this section, we collect some well–known results on S. We begin with some equivalent formulations of the statement '$x \in S$' which we prove for completeness.

Theorem 2. *For $[A] \in \mathbb{R}^{n\times n}$ and $[b] \in \mathbb{R}^n$ the following assertions are equivalent*:

(a) $x \in S$;

(b) $|\check{b} - \check{A}x| \leq (\mathrm{rad}[A]) \cdot |x| + \mathrm{rad}[b]$; (Oettli and Prager, 1964 [21])

(c) $\exists D \in \mathbb{R}^{n\times n}: \quad |D| \leq I \wedge \check{A}x - \check{b} = D((\mathrm{rad}[A])|x| + \mathrm{rad}[b])$; (Rohn, 1984 [22])

(d) $[b] \cap [A]x \neq \emptyset$; (Beeck, 1972 [10])

(e) $0 \in [b] - [A]x$; (Beeck, 1972 [10])

(f) $\underline{b}_i - \sum_{j=1}^{n} a_{ij}^{+} x_j \leq 0 \leq \overline{b}_i - \sum_{j=1}^{n} a_{ij}^{-} x_j, \quad i = 1, \ldots, n,$

where a_{ij}^{-}, a_{ij}^{+} are defined by $[a]_{ij} = \begin{cases} [a_{ij}^{-}, a_{ij}^{+}] & \text{if } x_j \geq 0, \\ [a_{ij}^{+}, a_{ij}^{-}] & \text{if } x_j < 0. \end{cases}$ (Hartfiel, 1980 [12])

Proof:

(a) $\Rightarrow$ (b)

Since $x \in S$ there are $A \in [A]$, $b \in [b]$ such that $Ax = b$. Therefore,

$$0 = Ax - b \begin{cases} \geq \check{A}x - (\text{rad}[A])|x| - \check{b} - \text{rad}[b], \\ \leq \check{A}\,x + (\text{rad}[A])|x| - \check{b} + \text{rad}[b], \end{cases}$$

whence $-((\text{rad}[A])|x| + \text{rad}[b]) \leq \check{b} - \check{A}x \leq (\text{rad}[A])|x| + \text{rad}[b]$. This implies (b).

(b) $\Rightarrow$ (c)

is seen immediately.

(c) $\Rightarrow$ (d)

Let D_x be such that $|D_x| = I$ and $D_x x = |x|$. From (c) follows $(\check{A} - D(\text{rad}[A])D_x)x = \check{b} + D(\text{ rad}[b]) \in [b]$. Since $\check{A} - D(\text{rad}[A])D_x \in [A]$, the assertion (d) follows.

(d) $\Rightarrow$ (e)

From the assumption (d) and Theorem 1 we obtain $A \in [A]$, $b \in [b]$ such that $Ax = b$ holds. Hence $0 = b - Ax \in [b] - [A]x$.

(e) $\Rightarrow$ (f)

From (e) and the definition of $a_{ij}^{\pm}$ it follows:

$$\inf([A]x - [b])_i = -\overline{b}_i + \sum_{j=1}^{n} a_{ij}^{-} x_j \leq 0 \leq -\underline{b}_i + \sum_{j=1}^{n} a_{ij}^{+} x_j = \sup([A]x - [b])_i, \quad i = 1, \ldots, n.$$

(f) $\Rightarrow$ (a)

From (f) we obtain:

$$\underline{([A]x)}_i = \sum_{j=1}^{n} a_{ij}^{-} x_j \leq \overline{b}_i, \quad \overline{([A]x)}_i = \sum_{j=1}^{n} a_{ij}^{+} x_j \geq \underline{b}_i.$$

Define $\alpha_{ij}(t) := t a_{ij}^{+} + (1-t) a_{ij}^{-}$, $\beta_i(t) := \sum_{j=1}^{n} \alpha_{ij}(t) x_j$, $t \in [0, 1]$, $i, j = 1, \ldots, n$. Since $\beta_i(t)$ is continuous it assumes for $0 \leq t \leq 1$ all values between $\beta_i(0)$ and $\beta_i(1)$. From $\beta_i(0) \leq \overline{b}_i$, $\underline{b}_i \leq \beta_i(1)$ and $\underline{b}_i \leq \overline{b}_i$ there is some value $t = t_i \in [0, 1]$

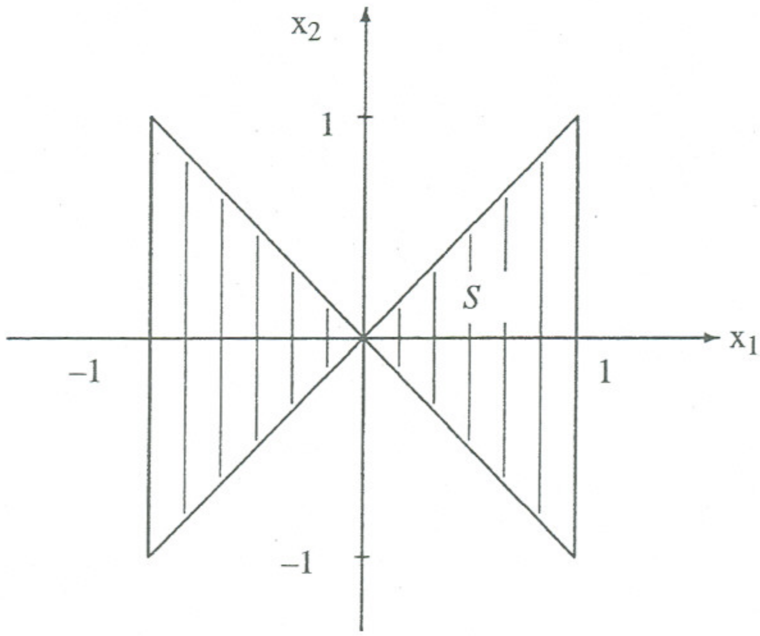

Fig. 1. The solution set S of Example 1

such that $\beta_i(t_i) \in [b]_i$. With $\hat{A} := A(t_1, \ldots, t_n) := (\alpha_{ij}(t_i))$, $\hat{b} := (\beta_i(t_i))$ we get $\hat{A}x = \hat{b}$, $\hat{A} \in [A]$, $\hat{b} \in [b]$, hence $x \in S$. □

Note that the equivalence $(a) \Leftrightarrow (b)$ is the famous Oettli–Prager criterion which was generalized by Fischer and Heindl in [13] (cf. also [14]). In the equivalence $(a) \Leftrightarrow (f)$ the coefficients of the inequalities remain fix as long as one stays in a fixed orthant. Therefore, this equivalence shows that the intersection of S with any fixed closed orthant O is the intersection of finitely many half–spaces. In particular, the boundary ∂S of S is composed of finitely many pieces of hyperplanes. As long as $[A]$ is regular, S is connected and compact which follows from the continuous mapping $f(A, b) = A^{-1}b$ on the connected and compact set $[A] \times [b]$. If $[A]$ contains a singular matrix, connectivity may be lost as Jansson's example $[A] = [-1, 1]$, $[b] = 1$, $S = (-\infty, -1] \cup [1, \infty)$ in [17] shows. On the other hand it can be present as the example $[A] = [-1, 1]$, $[b] = [0, 1]$, $S = \mathbb{R}$ illustrates. Compactness is always absent if $[A]$ is not regular and $S \neq \emptyset$. In this case S is unbounded even if the linear systems with singular matrices $A \in [A]$ are not solvable for any $b \in [b]$. Use Cramer's rule, e.g., to prove this statement. Since half–spaces are convex and intersections of convex sets share this property one obtains it for $S \cap O$. The following example shows, however, that convexity need not hold for the whole solution set S.

Example 1. Let

$$[A] = \begin{pmatrix} 1 & 0 \\ [-1, 1] & 1 \end{pmatrix}, \quad [b] = \begin{pmatrix} [-1, 1] \\ 0 \end{pmatrix}.$$

From Theorem 2, we get the inequalities $|x_2| \leq |x_1| \leq 1$ which characterize the set S completely. Its position in $\mathbb{R}^2$ can be seen from the subsequent Fig. 1.

4. The Symmetric Solution Set S_{sym}

4.1. Historical Remarks and Properties

In contrast to S the symmetric solution set S_{sym} as defined in (1) is much more difficult to characterize. It occurs when the coefficients of a fixed linear system with a symmetric matrix A is perturbed by symmetric matrices only. Since $S_{\mathrm{sym}} \subseteq S$ Neumaier advised in a letter to J. Rohn [23] dated on 23rd December, 1985, to consider this set for its own. In his book [20] which appeared in 1990 he mentioned S_{sym} in Example 3.4.2 . In the same year, Jansson gave a talk in Albena in which he presented inner and outer enclosures for S_{sym} (cf. Sect. 5 of the present paper). For more general dependencies such enclosures were given by Rump in his survey article [24] which appeared in 1994. In 1993 an interval version of the Cholesky method was introduced by the authors in [8] in order to enclose S_{sym}. Criteria of feasibility were proved there and a corresponding perturbation result was given in [9]. This method was recently extended to a block version by Schäfer in [26]. It was proved for the first time in 1995 in [9] that the boundary $\partial S_{\mathrm{sym}}$ can be curvilinear – in contrast to the boundary ∂S of S. The proof covered only 2×2 matrices and could unfortunately not be generalized to the n–dimensional case. By means of the Fourier–Motzkin elimination process of linear programming this was possible in 1996 in [2]; cf. also [3]. These results were even generalized two years later in [4] where now affine dependencies were allowed. In [7] the elimination process introduced in [2] and [4] was generalized in order to handle particular dependencies such as Toeplitz and Hankel matrices. While the approaches always took place in a fixed orthant it was possible in [19] to modify the elimination process by an approach similar to the equivalence (a) $\Leftrightarrow$ (e) in Theorem 2 due to Beeck; cf. Sect. 4.3 of the present paper.

We start by an instructive 2×2 example whose features remain true also in the $n \times n$ case.

Example 2. Let

$$[A] = \begin{pmatrix} 1 & [1,2] \\ [1,2] & [-1,0] \end{pmatrix}, \quad [b] = \begin{pmatrix} 4 \\ [1,2] \end{pmatrix}.$$

Considering $A^{-1}b$ one easily recognizes $S_{\mathrm{sym}} \subseteq S \subseteq O_1$. Anticipating Sect. 4.2, we list the inequalities by which S_{sym} is described:

$$\begin{aligned} -4 + x_1 + x_2 &\le 0, \\ 4 - x_1 - 2x_2 &\le 0, \\ -2 + x_1 - x_2 &\le 0, \\ 1 - 2x_1 &\le 0, \\ x_1^2 - 4x_1 + x_2 &\le 0, \\ x_1^2 + x_2^2 - 4x_1 + 2x_2 &\ge 0. \end{aligned}$$

The first four inequalities characterize S while the last two restrict x to be contained in S_{sym}.

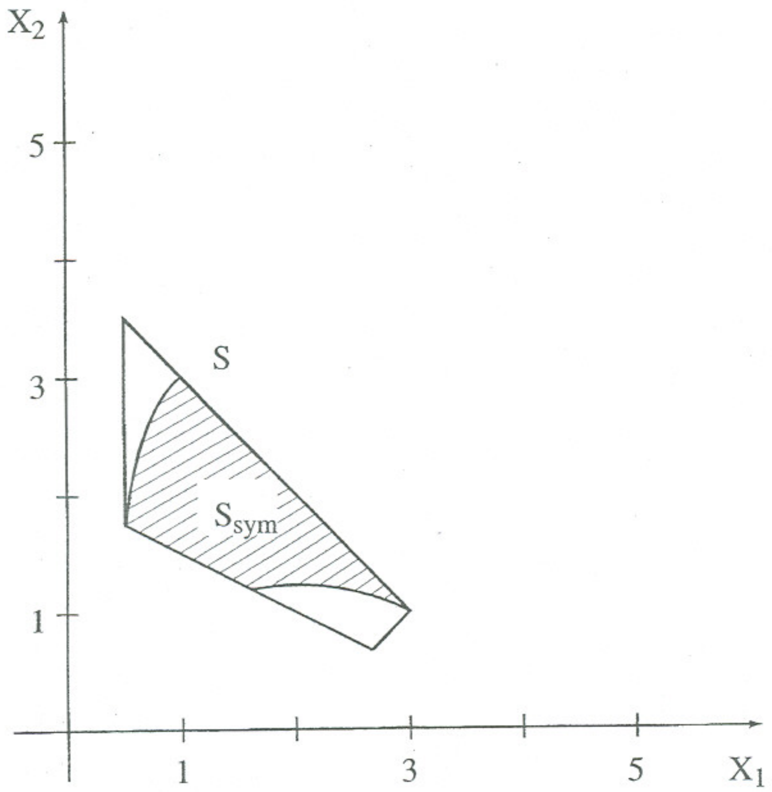

Fig. 2. The solution sets S and S_{sym} of Example 2

From Fig. 2, one sees that S_{sym} is connected and compact. This remains true also in the general case as long as $[A]$ is regular. The counterexamples for non-regular matrices are the same as in Sect. 3. As we shall see (Sects. 4.2 and 4.4) the boundary ∂S_{sym} of S_{sym} is composed by pieces of hyperplanes and quadrics; in particular, ∂S_{sym} can be curvilinear. In general, S_{sym} is not convex, and the same holds true for its intersection with an orthant. Up to now it is unknown whether $S_{sym} \cap O$ is always connected.

4.2. Characterization of S_{sym} by Means of a Fourier–Motzkin Elimination

In this section, we show that $S_{sym} \cap O$ can be characterized by means of quadrics and hyperplanes. To this end we aim at describing $S_{sym} \cap O$ by means of linear and quadratic inequalities whose coefficients consist of the bounds $\underline{b}_i$, $\overline{b}_i$ and $\underline{a}_{ij}$, $\overline{a}_{ij}$. Without loss of generality we choose the orthant $O = O_1$. We only indicate the algorithm by replacing b and a_{12} by $\underline{b}$, $\overline{b}$ and $\underline{a}_{12}$, $\overline{a}_{12}$, respectively. We get the following equivalences:

$$\begin{aligned} x \in S_{sym} \cap O_1 \Leftrightarrow & x \in O_1 \ \wedge \ \exists A = A^T \in [A],\ b \in [b] :\ Ax = b \\ \Leftrightarrow & x \in O_1 \ \wedge \ \exists a_{ij} \in \mathbb{R} \ \text{ for } i,j = 1,\ldots,n : \\ & \left(a_{ij} = a_{ji} \wedge \left\{ \begin{array}{c} \underline{b}_i \le \sum_{j=1}^{n} a_{ij}x_j \le \overline{b}_i \\ \underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij} \end{array} \right\} \right) \\ \Leftrightarrow & x \in S \cap O_1 \ \wedge \ \exists a_{ij} \in \mathbb{R} \ \textit{for } i,j = 1,\ldots,n : \end{aligned}$$

$$\left(a_{ij} = a_{ji} \wedge \left\{ \begin{array}{c} \underline{b}_i x_i \le \sum_{j=1}^{n} a_{ij} x_i x_j \le \overline{b}_i x_i \\ \underline{a}_{ij} x_i x_j \le a_{ij} x_i x_j \le \overline{a}_{ij} x_i x_j \end{array} \right\}\right). \tag{5}$$

Only the implication '$\Leftarrow$' of the last assertion requires a proof since $x_i = 0$ or $x_j = 0$ is possible. Let $A = A^T$ be the matrix whose entries fulfill, by assumption, the inequalities in (5). If $x_k \neq 0,\ x_l \neq 0$ then obviously $a_{kl} \in [a]_{kl}$. Otherwise nothing can be said on a_{kl} in this respect. From $x \in S \cap O_1$ we get $\hat{a}_{ij} \in [a]_{ij}$ such that $\sum_{j=1}^{n} \hat{a}_{ij} x_j = b_i \in [b]_i$. Define $\tilde{A}$ by

$$\tilde{a}_{kl} := \begin{cases} a_{kl} & \text{if } x_k \neq 0,\ x_l \neq 0\ , \\ \hat{a}_{kl} & \text{if } x_k = 0 \text{ and } x_l \neq 0\ , \\ \hat{a}_{lk} & \text{if } x_k \neq 0 \text{ and } x_l = 0, \\ \check{a}_{kl} & \text{if } x_k = 0 \text{ and } x_l = 0. \end{cases}$$

Since we assumed $[A] = [A]^T$ when considering S_{sym}, the midpoint $\check{A}$ of $[A]$ is symmetric, and the definition of $\tilde{A}$ shows that $\tilde{A}$ has this property, too. Moreover we get $\tilde{A} \in [A]$ and $\underline{b} \le \tilde{A}x \le \overline{b}$ for the vector x under consideration. Note that

(a) whenever a_{ij} occurs in (5) it has the same factor $x_i x_j$;

(b) trivial inequalities like $0 \le 0$ can be omitted;

(c) there are only linear inequalities (cf. Theorem 2 (f)) and quadratic ones;

(d) no additional multiplication will be needed in the further stages of the algorithm.

We next isolate the a_{12}–term:

$$x \in S_{\text{sym}} \cap O_1 \Leftrightarrow x \in S \cap O_1 \ \wedge\ \exists a_{ij} \in \mathbb{R} \text{ for } i, j = 1, \ldots, n :$$

$$\left(a_{ij} = a_{ji} \ \wedge \left\{ \begin{array}{l} \{\underline{b}_1 - \sum_{\substack{j=1 \\ j\neq 2}}^{n} a_{1j} x_j\} x_1 \le a_{12} x_1 x_2 \\ \qquad\qquad \le \{\overline{b}_1 - \sum_{\substack{j=1 \\ j\neq 2}}^{n} a_{1j} x_j\} x_1 \\ \{\underline{b}_2 - \sum_{j=2}^{n} a_{2j} x_j\} x_2 \le a_{12} x_1 x_2 \\ \qquad\qquad \le \{\overline{b}_2 - \sum_{j=2}^{n} a_{2j} x_j\} x_2 \\ \underline{a}_{12} x_1 x_2 \le a_{12} x_1 x_2 \le \overline{a}_{12} x_1 x_2 \\ \text{remaining} \quad a_{12} - \text{free inequalities} \end{array} \right\}\right). \tag{6}$$

Now those inequalities written down explicitly in (6) hold if and only if the maximum of their left–hand sides is less or equal than the minimum of their right–hand sides and this again holds if and only if each of these left–hand sides is less or equal than each of the right–hand sides. Therefore we get:

$$x \in S_{\text{sym}} \cap O_1 \Leftrightarrow x \in S \cap O_1 \wedge \exists a_{ij} \in \mathbb{R} \text{ for } i, j = 1, \ldots, n,\ (i,j) \notin \{(1,2),(2,1)\} :$$

$$\left(a_{ij} = a_{ji} \ \wedge \left\{ \begin{array}{c} \{\underline{b}_1 - \sum\limits_{\substack{j=1 \\ j \neq 2}}^{n} a_{1j}x_j\}x_1 \leq \overline{a}_{12}x_1x_2 \\ \{\underline{b}_2 - \sum\limits_{j=2}^{n} a_{2j}x_j\}x_2 \leq \overline{a}_{12}x_1x_2 \\ \{\underline{b}_1 - \sum\limits_{\substack{j=1 \\ j \neq 2}}^{n} a_{1j}x_j\}x_1 \leq \{\overline{b}_2 - \sum\limits_{j=2}^{n} a_{2j}x_j\}x_2 \\ \underline{a}_{12}x_1x_2 \leq \{\overline{b}_1 - \sum\limits_{\substack{j=1 \\ j \neq 2}}^{n} a_{1j}x_j\}x_1 \\ \underline{a}_{12}x_1x_2 \leq \{\overline{b}_2 - \sum\limits_{j=2}^{n} a_{2j}x_j\}x_2 \\ \{\underline{b}_2 - \sum\limits_{j=2}^{n} a_{2j}x_j\}x_2 \leq \{\overline{b}_1 - \sum\limits_{\substack{j=1 \\ j \neq 2}}^{n} a_{1j}x_j\}x_1 \\ \text{remaining } a_{12} - \text{free inequalities} \end{array} \right\} \right).$$

No a_{12}-term appears now any longer in the set of inqualities, and the isolation and elimination process can be repeated for the other entries a_{ij}. At the end we are left with a set of inequalities as indicated above. This process which was given in a slightly modified form in [2] must be repeated for any of the 2^n orthants, so the amount of inequalities can increase tremendously with n although sometimes inequalities can be combined and turn out to be contained in other ones.

We end this section with another example whose inequalities were derived by the method above.

Example 3. Let

$$[A] := \begin{pmatrix} 1 & [0,1] \\ [0,1] & [-4,-1] \end{pmatrix}, \quad [b] := \begin{pmatrix} [0,2] \\ [0,2] \end{pmatrix}.$$

Then $S \cap O_1$ is characterized by

$$0 \leq x_1 \leq 2, \quad 0 \leq x_2 \leq x_1,$$

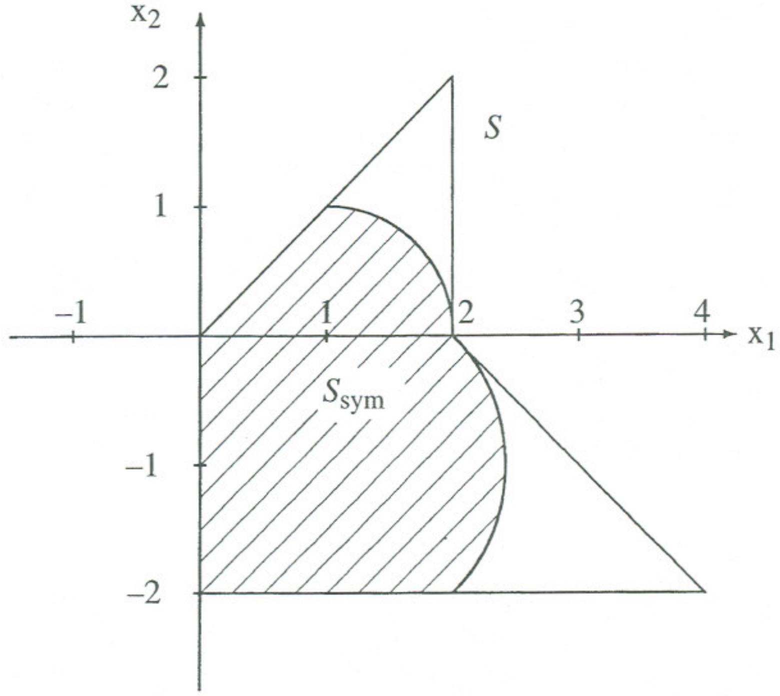

Fig. 3. The solution sets S and S_{sym} of Example 3

while $S \cap O_4$ with $O_4 := \{(x_1, x_2)^T \in \mathbb{R}^2 |\ x_1 \geq 0,\ x_2 \leq 0\}$ can be described by

$$-2 \leq x_2 \leq 0, \quad 0 \leq x_1 \leq 2 - x_2.$$

Note that we combined some of the inequalities omitting redundant ones and taking into account the respective orthant. For S_{sym} the following *additional* inequalities are needed

$$\begin{cases} (x_1 - 1)^2 + x_2^2 \leq 1 & \text{in the case } S_{\text{sym}} \cap O_1, \\ (x_1 - 1)^2 + (x_2 + 1)^2 \leq 2 & \text{in the case } S_{\text{sym}} \cap O_4, \end{cases}$$

where we omitted again redundant ones. Figure 3 shows S and S_{sym}.

4.3 Characterization of S_{sym} by Means of Sets

In this section, we do not stick to a fixed orthant but describe S_{sym} as a whole. This decreases the amount of relations drastically. For the algorithm we need a simple lemma which we formulate first.

Lemma 1.

(a) *Let* $[a]_i \in \mathbb{IR}$ *for* $i = 1, \ldots, n,\ n \geq 3$. *Then*

$$\bigcap_{i=1}^{n} [a]_i \neq \emptyset \ \Leftrightarrow\ [a]_i \cap [a]_j \neq \emptyset \quad \textit{for } i < j,\ i, j = 1, \ldots, n.$$

(b) *Let* $[a],\ [b],\ [c] \in \mathbb{IR}$. *Then*

$$([a] + [b]) \cap [c] \neq \emptyset \ \Leftrightarrow\ [a] \cap ([c] - [b]) \neq \emptyset. \qquad \square$$

The proof is easy. Like the whole approach in this section it can be found in [19]. As the referee pointed out part (a) of the lemma is the simplest case of Helly's theorem on intersections of convex sets; cf., e.g., Theorem 12 in [18], p. 166. He also mentioned that part (b) is often used in the equivalent form

$$([a] - [b]) \cap [c] \neq \emptyset \;\Leftrightarrow\; ([a] - [c]) \cap [b] \neq \emptyset.$$

Now we show how to replace the parameters $a_{ij},\ b_i$ by intersections involving $[a]_{ij}$, $[b]_j$.

$$x \in S_{\text{sym}} \Leftrightarrow \exists\, a_{ij} \in \mathbb{R} \text{ for } i,j = 1,\ldots,n:$$

$$\left(a_{ij} = a_{ji} \;\wedge\; \left\{ \begin{array}{c} \left\{ \sum_{j=1}^{n} a_{ij}x_j \right\} \cap [b]_i \neq \emptyset \\ \\ \{a_{ij}\} \cap [a]_{ij} \neq \emptyset \end{array} \right\} \right)$$

$$\Leftrightarrow x \in S \;\wedge\; \exists\, a_{ij} \in \mathbb{R} \text{ for } i,j = 1,\ldots,n:$$

$$\left(a_{ij} = a_{ji} \;\wedge\; \left\{ \begin{array}{c} \left\{ \sum_{j=1}^{n} a_{ij}x_{ij} \right\} \cap [b]_i x_i \neq \emptyset \\ \\ \{a_{ij}x_ix_j\} \cap [a]_{ij} x_i x_j \neq \emptyset \end{array} \right\} \right)$$

$$\Leftrightarrow x \in S \;\wedge\; \exists\, a_{ij} \in \mathbb{R} \text{ for } i,j = 1,\ldots,n:$$

$$\left(a_{ij} = a_{ji} \;\wedge\; \left\{ \begin{array}{c} \{a_{12}x_1x_2\} \cap \Big([b]_1 x_1 - \sum_{\substack{j=1 \\ j\neq 2}}^{n} a_{1j}x_1x_j\Big) \neq \emptyset \\ \\ \{a_{12}x_1x_2\} \cap \Big([b]_2 x_2 - \sum_{j=2}^{n} a_{2j}x_2x_j\Big) \neq \emptyset \\ \\ \{a_{12}x_1x_2\} \cap [a]_{12}x_1x_2 \neq \emptyset \\ \\ \text{remaining } a_{12} - \text{free intersections} \end{array} \right\} \right)$$

$$\Leftrightarrow x \in S \;\wedge\; \exists\, a_{ij} \in \mathbb{R} \text{ for } i,j = 1,\ldots,n,\ (i,j) \notin \{(1,2),(2,1)\}:$$

$$\left(a_{ij} = a_{ji} \wedge \left\{ \begin{array}{c} [a]_{12}x_1x_2 \cap \Big([b]_1 x_1 - \sum_{\substack{j=1 \\ j\neq 2}}^{n} a_{1j}x_1x_j\Big) \\ \\ \cap \Big([b]_2 x_2 - \sum_{j=2}^{n} a_{2j}x_2x_j\Big) \neq \emptyset \\ \\ \text{remaining} \quad a_{12} - -\text{free intersections} \end{array} \right\} \right).$$

The multiplication by x_i and x_j, respectively, is used as in Sect. 4.2. No additional multiplication is needed in the further course of the algorithm. In order to continue it is necessary to resolve the triple intersection by intersections with two operands. The key to this is given by Lemma 1 . We leave the details to the reader. The quadrics are seen here only at a second glance. They are hidden behind the intersections. However, since these intersections can be expressed by inequalities the approach in this section can be reformulated by formulas as in Sect. 4.2. Both approaches lack, however, a generalization to more than dependencies as they occur in symmetric ($a_{ij} = a_{ji}$), skew–symmetric ($a_{ij} = -a_{ji}$), and persymmetric matrices ($a_{ij} = a_{n+1-j,n+1-i}$), respectively. In order to generalize the elimination process in Sect. 4.2 we need a modification presented in Sect. 4.4.

We conclude the present section by the general 3×3 case.

Example 4. Let $[A] = [A]^T \in \mathrm{I}\mathbb{R}^{3\times 3}$, $[b] \in \mathrm{I}\mathbb{R}^3$ and consider again S_{sym}. Eliminating b_i, a_{ii} for $i = 1, 2, 3$ and a_{12} leads to

$$x \in S_{\mathrm{sym}} \Leftrightarrow x \in S \ \wedge \ \exists\ a_{13},\ a_{23} \in \mathbb{R} : \Bigg(a_{13} = a_{31} \ \wedge \ a_{23} = a_{32}$$

$$\wedge \left\{ \begin{array}{rl} [a]_{12} x_1 x_2 & \cap \ ([b]_1 x_1 - [a]_{11} x_1^2 - a_{13} x_1 x_3) \\ & \cap \ ([b]_2 x_2 - [a]_{22} x_2^2 - a_{23} x_2 x_3) \neq \emptyset \\ \{a_{13} x_1 x_3\} & \cap \ ([b]_3 x_3 - a_{23} x_2 x_3 - [a]_{33} x_3^2) \neq \emptyset \\ \{a_{13} x_1 x_3\} & \cap \ [a]_{13} x_1 x_3 \neq \emptyset \\ \{a_{23} x_2 x_3\} & \cap \ [a]_{23} x_2 x_3 \neq \emptyset \end{array} \right\} \Bigg). \tag{7}$$

After having resolved the triple intersection in (7) into simple intersections the isolation of a_{13} and its elimination yield to

$$x \in S_{\mathrm{sym}} \Leftrightarrow \ x \in S \ \wedge \ \exists \ \ a_{23} \in \mathbb{R} : \ \Bigg(\ a_{23} = a_{32}$$

$$\wedge \left\{ \begin{array}{rl} [a]_{13} x_1 x_3 & \cap \ ([b]_1 x_1 - [a]_{11} x_1^2 - [a]_{12} x_1 x_2) \\ & \cap \ ([b]_3 x_3 - a_{23} x_2 x_3 - [a]_{33} x_3^2) \\ & \cap \ ([b]_1 x_1 - [b]_2 x_2 - [a]_{11} x_1^2 + [a]_{22} x_2^2 + a_{23} x_2 x_3) \neq \emptyset \\ \{a_{23} x_2 x_3\} & \cap \ ([b]_2 x_2 - [a]_{12} x_1 x_2 - [a]_{22} x_2^2) \neq \emptyset \\ \{a_{23} x_2 x_3\} & \cap \ [a]_{23} x_2 x_3 \neq \emptyset \end{array} \right\} \Bigg).$$

Here again we omitted an intersection which can be deduced from $x \in S$. Eliminating the last parameter a_{23} the fourfold intersection produces $\binom{4}{2}$ simple intersections among which only one is a_{23}–free. This one turns out to be a consequence of $x \in S$. Therefore, it can be omitted, and we are finally left with the equivalence

$$\begin{aligned}
x \in S_{\text{sym}} \;\Leftrightarrow\; & x \in S \\
\wedge\; [a]_{23}x_2x_3 \;\cap\; & ([b]_2x_2 - [a]_{12}x_1x_2 - [a]_{22}x_2^2) \\
\cap\; & ([b]_3x_3 - [a]_{13}x_1x_3 - [a]_{33}x_3^2) \\
\cap\; & (-[b]_1x_1 + [b]_2x_2 + [a]_{11}x_1^2 - [a]_{22}x_2^2 + [a]_{13}x_1x_3) \\
\cap\; & (-[b]_1x_1 + [b]_2x_2 + [b]_3x_3 + [a]_{11}x_1^2 - [a]_{22}x_2^2 - [a]_{33}x_3^2)/2 \\
\cap\; & (-[b]_1x_1 + [b]_3x_3 + [a]_{11}x_1^2 + [a]_{12}x_1x_2 - [a]_{33}x_3^2) \\
\cap\; & \{([b]_1 - [b]_1)x_1 + [b]_2x_2 + ([a]_{11} - [a]_{11})x_1^2 \\
& - [a]_{12}x_1x_2 - [a]_{22}x_2^2\} \\
\neq\; & \emptyset.
\end{aligned}$$

Using Lemma 1 (a) this multiple intersection can equivalently be written as $\binom{7}{2} = 21$ simple ones. Two of them can be deduced from the requiry $x \in S$. The remaining 19 ones together with the three occuring from Beeck's criterion in Theorem 2 (d) result in a total of 22 which all can be rewritten in the form "$0 \in \ldots$". Introducing the interval bounds, fixing an orthant O and taking into account

$$0 \in [a] \Leftrightarrow \underline{a} \leq 0 \leq \overline{a}$$

finally yields to 44 inequalities for the characterization of $S_{\text{sym}} \cap O$ as was already mentioned without proof in [2].

4.4. A Generalizable Approach for Characterizing S_{sym}

The subsequent modification of Sect. 4.2 is based on the following theorem which is proved in [7].

Theorem 3. *Let $f_{\lambda\mu}$, g_λ, $\lambda = 1, \ldots, k$ (≥ 2), $\mu = 1, \ldots, m$, be real valued functions of $x = (x_1, \ldots, x_n)^T$ on some subset $D \subseteq \mathbb{R}^n$. Assume that there is a positive integer $k_1 < k$ such that*

$$f_{\lambda 1}(x) \not\equiv 0 \text{ for all } \lambda \in \{1, \ldots, k\}, \tag{8}$$

$$f_{\lambda 1}(x) \geq 0 \text{ for all } x \in D \text{ and all } \lambda \in \{1, \ldots, k\}, \tag{9}$$

$$\begin{aligned}&\text{for each } x \in D \text{ there is an index } \beta^* = \beta^*(x) \in \{1, \ldots, k_1\} \text{ with } f_{\beta^* 1}(x) > 0 \\ &\text{and an index } \gamma^* = \gamma^*(x) \in \{k_1 + 1, \ldots, k\} \text{ with } f_{\gamma^* 1}(x) > 0.\end{aligned} \tag{10}$$

For m parameters $u_1, \ldots, u_m$ varying in $\mathbb{R}$ and for x varying in D define the sets S_1, S_2 by

$$S_1 := \{x \in D | \exists u_\mu \in \mathbb{R},\ \ \mu = 1, \ldots, m:\ (11),\ (12)\ hold\},$$
$$S_2 := \{x \in D | \exists u_\mu \in \mathbb{R},\ \ \mu = 2, \ldots, m:\ (13)\ holds\},$$

where the inequalities (11), (12) and (13), respectively, are given by

$$g_\beta(x) + \sum_{\mu=2}^{m} f_{\beta\mu}(x)u_\mu \leq f_{\beta 1}(x)u_1, \quad \beta = 1, \ldots, k_1, \tag{11}$$

$$f_{\gamma 1}(x)u_1 \leq g_\gamma(x) + \sum_{\mu=2}^{m} f_{\gamma\mu}(x)u_\mu, \quad \gamma = k_1 + 1, \ldots, k, \tag{12}$$

and

$$g_\beta(x)f_{\gamma 1}(x) + \sum_{\mu=2}^{m} f_{\beta\mu}(x)f_{\gamma 1}(x)u_\mu \leq g_\gamma(x)f_{\beta 1}(x) + \sum_{\mu=2}^{m} f_{\gamma\mu}(x)f_{\beta 1}(x)u_\mu, \tag{13}$$
$$\beta = 1, \ldots, k_1,\ \gamma = k_1 + 1, \ldots, k.$$

(Trivial inequalities such as $0 \leq 0$ *can be omitted.) Then*

$$S_1 = S_2. \qquad \square$$

The assertion of Theorem 3 even holds if $f_{\beta 1}(x)$, $f_{\gamma 1}(x)$ are replaced in (13) (but not in (11), (12)) by $\tilde{f}_{\beta 1}(x)$ and $\tilde{f}_{\gamma 1}(x)$, respectively, where

$$f_{\beta 1}(x) = h_{\beta\gamma}(x)\tilde{f}_{\beta 1}(x), \quad f_{\gamma 1}(x) = h_{\beta\gamma}(x)\tilde{f}_{\gamma 1}(x)$$

with nonnegative functions $\tilde{f}_{\beta 1}$, $\tilde{f}_{\gamma 1}$, $h_{\beta\gamma}$ defined on D.

In contrast to the procedure in Sect. 4.2 there is no need in Theorem 3 to find factors such that the expressions to be eliminated have the same shape in *all* inequalities after multiplication. This allows the application of this theorem also for more general dependencies of the entries of A and b, respectively. We apply now the theorem to symmetric matrices with $D := O_1$ starting as in Sect. 4.2. Just for illustration, we choose a_{12} as the entry to be eliminated.

$$x \in S_{\text{sym}} \cap O_1 \Leftrightarrow x \in O_1 \ \wedge\ \exists\ a_{ij} \in \mathbb{R} \text{ for } i, j = 1, \ldots, n:$$
$$\left(a_{ij} = a_{ji} \wedge \left\{ \begin{array}{c} \underline{b}_i \leq \sum\limits_{j=1}^{n} a_{ij}x_j \leq \overline{b}_i \\ \underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij} \end{array} \right\} \right)$$

$$\Leftrightarrow x \in O_1 \ \wedge \ \exists\, a_{ij} \in \mathbb{R} \text{ for } i,j = 1,\ldots,n:$$

$$\left(a_{ij} = a_{ji} \ \wedge \ \left\{ \begin{array}{rl} \underline{b}_1 - \sum\limits_{\substack{j=1\\ j\neq 2}}^{n} a_{1j}x_j \leq & a_{12}x_2 \\ & a_{12}x_2 \leq \overline{b}_1 - \sum\limits_{\substack{j=1\\ j\neq 2}}^{n} a_{1j}x_j \\ \underline{b}_2 - \sum\limits_{j=2}^{n} a_{2j}x_j \leq & a_{12}x_1 \\ & a_{12}x_1 \leq \overline{b}_2 - \sum\limits_{j=2}^{n} a_{2j}x_j \\ \underline{a}_{12} \leq & a_{12} \\ & a_{12} \leq \overline{a}_{12} \\ \multicolumn{2}{c}{\text{remaining } a_{12} - \text{free inequalities}} \end{array} \right\} \right)$$

$$\Leftrightarrow x \in O_1 \ \wedge \ \exists\, a_{ij} \in \mathbb{R} \text{ for } i,j = 1,\ldots,n, \ (i,j) \notin \{(1,2),(2,1)\}:$$

$$\left(a_{ij} = a_{ji} \ \wedge \ \left\{ \begin{array}{c} \underline{b}_1 - \sum\limits_{\substack{j=1\\ j\neq 2}}^{n} a_{1j}x_j \leq \overline{a}_{12}x_2, \\ \underline{a}_{12}x_2 \leq \overline{b}_1 - \sum\limits_{\substack{j=1\\ j\neq 2}}^{n} a_{1j}x_j, \\ \underline{b}_2 - \sum\limits_{j=2}^{n} a_{2j}x_j \leq \overline{a}_{12}x_1, \\ \underline{a}_{12}x_1 \leq \overline{b}_2 - \sum\limits_{j=2}^{n} a_{2j}x_j, \\ \underline{b}_1x_1 - \sum\limits_{\substack{j=1\\ j\neq 2}}^{n} a_{1j}x_1x_j \leq \overline{b}_2x_2 - \sum\limits_{j=2}^{n} a_{2j}x_2x_j, \\ \underline{b}_2x_2 - \sum\limits_{j=2}^{n} a_{2j}x_2x_j \leq \overline{b}_1x_1 - \sum\limits_{\substack{j=1\\ j\neq 2}}^{n} a_{1j}x_1x_j, \\ \text{remaining } a_{12} - \text{free inequalities} \end{array} \right\} \right)$$

The first four inequalities coincide with those in the elimination process for $S \cap O_1$. The next two are new; they are apparently caused by the symmetry and contain quadratic polynomials. When eliminating a_{ij} for $(i,j) \notin \{(1,2),(2,1)\}$

according to the remark below Theorem 3, no additional multiplication is needed in inequalities which contain quadratic polynomials. This is true because the function $f_{\lambda\mu}$ in front of a_{ij} reads $f_{\lambda\mu}(x) = x_i x_j$ in these inequalities, and in the remaining (non–quadratic) inequalities they are given by $f_{\lambda\mu}(x) = x_i$, $f_{\lambda\mu}(x) = x_j$ and $f_{\lambda\mu}(x) = 1$, respectively. Note that the sign of these functions remains constant over the orthant O_1. This is the reason, why no splitting is needed for $D = O_1$ during the elimination process. Pursuing this process shows that the final inequalities for $S_{\text{sym}} \cap O_1$ consist of the inequalities which characterize $S \cap O_1$, and quadratic inequalities.

5. Enclosures for S_{sym}

As the preceding sections show it is generally not easy to decide whether a given vector x belongs to S_{sym} for given $[A] = [A]^T \in \mathbb{IR}^{n\times n}$, $[b] \in \mathbb{R}$. Therefore, it is interesting to look for bounds of S_{sym}. We restrict ourselves to bounds which are not at the same time bounds of S, i.e., we look for bounds which should be better than the latter ones. Up to now there are essentially two methods with this property: a direct one and an iterative one. The first generalizes the well–known Cholesky method and uses the definitions

$$[a]^2 := \{a^2 | a \in [a]\}, \quad \sqrt{[a]} := [a]^{\frac{1}{2}} := \{\sqrt{a} | a \in [a]\},$$

where $\underline{a} \geq 0$ is assumed for the square root. The algorithm which was first considered in [8] reads

Step 1: "LL^T–decomposition"

$$\begin{aligned}
&\text{for } j := 1 \text{ to } n \text{ do} \\
&[l]_{jj} := \left([a]_{jj} - \sum_{k=1}^{j-1} [l]_{jk}^2 \right)^{1/2}; \\
&\text{for } i := j+1 \text{ to } n \text{ do} \\
&[l]_{ij} := \left([a]_{ij} - \sum_{k=1}^{j-1} [l]_{ik}[l]_{jk} \right) / [l]_{jj};
\end{aligned}$$

Step 2: Forward substitution‘

$$\begin{aligned}
&\text{for } i := 1 \text{ to } n \text{ do} \\
&[y]_i := \left([b]_i - \sum_{j=1}^{i-1} [l]_{ij}[y]_j \right) / [l]_{ii};
\end{aligned}$$

Step 3: Backward substitution

$$\begin{aligned}
&\text{for } i := n \text{ downto } 1 \text{ do} \\
&[x]_i^C := \left([y]_i - \sum_{j=i+1}^{n} [l]_{ji}[x]_j^C \right) / [l]_{ii}.
\end{aligned}$$

It is easy to see by the inclusion monotonicity of the interval arithmetic operations that $S_{\text{sym}} \subseteq [x]^C$ holds. One also sees directly that the feasibility of the preceding interval Cholesky method does not depend on $[b]$ and requires that each matrix $A \in [A]$ is symmetric and positive definite. Unfortunately, this property is not sufficient. As was shown in [8] the feasibility is guaranteed if $[A]$ is a symmetric H matrix satisfying $\underline{a}_{ii} > 0$ for $i = 1, \ldots, n$. Interval matrices $[A]$ with a strictly diagonally dominant comparison matrix $\langle [A] \rangle$ and $\underline{a}_{ii} > 0$, $i = 1, \ldots, n$, belong to this class. Moreover, if the algorithm is feasible for $[A]$ and if $\rho(|[A]^C| q([A], [B])) < 1$ then it is feasible for $[B]$, too, as was proved in [9]. Here, $|[A]^C| := \langle [L]^T \rangle^{-1} \langle [L] \rangle^{-1}$ where $[L] = ([l]_{ij})$ is the lower triangular interval matrix whose non zero entries $[l]_{ij}$, $i \geq j$, are defined by the algorithm above. Replacing $[A]$, $[B]$ by $\check{A}$, $[A]$, respectively, proves that the interval Cholesky method is feasible whenever $\check{A}$ is positive definite and satisfies $\rho(|\check{A}^C| \operatorname{rad}([A])) < 1$. The following example shows that the interval Cholesky method is tailored to enclose S_{sym}, but not S. However, it does not always yield to enclosures for S_{sym} which are better than general enclosures for S. In particular, for larger n it may show a bad behavior due to rounding errors and data dependencies.

Example 5. Let

$$[A] := \begin{pmatrix} 4 & [-1, 1] \\ [-1, 1] & 4 \end{pmatrix}, \quad [b] := \begin{pmatrix} 6 \\ 6 \end{pmatrix}.$$

Setting $A := \begin{pmatrix} 4 & \alpha \\ \beta & 4 \end{pmatrix}$ for $A \in [A]$, we get

$$A^{-1} b = \frac{6}{16 - \alpha\beta} \begin{pmatrix} 4 - \alpha \\ 4 - \beta \end{pmatrix} \text{ with } \alpha, \beta \in [-1, 1].$$

Since $\beta = \alpha$ in the case $A = A^T \in [A]$ one obtains

$$\begin{aligned} \square S_{sym} &= ([\tfrac{18}{15}, 2], [\tfrac{18}{15}, 2])^T, & \square S &= ([\tfrac{18}{17}, 2], [\tfrac{18}{17}, 2])^T, \\ [x]^C &= ([1, 2], [\tfrac{18}{16}, 2])^T, & [x]^G &= ([1, 2], [\tfrac{18}{17}, 2])^T, \end{aligned}$$

where $[x]^G$ denotes the vector resulting from the interval Gaussian algorithm (cf. [1]). The sets

$$S_{sym} = \left\{ \frac{6}{4 + \alpha} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \middle| -1 \leq \alpha \leq 1 \right\} = \left\{ \gamma \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \middle| \frac{6}{5} \leq \gamma \leq 2 \right\}$$

and (see [12])

$$S = \text{convex hull} \left(\{ (\frac{6}{5}, \frac{6}{5})^T, (2, 2)^T, (\frac{18}{17}, \frac{30}{17})^T, (\frac{30}{17}, \frac{18}{17})^T \} \right)$$

can be seen in Fig. 4.

Example 5 illustrates that the following properties can occur.

(i) $\Box S_{sym} \neq \Box S$ (cf. also [20], p. 95),

(ii) $\Box S_{sym} \neq [x]^C$,

(iii) $\Box S \neq [x]^G$,

(iv) $S \not\subseteq [x]^C$ (but $S_{sym} \subseteq [x]^C$),

(v) $[x]^C \subseteq [x]^G$ *with* $[x]^C \neq [x]^G$.

Unfortunately, $[x]^G \subseteq [x]^C$ with $[x]^G \neq [x]^C$ can also occur as was shown by an example in [8].

The second method which we want to recall was introduced by Jansson [15], [16] and is a modification of the following well–known iterative method for general linear interval systems

$$[x]^{k+1} = \tilde{x} + C([b] - [A]\tilde{x}) + (I - C[A])([x]^k - \tilde{x}), \quad k = 0, 1, \ldots,$$

where $C \in \mathbb{R}^{n \times n}$ is any preconditioning matrix and $\tilde{x}$ is any fixed vector from $\mathbb{R}^n$. Usually, C is an approximate inverse of the midpoint $\check{A}$ and $\tilde{x}$ is an approximate solution of the midpoint equation $\check{A}x = \check{b}$. Exploiting the symmetry of $A \in [A]$ and of $[A] = [A]^T$ and substituting $[x]_\Delta^k := [x]^k - \tilde{x}$ finally yields to

$$[x]_\Delta^{k+1} = [z]^{\mathrm{sym}} + (I - C[A])[x]_\Delta^k, \quad k = 0, 1, \ldots \tag{14}$$

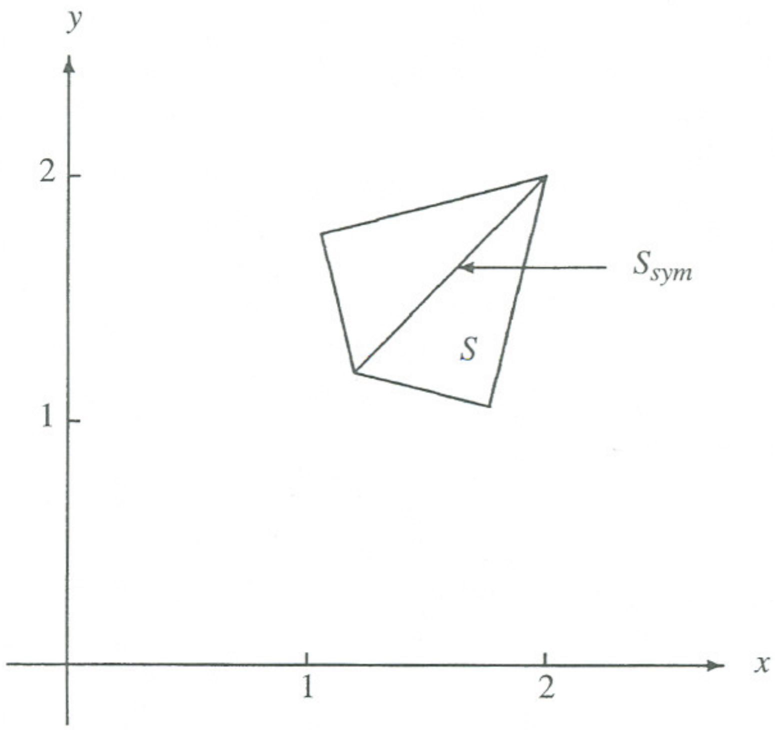

Fig. 4. The solution sets S and S_{sym} of Example 5

with

$$[z]_i^{\text{sym}} := \sum_{j=1}^{n} c_{ij}([b]_j - [a]_{jj}\tilde{x}_j) - \sum_{j=1}^{n}\sum_{l=1}^{j-1}(c_{ij}\tilde{x}_l + c_{il}\tilde{x}_j)[a]_{jl}$$
$$= \{(C(b - A\tilde{x}))_i | A = A^T \in [A], b \in [b]\} \quad \text{for } i = 1, \ldots, n.$$

Note that the last equality holds by virtue of Theorem 1 . Thus $[z]^{\text{sym}}$ is the interval hull of the set $\{C(b - A\tilde{x}) | A = A^T \in [A],\ b \in [b]\}$ and is therefore optimal. The subsequent results were proved in [15], [16].

Theorem 4. *Let $(S_{\text{sym}})_i$ denote the projection of S_{sym} onto the x_i coordinate axis and let $[\Delta]^k := (I - C[A])[x]_\Delta^k$. If*

$$([x]_\Delta^{k_0+1})_i \subsetneq ([x]_\Delta^{k_0})_i \quad \textit{for } i = 1, \ldots, n$$

then the following assertions hold.

(a) *$[A]$ and C are regular.*

(b) $S_{\text{sym}} \subseteq \tilde{x} + [x]_\Delta^{k+1} \subseteq \tilde{x} + [x]_\Delta^k$ *for $k \geq k_0$.*

(c) $\tilde{x}_i + \underline{z}_i^{\text{sym}} + \underline{\Delta}_i^{k_0} \leq \min(S_{\text{sym}})_i \leq \tilde{x}_i + \underline{z}_i^{\text{sym}} + \overline{\Delta}_i^{k_0}$,

$\tilde{x}_i + \overline{z}_i^{\text{sym}} + \underline{\Delta}_i^{k_0} \leq \max(S_{\text{sym}})_i \leq \tilde{x}_i + \overline{z}_i^{\text{sym}} + \overline{\Delta}_i^{k_0}$. □

By Theorem 4 one sees at once that the relations

$$\inf S_{\text{sym}} \in \tilde{x} + \underline{z}^{\text{sym}} + [\Delta]^{k_0},$$
$$\sup S_{\text{sym}} \in \tilde{x} + \overline{z}^{\text{sym}} + [\Delta]^{k_0}$$

hold. Therefore, if $\text{rad}[\Delta]^{k_0}$ is small then the enclosure for S_{sym} is good. This can be expected, in particular, if $\text{rad}[A], \text{rad}[b]$ are small, $C \approx \check{A}^{-1}$ and $\tilde{x} \approx \check{A}^{-1}\check{b}$. Then $\|\text{rad}[x]_\Delta^{k_0}\|_\infty \ll 1$ is possible whence $[\Delta]^{k_0}$ is quadratically small.

6. The Symmetric Eigenpair Set

Our final section is devoted to the eigenpair sets E and E_{sym} from (3) and (4), respectively. In order to characterize E we apply the Oettli–Prager theorem to the equation $(A - \lambda I)x = 0$ which leads to

$$(x^T, \lambda)^T \in E \Leftrightarrow |(\check{A} - \lambda I)x| \leq (\text{rad}[A])|x| \wedge x \neq 0. \tag{15}$$

There is a slight difficulty in comparison with Sect. 3 in so far as the matrix $A - \lambda I$ is not fixed but depends on the last component of the vector $(x^T, \lambda)^T$ tested to belong to E. This difficulty can be overcome thinking λ to be arbitrary but fixed.

Then $A - \lambda I$ is a fixed matrix for which Theorem 2 can be applied without any restrictions. Rewriting (15) as in Theorem 2(f) by means of inequalities shows that $E \cap O$ can be characterized by a variety of inequalities which – by virtue of the λx term are at most quadratic. That means that E is the union of finitely many intersections of sets whose boundaries are pieces of hyperplanes and quadrics. We mention that the inequalities for E can also be obtained by applying Theorem 3 . Using any approach of Sect. 4 shows that the symmetric eigenpair set E_{sym} can be described by means of inequalities with polynomials of order three at most. The starting point is now formed by the double inequality

$$(A - \lambda I)x \le 0 \le (A - \lambda I)x$$

which is equivalent to $(A - \lambda I)x = 0$. For details we refer to [6]. We conclude our paper with an illustrative example for E and E_{sym}.

Example 6. Let

$$[A] = \begin{pmatrix} 1 & [-1,1] \\ [-1,1] & 1 \end{pmatrix}.$$

Then E is characterized by

$$x_1 \cdot x_2 \ne 0 \ \wedge \ \begin{cases} x_1 - |x_2| \le \lambda x_1 \le x_1 + |x_2| \\ -|x_1| + x_2 \le \lambda x_2 \le |x_1| + x_2 \end{cases}.$$

If $\lambda = 1$ then any vector $(x_1, x_2)^T \ne 0$ occurs as an eigenvector hence $E \subseteq \mathbb{R}^3$ contains the plane $\lambda = 1$ punctured at $(0,0,1)^T$. If $\lambda \ne 1$ then $x_1 \cdot x_2 \ne 0$ and $(x^T, \lambda)^T \in E$ if and only if the subsequent statement holds.

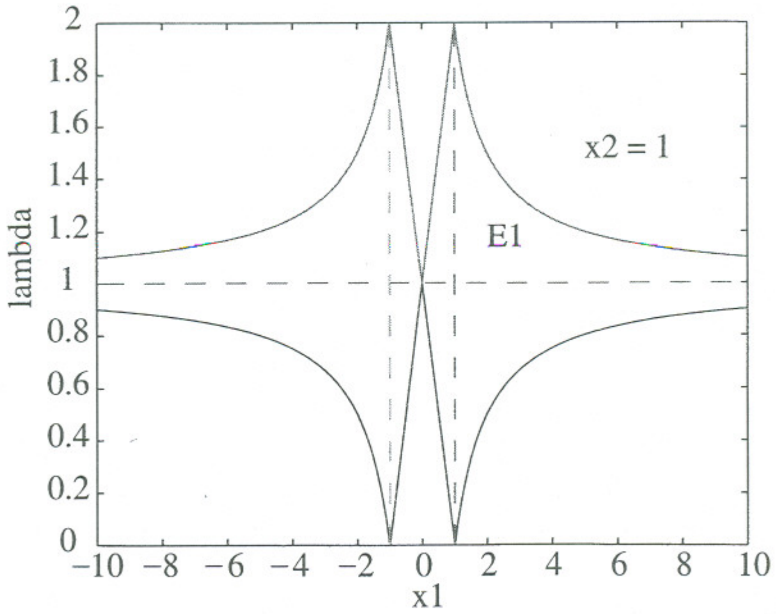

Fig. 5. $E_1 := E \cap P_1$ and $E_{\mathrm{sym}} \cap P_1$ (dashed) of Example 6

$$\lambda \neq 1 \wedge x_1 \cdot x_2 \neq 0 \wedge \begin{cases} 1 - \left|\frac{x_2}{x_1}\right| \leq \lambda \leq 1 + \left|\frac{x_2}{x_1}\right| \\ 1 - \left|\frac{x_1}{x_2}\right| \leq \lambda \leq 1 + \left|\frac{x_1}{x_2}\right| \end{cases}.$$

Restricting $A \in [A]$ to be symmetric yields to any pair $(x_1, x_2)^T \neq 0$ as eigenvector in the case $\lambda = 1$. This is true by virtue of $I = I^T \in [A]$. If $\lambda \neq 1$ then E_{sym} is described by $|x_1| = |x_2| > 0$, $0 \leq \lambda \leq 2$, $\lambda \neq 1$, which together with E leads to Fig. 5 in which we intersected these sets with the plane $P_1 : x_2 = 1$.

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