Modifications of the Oettli–Prager Theorem with Application to the Eigenvalue Problem

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Dedicated to Prof. Dr. Jürgen Herzberger on the occasion of his 60th birthday.

1 Introduction

In this paper we consider the eigenpair set

$$E_{\mathcal{P}} := \{ (x, \lambda) \mid Ax = \lambda x,\ x \neq 0,\ A \in [A],\ A \text{ with property } \mathcal{P} \}, \tag{1}$$

where $[A]$ is a given real $n \times n$ interval matrix (cf. Alefeld and Herzberger (1983), e.g., for interval analysis) and $\mathcal{P}$ is some fixed property such as symmetry, Toeplitz form, etc.. Before we study this set in greater detail we mention other ones which are related to it: When dealing with systems of linear equations

$$\check{A}x = \check{b}, \quad \check{A} \in \mathbb{R}^{n\times n},\ \check{b} \in \mathbb{R}^n \tag{2}$$

($\mathbb{R}^{n\times n}$ set of real $n \times n$ matrices, $\mathbb{R}^n$ set of real vectors with n components) there sometimes occurs the problem of varying the input data $\check{A}$, $\check{b}$ within certain tolerances and looking for the set S of the resulting solutions x^*. Examples of this problem are Wilkinson's backward analysis when solving linear systems on a computer (Wilkinson 1963) and an input–output model in economics which is regulated by (2) with input parameters $\check{A}$, $\check{b}$ and output x (Maier 1985). In the first example one solves (2) on a computer (assuming $\check{A}$ to be nonsingular). Due to rounding errors one normally does not obtain the exact solution x^* but another vector $\tilde{x}$. One accepts $\tilde{x}$ as a good approximation of the exact solution x^* if it can be interpreted as a solution of a nearby system $Ax = b$, where 'nearby' means $|A-\check{A}| \leq \Delta$, $|b-\check{b}| \leq \delta$ with given tolerances $O \leq \Delta \in \mathbb{R}^{n\times n}$, $0 \leq \delta \in \mathbb{R}^n$. (Here and in the sequel, the absolute value $|\cdot|$ and the inequality sign '$\leq$' are understood entrywise.) In other words, one considers $\tilde{x}$ as a good approximation for x^* if and only if it belongs to the *solution set*

$$S := \{ x \in \mathbb{R}^n \mid Ax = b,\ A \in \mathbb{R}^{n\times n},\ b \in \mathbb{R}^n,\ |A - \check{A}| \leq \Delta,\ |b - \check{b}| \leq \delta \} \tag{3}$$

where $\check{A}$, Δ, $\check{b}$, δ are given. Instead of writing $|A - \check{A}| \leq \Delta A$ one often prefers the shorter notation $A \in [A]$ where the bracketed letter denotes the real $n \times n$ interval matrix

$$[A] := [\check{A} - \Delta, \check{A} + \Delta] = [\underline{A}, \overline{A}] = ([a]_{ij}) = ([\underline{a}_{ij}, \overline{a}_{ij}]).$$

Similarly, $|b - \check{b}| \leq \delta$ is replaced by $b \in [b]$ with the interval vector

$$[b] := [\check{b} - \delta, \check{b} + \delta] = [\underline{b}, \overline{b}] = ([b]_i) = ([\underline{b}_i, \overline{b}_i]).$$

The second example – which we called input–output model – can be viewed under two aspects: Firstly, one can vary directly the original input data $\check{A}$, $\check{b}$ within given tolerances Δ, δ and consider all solutions outgrowing from the modified systems $Ax = b$ with $A \in [A]$, $b \in [b]$. This means that we look again for the solution set S in (3). Secondly, one can measure the output $x = \tilde{x}$ and ask whether it can be expected to be generated by input data A, b 'nearby' $\check{A}$ and $\check{b}$. Again one is led to the problem '$\tilde{x} \in S$?' this time having a sort of inverse problem as in Wilkinson's backward analysis.

A description of S was given in the sixties by Oettli and Prager (1964) in the form

$$x \in S \;\Leftrightarrow\; |\check{b} - \check{A}x| \leq \Delta \cdot |x| + \delta \tag{4}$$

which is known in the literature as Oettli–Prager Theorem and which was reformulated since then several times (see Theorem 1 below).

The solution set S covers *all* matrices from $[A]$. Often the matrix $\check{A}$ in (2) exhibits some particular structure, which should be kept when considering the perturbed systems $Ax = b$; cf. for instance Jansson (1991a,b), Rump (1994) or Alefeld and Mayer (1995). This leads to the modified solution set $S_{\mathcal{P}}$ with some given property $\mathcal{P}$ for A, i.e.,

$$S_{\mathcal{P}} := \{\, x \in \mathbb{R}^n \mid Ax = b,\; A \in [A],\; b \in [b],\; A \text{ with property } \mathcal{P} \,\} \subseteq S. \tag{5}$$

We will present a way how to describe $S_{\mathcal{P}}$ by means of inequalities which involve the bounds $\underline{A}$, $\overline{A}$, $\underline{b}$, $\overline{b}$ of the tolerance intervals. If there are no restrictions on $A \in [A]$ it will turn out that these inequalities reduce to (4). This justifies the title of our paper.

We now come back to the eigenvalue problem

$$Ax = \lambda x, \quad x \neq 0, \tag{6}$$

which we restrict to λ being real. This restriction is not substantial but simplifies matters. When perturbing $A \in \mathbb{R}^{n\times n}$ such that $A \in [A]$ is allowed we are led to the *eigenpair set*

$$E := \{\, (x, \lambda) \in \mathbb{R}^{n+1} \mid Ax = \lambda x,\; x \neq 0,\; A \in [A] \,\}. \tag{7}$$

If we are interested in matrices $A \in [A]$ sharing some property $\mathcal{P}$ we end up with the set $E_{\mathcal{P}}$ which we defined in (1) at the beginning of this section. In order to describe E and $E_{\mathcal{P}}$, respectively, the Oettli–Prager Theorem and its modifications will play a crucial role. As for S and $S_{\mathcal{P}}$ the Fourier–Motzkin elimination process of linear programming forms the basis. We will show that in each orthant of $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$, respectively, the boundary of the eigenpair set E as well as the *symmetric solution set*

$$S_{\text{sym}} := \{\, x \in \mathbb{R}^n \mid Ax = b,\ A = A^T \in [A],\ b \in [b] \,\} \subseteq S \tag{8}$$

can be described by means of hyperplanes and quadrics. For the *symmetric eigenpair set*

$$E_{\text{sym}} := \{\, (x, \lambda) \in \mathbb{R}^{n+1} \mid Ax = \lambda x,\ x \neq 0,\ A = A^T \in [A] \,\} \tag{9}$$

one has to enlarge this variety of geometric objects by algebraic surfaces of order 3. If $\mathcal{P}$ means A being skew–symmetric ($A = -A^T$, i.e., $a_{ij} = -a_{ji}$) or persymmetric (A symmetric with respect to the counter–diagonal, i.e., $a_{ij} = a_{n+1-j,n+1-i}$) the boundaries of the corresponding solution sets can be described by the same kind of objects. If one admits more general dependencies in the entries such as A being a Toeplitz matrix (A has constant values along each of its diagonals, i.e., $a_{ij} = c_{j-i}$ with some constants c_k, $k = -(n-1), \ldots, n-1$) or a Hankel matrix (A has constant values along each of its counter–diagonals, i.e., $a_{ij} = c_{i+j-2}$ with some constants c_k, $k = 0, \ldots, 2n-2$) details are more complicated. In these cases – as for general linear dependencies – one can only show that the boundary of $S_{\mathcal{P}}$ and $E_{\mathcal{P}}$, respectively, can be described by means of algebraic equations whose order is unknown up to now (Alefeld et al. 1998).

We have arranged our paper as follows: In Section 2 we shortly describe the Fourier–Motzkin elimination process, in Section 3 we consider the solution set $S_{\mathcal{P}}$ and in Section 4 we study the eigenpair set $E_{\mathcal{P}}$.

2 The Fourier–Motzkin Elimination Process

The Fourier–Motzkin elimination process eliminates parameters in inequalities. We will shortly describe the principle by executing one step when deriving the set of inequalities for $S_{\text{sym}} \cap O_1$. Here, O_1 denotes the closed first orthant of $\mathbb{R}^n$, i.e., $O_1 := \{\, x \in \mathbb{R}^n \mid x \geq 0 \,\}$. For $x \in O_1$ one starts with the trivial equivalences

$$\begin{aligned}
x \in S_{\text{sym}} \Leftrightarrow\ & \exists\, A = A^T \in [A],\ b \in [b] \,:\, Ax = b \\
\Leftrightarrow\ & \exists\, a_{ij} \in \mathbb{R} \quad (i,j = 1, \ldots, n) \,: \\
& \underline{b}_i \leq \sum_{j=1}^{n} a_{ij} x_j \leq \overline{b}_i, \quad \underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij}, \quad a_{ij} = a_{ji} \\
\Leftrightarrow\ & \exists\, a_{ij} \in \mathbb{R} \quad (i,j = 1, \ldots, n) \,: \\
& \underline{b}_1 - \sum_{\substack{j=1 \\ j \neq 2}}^{n} a_{1j} x_j \leq a_{12} x_2 \leq \overline{b}_1 - \sum_{\substack{j=1 \\ j \neq 2}}^{n} a_{1j} x_j, \\
& \underline{b}_2 - \sum_{j=2}^{n} a_{2j} x_j \leq a_{12} x_1 \leq \overline{b}_2 - \sum_{j=2}^{n} a_{2j} x_j, \\
& \underline{a}_{12} \leq a_{12} = a_{21} \leq \overline{a}_{12}, \\
& \text{inequalities without } a_{12},\ a_{21}.
\end{aligned}$$

If $x_1 > 0$, $x_2 > 0$ this is equivalent to

$$\begin{aligned}
&\exists\, a_{ij} \in \mathbb{R} \quad (i,j = 1, \ldots, n) : \\
&\{\underline{b}_1 - \sum_{\substack{j=1\\ j\neq 2}}^{n} a_{1j}x_j\}/x_2 \le a_{12} \le \{\overline{b}_1 - \sum_{\substack{j=1\\ j\neq 2}}^{n} a_{1j}x_j\}/x_2, \\
&\{\underline{b}_2 - \sum_{j=2}^{n} a_{2j}x_j\}/x_1 \le a_{12} \le \{\overline{b}_2 - \sum_{j=2}^{n} a_{2j}x_j\}/x_1, \\
&\underline{a}_{12} \le a_{12} = a_{21} \le \overline{a}_{12}, \\
&\text{inequalities without } \ a_{12},\ a_{21}.
\end{aligned}$$

Here, the first three double inequalities hold if and only if the *maximum* of the first three left–hand sides is less or equal than the *minimum* of the first three right–hand sides. This, however, is true if and only if *each* of the three left–hand sides is less or equal than *each* of the three right–hand sides which results in the following equivalent inequalities:

$$\begin{aligned}
&\exists\, a_{ij} \in \mathbb{R} \quad (i,j = 1, \ldots, n,\ (i,j) \neq (1,2),\ (i,j) \neq (2,1)\,) : \\
&\{\underline{b}_1 - \sum_{\substack{j=1\\ j\neq 2}}^{n} a_{1j}x_j\}/x_2 \le \{\overline{b}_2 - \sum_{j=2}^{n} a_{2j}x_j\}/x_1, \qquad \{\underline{b}_1 - \sum_{\substack{j=1\\ j\neq 2}}^{n} a_{1j}x_j\}/x_2 \le \overline{a}_{12}, \\
&\{\underline{b}_2 - \sum_{j=2}^{n} a_{2j}x_j\}/x_1 \le \{\overline{b}_1 - \sum_{\substack{j=1\\ j\neq 2}}^{n} a_{1j}x_j\}/x_2, \qquad \{\underline{b}_2 - \sum_{j=2}^{n} a_{2j}x_j\}/x_1 \le \overline{a}_{12}, \\
&\underline{a}_{12} \le \{\overline{b}_1 - \sum_{\substack{j=1\\ j\neq 2}}^{n} a_{1j}x_j\}/x_2, \qquad \underline{a}_{12} \le \{\overline{b}_2 - \sum_{j=2}^{n} a_{2j}x_j\}/x_1, \\
&\text{inequalities without } \ a_{12},\ a_{21}
\end{aligned}$$

which are equivalent to

$$\begin{aligned}
&\exists\, a_{ij} \in \mathbb{R} \quad (i,j = 1, \ldots, n,\ (i,j) \neq (1,2),\ (i,j) \neq (2,1)\,) : \\
&\{\underline{b}_1 - \sum_{\substack{j=1\\ j\neq 2}}^{n} a_{1j}x_j\}x_1 \le \{\overline{b}_2 - \sum_{j=2}^{n} a_{2j}x_j\}x_2, \qquad \underline{b}_1 - \sum_{\substack{j=1\\ j\neq 2}}^{n} a_{1j}x_j \le \overline{a}_{12}x_2, \\
&\{\underline{b}_2 - \sum_{j=2}^{n} a_{2j}x_j\}x_2 \le \{\overline{b}_1 - \sum_{\substack{j=1\\ j\neq 2}}^{n} a_{1j}x_j\}x_1, \qquad \underline{b}_2 - \sum_{j=2}^{n} a_{2j}x_j \le \overline{a}_{12}x_1, \\
&\underline{a}_{12}x_2 \le \overline{b}_1 - \sum_{\substack{j=1\\ j\neq 2}}^{n} a_{1j}x_j, \qquad \underline{a}_{12}x_1 \le \overline{b}_2 - \sum_{j=2}^{n} a_{2j}x_j, \\
&\text{inequalities without } \ a_{12},\ a_{21}.
\end{aligned}$$

By inspecting the particular cases $x_1 = 0$ and $x_2 = 0$, respectively, one can see that $x \in S_{\text{sym}}$ remains equivalent to this latter set of inequalities provided that x is restricted to O_1. We thus have described $S_{\text{sym}} \cap O_1$ by a set of inequalities which no longer contain b_1, a_{12} and a_{21}. This process of eliminating successively the parameters $b_i \in [b]_i$, $a_{ij} \in [a]_{ij}$ is the famous Fourier–Motzkin elimination process as presented, e.g., by Schrijver (1986). It finally results in a set of inequalities which contain the components of x at most quadratically, and only the bounds of $[b]_i$ and $[a]_{ij}$. It describes $S_{\text{sym}} \cap O_1$ completely and characterizes halfspaces and sets whose boundaries are quadrics. Analogous statements can be made for the remaining orthants. We refer to Alefeld et al. (1999) for a more general description of the elimination process.

3 The Solution Set $S_{\mathcal{P}}$

We want to describe now – without proof – several features of the set $S_{\mathcal{P}}$ from (5). We first list some equivalences of the Oettli–Prager Theorem whose proofs can be found in the literature listed below.

Theorem 1 *For a real $n \times n$ interval matrix $[A] = [\check{A} - \Delta, \check{A} + \Delta]$ with $O \leq \Delta \in \mathbb{R}^{n\times n}$ and for a real interval vector $[b] = [\check{b} - \delta, \check{b} + \delta]$ with n components and with $0 \leq \delta \in \mathbb{R}^n$ the following properties are equivalent:*

a) $x \in S$;

b) $[b] \cap [A]x \neq \emptyset$; *(Beeck 1972)*

c) $0 \in [b] - [A]x$; *(Beeck 1972)*

d) $|\check{b} - \check{A}x| \leq \Delta \cdot |x| + \delta$; *(Oettli and Prager 1964)*

e) $\exists\, D \in \mathbb{R}^{n\times n} : |D| \leq I \wedge \check{b} - \check{A}x = D\,(\Delta\,|x| + \delta)$; *(Rohn 1984)*

f) $\underline{b}_i - \sum_{j=1}^{n} a_{ij}^{+} x_j \leq 0 \leq \overline{b}_i - \sum_{j=1}^{n} a_{ij}^{-} x_j, \qquad i = 1, \ldots, n,$

where a_{ij}^{-}, a_{ij}^{+} are defined by $[a]_{ij} = \begin{cases} [a_{ij}^{-}, a_{ij}^{+}] & \text{if } x_j \geq 0 \\ [a_{ij}^{+}, a_{ij}^{-}] & \text{if } x_j < 0 \end{cases}$.

(Hartfiel 1980)

In b), c) real interval arithmetic has to be used as introduced, e.g., by Alefeld and Herzberger (1983). It is the representation in f) which can be derived directly by means of the Fourier–Motzkin elimination process. Therefore, in the general case, i.e., if $\mathcal{P}$ means no restriction, the Hartfiel description of S fits into our way of describing $S_{\mathcal{P}}$. If $\mathcal{P}$ means 'A is symmetric' the second and the last three inequalities of the last equivalence in Section 2 indicate that those in f) for S reappear in those for S_{sym}. This expresses the trivial property $S_{\text{sym}} \subseteq S$.

Part a) of the following theorem is a direct consequence of Theorem 1 f) while part b) summarizes the remarks on S_{sym} in the Sections 1 and 2.

Theorem 2 *Let $[A]$ be a real $n \times n$ interval matrix and let $[b]$ be a real interval vector with n components.*

a) *In each orthant of* $\mathbb{R}^n$ *the solution set* S *can be represented as intersection of finitely many halfspaces.*

b) *In each orthant* O_i *of* $\mathbb{R}^n$ *the symmetric solution set* S_{sym} *can be represented as the intersection of the solution set* $S \cap O_i$ *and sets with quadrics as boundaries.*

Theorem 2 b) holds analogously for persymmetric matrices and skew–symmetric matrices, respectively. For details see Alefeld et al. (1997). It should be noted that the inequalities for these solution sets remain fixed if the orthant is fixed. For Hankel and Toeplitz matrices Alefeld et al. (1999) showed that the inequalities may change within an orthant. Thus for Hankel matrices from

$$[A] = \begin{pmatrix} 0 & [s] & [d] \\ [s] & [d] & 0 \\ [d] & 0 & 0 \end{pmatrix} \qquad ([s],\ [d] \text{ given real intervals})$$

and for right–hand sides b from some given interval vector $[b]$ the elimination process reveals such a change in O_1 depending on $x \in C \cap O_1$ and $x \in O_1 \backslash C$, respectively, where C denotes the cone $x_1x_3 - x_2^2 \geq 0$. For details see again Alefeld et al. (1999). It is an open question what is going on in the general case of Toeplitz or Hankel matrices with perturbations. It is also unknown up to now whether there is a bound for the degree of the algebraic inequalities needed to describe the solution set S_{Toep} and S_{Hank}, respectively.

The example

$$[A] = \begin{pmatrix} [1,2] & 0 & 0 \\ [-4,-2] & [1,2] & 0 \\ [-8,-4] & [-4,-2] & [1,2] \end{pmatrix}, \quad [b] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

shows that S_{Toep} can be described by the inequalities

$$\frac{1}{2} \leq x_1 \leq 4, \quad 2x_1^2 \leq x_2 \leq 4x_1^2, \quad 4x_1^3 \leq x_1x_3 - x_2^2 \leq 8x_1^3$$

which reveals that $S_{\text{Toep}} \subseteq O_1$ with its boundary partly contained in the two algebraic surfaces

$$x_1x_3 - x_2^2 - 4x_1^3 = 0, \quad x_1x_3 - x_2^2 - 8x_1^3 = 0.$$

These surfaces are of order three which means, in particular, that Theorem 2 can no longer hold for S_{Toep} and S_{Hank}, respectively.

All particular solution sets $S_{\mathcal{P}}$ we considered up to now can be transformed into systems of linear equations $Ax = b$ with

$$\left.\begin{array}{rcl} a_{ij} & := & -a_{ij,0} + \sum_{k=1}^{p} a_{ij,k} f_k \\ b_i & := & b_{i,0} + \sum_{k=1}^{p} b_{i,k} f_k \end{array}\right\} \quad i,j = 1,\ldots,n, \qquad (10)$$

where $p \in \mathbb{N}_0$, where f_k varies in given intervals $[f]_k$ and where $a_{ij,k}$, $b_{i,k}$ are appropriate coefficients. For the solution set of any linear system subject to (10) it was shown by Alefeld et al. (1998) that it is semialgebraic, i.e., it is a finite union of subsets each of which is defined by a finite system of polynomial equations $P_r(x_1, \dots, x_n) = 0$ and inequalities of the type $P_s(x_1, \dots, x_n) > 0$ and $P_t(x_1, \dots, x_n) \geq 0$ for some polynomials P_r, P_s, P_t.

4 The Eigenpair Set E_P

In order to describe $E_{\mathcal{P}}$ from (1) we first omit any restriction $\mathcal{P}$, i.e., we consider E from (7). Since $Ax = \lambda x$ is equivalent to $(A - \lambda I)x = 0$ we can apply the Oettli–Prager Theorem on the matrix $A - \lambda I$ (assuming that λ is a fixed real number for the moment) and on the right–hand side $b = 0$. This was already done by Deif (1991) ending up with

Theorem 3 *Let $[A] = [\check{A} - \Delta, \check{A} + \Delta]$ be a real $n \times n$ interval matrix with $O \leq \Delta \in \mathbb{R}^{n \times n}$. Then*

$$(x, \lambda) \in E \iff |\check{A}x - \lambda x| \leq \Delta \cdot |x| \text{ and } x \neq 0.$$

In particular, the boundary of E consists of parts of hyperplanes and quadrics. A shortened analogue of Theorem 1 reads

Theorem 4 *For a real $n \times n$ interval matrix $[A] = [\check{A} - \Delta, \check{A} + \Delta]$ with $O \leq \Delta \in \mathbb{R}^{n \times n}$ and for $x \in \mathbb{R}^n \backslash \{0\}$, $\lambda \in \mathbb{R}$ the following properties are equivalent:*

a) $(x, \lambda) \in E$;

b) $[A]x \cap \{\lambda x\} \neq \emptyset$;

c) $|\check{A}x - \lambda x| \leq \Delta \cdot |x|$;

d) $\sum_{j=1}^{n} a_{ij}^- x_j \leq \lambda x_i \leq \sum_{j=1}^{n} a_{ij}^+ x_j, \quad i = 1, \dots, n,$

where a_{ij}^- and a_{ij}^+ are defined by $[a]_{ij} = \begin{cases} [a_{ij}^-, a_{ij}^+] & \text{if } x_j \geq 0 \\ [a_{ij}^+, a_{ij}^-] & \text{if } x_j < 0 \end{cases}$.

Here, d) can form the starting point for describing the eigenvalue set if the entries of $A \in [A]$ are subject to dependencies. Thus the following analogue of Theorem 2 can again be seen from the Fourier–Motzkin elimination process.

Theorem 5 *Let $[A]$ be a real $n \times n$ interval matrix.*

a) In each orthant of $\mathbb{R}^{n+1}$ the eigenpair set E can be represented as intersection of finitely many sets whose boundaries are hyperplanes or quadrics.

b) In each orthant of $\mathbb{R}^{n+1}$ the symmetric eigenpair set E_{sym} can be represented as intersection of finitely many sets whose boundaries are described by algebraic equations of order ≤ 3.

An analogous statement holds for skew–symmetric and persymmetric matrices. We will illustrate Theorem 5 by the following example:

Let $[A] = \begin{pmatrix} 1 & [-1,1] \\ [-1,1] & 1 \end{pmatrix}$. Then the eigenpair set E is completely described by the inequalities

$$|1-\lambda| \cdot |x_1| \leq |x_2|, \qquad |1-\lambda| \cdot |x_2| \leq |x_1| \tag{11}$$

Thus if $\lambda = 1$ then any vector $(x_1, x_2, 1)^T \neq (0,0,1)^T$ belongs to E, i.e., E contains the plain $\lambda = 1$ punctured at $(0,0,1)^T$. For $\lambda \neq 1$ we have $x_1 \cdot x_2 \neq 0$, and (11) is equivalent to

$$1 - \left|\frac{x_2}{x_1}\right| \leq \lambda \leq 1 + \left|\frac{x_2}{x_1}\right|, \qquad 1 - \left|\frac{x_1}{x_2}\right| \leq \lambda \leq 1 + \left|\frac{x_1}{x_2}\right|,$$

whence

$$1 - \min\left\{\left|\frac{x_2}{x_1}\right|, \left|\frac{x_1}{x_2}\right|\right\} \leq \lambda \leq 1 + \min\left\{\left|\frac{x_2}{x_1}\right|, \left|\frac{x_1}{x_2}\right|\right\}.$$

The symmetric eigenpair set $E_{\text{sym}} \subseteq E$ consists of the plane $\lambda = 1$ punctured at $(x_1, x_2, \lambda)^T = (0,0,1)^T$ and of the vectors $(x_1, x_2, \lambda)^T$ satisfying $\lambda \neq 1$, $0 \leq \lambda \leq 2$, $|x_1| = |x_2| > 0$. In order to get an impression of the situation we visualize the intersection $E \cap P_1$ and $E_{\text{sym}} \cap P_1$ in Fig. 1, where P_α denotes the plane $x_2 = \alpha$. We obtain

$$E \cap P_1 = \{\, (x_1, 1, 1)^T \mid x_1 \in \mathbb{R} \,\}$$
$$\cup \left\{ (x_1, 1, \lambda)^T \;\middle|\; 1 - \min\{\frac{1}{|x_1|}, |x_1|\} \leq \lambda \leq 1 + \min\{\frac{1}{|x_1|}, |x_1|\},\ x_1 \in \mathbb{R}\backslash\{0\} \right\}$$

and

$$E_{\text{sym}} \cap P_1 = \{\, (x_1, 1, 1)^T \mid x_1 \in \mathbb{R} \,\} \cup \begin{pmatrix} 1 \\ 1 \\ [0,2] \end{pmatrix} \cup \begin{pmatrix} -1 \\ 1 \\ [0,2] \end{pmatrix}.$$

As one can see at once from (11) the maximal domain $[0,2]$ for λ is attained for $|x_1| = |x_2| \neq 0$ and nowhere else. For $\alpha \neq 0$ the intersections $E \cap P_\alpha$ and $E_{\text{sym}} \cap P_\alpha$ look similar as for $\alpha = 1$. For $\alpha = 0$ they reduce to

$$E \cap P_0 = E_{\text{sym}} \cap P_0 = \{\, (x_1, 0, 1)^T \mid x_1 \in \mathbb{R}\backslash\{0\} \,\}$$

i.e., to a punctured straight line.

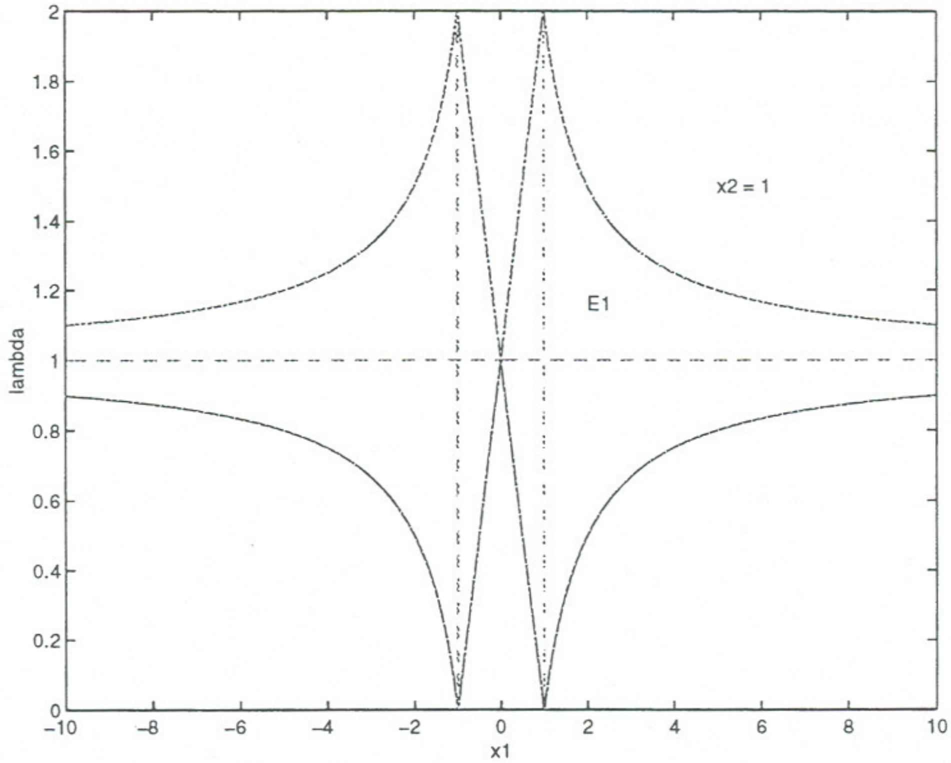

Fig. 1. $E1 := E \cap P_1$ and $E_{\mathrm{sym}} \cap P_1$ (dashed) for the example.

There is no difficulty to apply the ideas of Section 3 to the eigenvalue problem if $A \in [A]$ is subject to more general dependencies.

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