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# Regular Splittings and Monotone Iteration Functions

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**Summary.** In this paper we introduce the set of so-called monotone iteration functions (MI-functions) belonging to a given function. We prove necessary and sufficient conditions in order that a given MI-function is (in a precisely defined sense) at least as fast as a second one.

Regular splittings of a function which were initially introduced for linear functions by R.S. Varga in 1960 are generating MI-functions in a natural manner.

For linear functions every MI-function is generated by a regular splitting. For nonlinear functions, however, this is generally not the case.

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## 0. Introduction

In his famous book "Matrix Iterative Analysis" Varga [8] has introduced the concept of a regular splitting of a matrix. See also [7]. Until now regular splittings of matrices remained a fundamental tool for the investigation and comparison of iteration methods for linear equations in finite dimensional spaces.

In 1972 one of the authors [1] generalized the concept of a regular splitting to nonlinear mappings in partially ordered Banach spaces. A series of results for iteration methods generated by nonlinear regular splittings could be proved. Among these results are, for example, convergence theorems for the nonlinear total step method and single step method, respectively, if these methods are used to compute zeroes of M-functions. M-functions are important nonlinear generalizations of M-matrices. They were introduced by Rheinboldt in [5]. See also [4].

In the present paper we consider the *monotone* behaviour of iteration methods which was discussed several times in the past. See, for example, the monography by Schröder [6]. It turns out that regular splittings are playing again a fundamental role.

The paper is organized as follows:

In Sect. 1–4 we discuss some preliminaries. In Sect. 5 we introduce the concept of a monotone iteration function (MI-function). Theorems 1 and 2 contain results on the monotone behaviour of sequences produced by MI-functions. Theorem 3 contains a necessary and sufficient condition for the global convergence of the iteration method belonging to an MI-function.

In Sect. 6 we define in a natural way a relation $h \succeq g$ for two MI-functions ("The iteration method belonging to $h$ is at least as fast as the iteration method belonging to $g$"). Theorem 4 contains necessary and sufficient conditions for $h \succeq g$ to hold.

In Sect. 7 the definition of a regular splitting for a nonlinear mapping from [1] is repeated. Theorem 5 shows that given a regular splitting we can define an MI-function in a natural way.

Section 8 repeats the definition of a regular splitting of a matrix introduced by Varga [8] and demonstrates that it is covered by the nonlinear definition.

In Theorem 6 of Sect. 9 we prove necessary and sufficient conditions for $h \succeq g$ if $h$ and $g$ are MI-functions generated by regular splittings.

In Sect. 10 so-called additively composed regular splittings are introduced. In this case Theorem 7 contains again necessary and sufficient conditions for $h \succeq g$. Using results of Woźnicki [9] and of Csordas and Varga [2] we can show that a sufficient condition from [1] for $h \succeq g$ is not necessary.

In Theorem 8 of Sect. 11 we prove that in the finite dimensional case with the natural partial ordering every *linear* MI-function is generated by a regular splitting of some linear function.

The final example in Sect. 12 shows on the other hand that for *nonlinear* MI-functions this is in general not the case.

## 1. Partial Ordering

Assume that $X$ is a set in which a partial ordering is introduced via a (reflexive, antisymmetric and transitive) relation "$\geqq$". We assume that $X$ is directed downwards and upwards. By this we mean that given $a, b \in X$ there exist $\underline{c}, \bar{c} \in X$ such that

$$\underline{c} \leqq a \leqq \bar{c}, \quad \underline{c} \leqq b \leqq \bar{c}$$

hold.

Assume that a sequence $(a_k)$ in $X$ is monotone increasing, that is

$$a_0 \leqq a_1 \leqq a_2 \leqq \ldots .$$

Then we write $a_k \uparrow$. Analogously we write $a_k \downarrow$ if the sequence is monotone decreasing.

## 2. Convergence

We assume that in $X$ a convergence definition "$\rightarrow$" is given which is in the following sense compatible with the order relation "$\geqq$": If $(a_k)$ is a sequence in

$X$ then there is at most one $a \in X$ such that $a_k \to a$. Furthermore the properties (C1)–(C6) below have to hold. In formulating these properties we use the following notation: If $a_k \to a$ and $a_k \uparrow$ then we write $a_k \uparrow a$. Similarly $a_k \downarrow a$ means that $a_k \to a$ and $a_k \downarrow$.

(C1) If $a_k \leqq b$, $k=0,1,2,\ldots$, and $a_k \uparrow$ then there exists some $a \in X$ for which $a_k \uparrow a$.
(C2) If $a_k \geqq b$, $k=0,1,2,\ldots$, and $a_k \downarrow$ then there exists some $a \in X$ for which $a_k \downarrow a$.
(C3) If $a_k \uparrow a$ then $a_k \leqq a$, $k=0,1,2,\ldots$.
(C4) If $a_k \downarrow a$ then $a_k \geqq a$, $k=0,1,2,\ldots$.
(C5) If $\underline{a}_k \leqq a_k \leqq \bar{a}_k$, $k=0,1,2,\ldots$, $\underline{a}_k \uparrow a$ and $\bar{a}_k \downarrow a$ then $a_k \to a$.
(C6) If $a_k \leqq b_k$, $k=0,1,2,\ldots$, $a_k \to a$, $b_k \to b$ then $a \leqq b$.

From our assumptions the following remarks can be concluded.

*Remark 1.* A sequence $(a_k)$ for which $a_k = a$, $k=0,1,2,\ldots$, holds is convergent to $a$.

*Remark 2.* Every subsequence of a monotone convergent sequence is convergent (to the same limit as the original sequence).

## 3. Example

Suppose that $X$ is a real Banach space and that $K$ is a reproducing (i.e. $K-K=X$) and (closed) cone. $K$ is assumed to be regular in the sense of Krasnoselskii [3]. Let "$\leqq$" be defined as usual via the cone and let "$\to$" denote norm convergence. Then (C1)–(C6) hold.

## 4. Special Case

$X=\mathbb{R}^n$. $K=\mathbb{R}^n_+=\{x=(x^\nu) \mid x^\nu \geqq 0, \nu=1,2,\ldots,n\}$.

## 5. Monotone Iteration Functions (MI-Functions)

Assume that a set $X$ is given which is equipped with a partial ordering and a convergence definition as described above. Besides $X$ we consider a partially ordered set $Y$ which is directed downwards and upwards.

Let now a function

$$f\colon X \to Y$$

be given. Then a function

$$g\colon X \times Y \to X$$

is called a *monotone iteration function (MI-function) belonging to f* if $g$ has the following properties:

(M1) $g$ is (weakly) monotone increasing with respect to both arguments;
(M2) $g(x, f(x)) = x$ for $x \in X$;

(M3) $\left\{\begin{array}{l} x \leqq g(x,y) \Rightarrow f(g(x,y)) \leqq y \\ x \geqq g(x,y) \Rightarrow f(g(x,y)) \geqq y \end{array}\right\}$ for $x \in X$, $y \in Y$;

(M4) If $x_k \uparrow x$ or $x_k \downarrow x$ then for all $y \in Y$ it follows that $g(x_k, y) \to g(x, y)$.

The set of all MI-functions belonging to $f$ is denoted by $\mathrm{MI}(f)$. The following result holds for MI-functions.

**Theorem 1.** *Assume that $f: X \to Y$ is given and let $g \in \mathrm{MI}(f)$. Assume that for $x \in X$ and $y \in Y$ a sequence $(a_k)$ in $X$ is defined by the iteration method*

$$\begin{aligned} &a_0 = x, \\ &a_{k+1} = g(a_k, y), \qquad k = 0, 1, 2, \ldots. \end{aligned} \tag{5.1}$$

*Then the following hold:*

$$f(x) \leqq y \Rightarrow f(a_k) \leqq y, \qquad k = 0, 1, 2, \ldots, a_k \uparrow. \tag{5.2}$$

$$f(x) \geqq y \Rightarrow f(a_k) \geqq y, \qquad k = 0, 1, 2, \ldots, a_k \downarrow. \tag{5.3}$$

*Proof.* We prove (5.2). Assume therefore that $f(x) = f(a_0) \leqq y$. Then it suffices to conclude that (for arbitrary $k \geqq 0$)

$$a_k \leqq a_{k+1} \tag{5.4}$$

and

$$f(a_{k+1}) \leqq y \tag{5.5}$$

follow from

$$f(a_k) \leqq y. \tag{5.6}$$

By (5.6) and (M1) we have

$$a_{k+1} = g(a_k, y) \geqq g(a_k, f(a_k)) = a_k$$

where the last $=$-sign follows from (M2). Hence (5.4) holds. From the last inequality we have $a_k \leqq g(a_k, y)$ and the first part of (M3) yields

$$f(g(a_k, y)) \leqq y$$

which – using the definition of $a_{k+1}$ – states that (5.5) holds. The proof of (5.3) can be performed similarly. □

In the next Theorem we consider two elements $\underline{x} \in X$ and $\bar{x} \in X$ for which for some $y \in Y$ the left-hand side of (5.2) and (5.3), respectively, hold. In this case the corresponding sequences $(\underline{a}_k)$ and $(\bar{a}_k)$ are monotonically converging to solutions $\underline{a}$ and $\bar{a}$ of $f(x) = y$, respectively.

**Theorem 2.** *Assume that $f: X \to Y$ is given and let $g \in \mathrm{MI}(f)$. Furthermore assume that for $\underline{x}, \bar{x} \in X$ and $y \in Y$ it holds that*

$$\underline{x} \leqq \bar{x}, \quad f(\underline{x}) \leqq y \leqq f(\bar{x}). \tag{5.7}$$

*Let the sequences $(\underline{a}_k)$ and $(\bar{a}_k)$ in $X$ be defined by*

$$\underline{a}_0 = \underline{x}, \quad \underline{a}_{k+1} = g(\underline{a}_k, y) \tag{5.8}$$

*and*

$$\bar{a}_0=\bar{x}, \qquad \bar{a}_{k+1}=g(\bar{a}_k, y), \tag{5.9}$$

*respectively. Then*

$$\underline{a}_k \leqq \bar{a}_k, \qquad k=0,1,2,\ldots, \tag{5.10}$$

$$\underline{a}_k \uparrow \underline{a},\ \bar{a}_k \downarrow \bar{a}, \qquad \underline{a} \leqq \bar{a} \tag{5.11}$$

*and*

$$f(\underline{a})=f(\bar{a})=y. \tag{5.12}$$

*Proof.* By (5.7) the statement (5.10) holds for $k=0$. Assume that (5.10) is true for some $k \geqq 0$. Then by (M1) it follows that

$$\underline{a}_{k+1}=g(\underline{a}_k, y) \leqq g(\bar{a}_k, y)=\bar{a}_{k+1}.$$

Therefore (5.10) is proved by mathematical induction. From Theorem 1 we have

$$\underline{a}_k \uparrow,\ \bar{a}_k \downarrow. \tag{5.13}$$

From (5.10) and (5.13) it follows that

$$\underline{a}_k \leqq \bar{a}_l, \qquad k, l=0,1,2,\ldots.$$

Therefore using (C1) and (C2) it follows that the first two parts of (5.11) hold. The third part follows from (5.10) and (C6).

Using (M4) in (5.8) and (5.9) the equations

$$\underline{a}=g(\underline{a}, y), \qquad \bar{a}=g(\bar{a}, y)$$

follow. Applying (M3) we get (5.12). □

We add some *remarks* to the preceding result:

Assume that (5.7) hold and that instead of the sequences $(\underline{a}_k)$ and $(\bar{a}_k)$ defined by (5.8) and (5.9) we have two sequences $(\underline{x}_k)$ and $(\bar{x}_k)$ for which

$$\left.\begin{aligned} \underline{x}_0 \leqq \underline{x},\ \underline{x}_{k+1} \leqq g(\underline{x}_k, y) \\ \bar{x}_0 \geqq \bar{x},\ \bar{x}_{k+1} \geqq g(\bar{x}_k, y) \end{aligned}\right\} \ k=0,1,2,\ldots. \qquad \begin{aligned} &(5.8') \\ &(5.9') \end{aligned}$$

Such sequences could, for example, be produced by systematically rounding downwards and upwards if the iteration methods (5.8) and (5.9) are performed on a computer using a fixed length floating point number system.

We prove that

$$\underline{x}_k \leqq \underline{a}_k,\ \bar{a}_k \leqq \bar{x}_k, \qquad k=0,1,2,\ldots. \tag{5.14}$$

Because of the first parts in (5.8′) and (5.9′), respectively, (5.14) is certainly true for $k=0$. If (5.14) holds for some $k \geqq 0$ then by (5.8′), (M1) and (5.8)

$$\underline{x}_{k+1} \leqq g(\underline{x}_k, y) \leqq g(\underline{a}_k, y)=\underline{a}_{k+1}.$$

Similarly $\bar{a}_{k+1} \leqq \bar{x}_{k+1}$ follows.

By (C3) and (C4) we have from (5.11) $\underline{a}_k \leqq \underline{a}$, $\bar{a} \leqq \bar{a}_k$. Therefore we conclude that

$$\underline{x}_k \leqq \underline{a} \leqq \bar{a} \leqq \bar{x}_l, \qquad k, l = 0, 1, 2, \dots$$

and we have the

**Corollary 1.** *Assume that for the sequences $(\underline{x}_k)$ and $(\bar{x}_k)$ the inequalities (5.8′) and (5.9′) hold. Then there exists at least one solution of the equation $f(x) = y$ between $\underline{x}_k$ and $\bar{x}_l$ where $k$ and $l$ are arbitrary nonnegative integers.* □

The next Theorem delivers a global convergence result for MI-functions.

**Theorem 3.** *Assume that $f\colon X \to Y$ is bijective and let $g \in \mathrm{MI}(f)$. Then $f^{-1}\colon Y \to X$ is monotone increasing if and only if the sequence $(a_k)$ computed by the iteration method*

$$\begin{aligned} a_0 &= x \\ a_{k+1} &= g(a_k, y), \qquad k = 0, 1, 2, \dots, \end{aligned} \tag{5.16}$$

*is convergent to $f^{-1}(y)$ for arbitrary $x \in X$, $y \in Y$:*

$$a_k \to f^{-1}(y). \tag{5.17}$$

*Proof.* a) Assume that $f^{-1}$ is monotone increasing. Choose $x \in X$, $y \in Y$ arbitrarily, but fixed. Since the partial ordering in $Y$ is directed downwards and upwards there exist elements $\underline{y}, \bar{y} \in Y$ such that

$$\underline{y} \leqq f(x) \leqq \bar{y} \tag{5.18}$$

and

$$\underline{y} \leqq y \leqq \bar{y}. \tag{5.19}$$

For $\underline{x} = f^{-1}(\underline{y})$, $\bar{x} = f^{-1}(\bar{y})$ it follows from (5.18) that

$$\underline{x} \leqq x \leqq \bar{x}. \tag{5.20}$$

Besides of the sequence (5.16) we consider the two sequences $(\underline{a}_k)$ and $(\bar{a}_k)$ which are computed by

$$\underline{a}_0 = \underline{x}, \ \underline{a}_{k+1} = g(\underline{a}_k, y), \qquad k = 0, 1, 2, \dots$$

and

$$\bar{a}_0 = \bar{x}, \ \bar{a}_{k+1} = g(\bar{a}_k, y), \qquad k = 0, 1, 2, \dots .$$

Because of (5.19) it follows that $f(\underline{x}) \leqq y \leqq f(\bar{x})$ which together with (5.20) means that (5.7) of Theorem 2 holds. Therefore, by Theorem 2,

$$\underline{a}_k \uparrow \underline{a}, \qquad \bar{a}_k \downarrow \bar{a} \tag{5.11}$$

and

$$f(\underline{a}) = f(\bar{a}) = y.$$

Since $f$ is bijective we have $\underline{a} = \bar{a} = f^{-1}(y)$.

We now show that

$$\underline{a}_k \leqq a_k \leqq \bar{a}_k, \qquad k = 0, 1, 2, \dots, \tag{5.21}$$

from which because of (5.11) and (C5) the convergence $a_k \to f^{-1}(y)$ follows. We now prove (5.21). (5.21) is certainly true for $k=0$ because of (5.20). If (5.21) holds for some $k \geqq 0$ then, using (M1),

$$\underline{a}_{k+1} = g(\underline{a}_k, y) \leqq g(a_k, y) \leqq g(\bar{a}_k, y) = \bar{a}_{k+1}$$

and therefore (5.21) holds since $a_{k+1} = g(a_k, y)$.

b) Assume now that for arbitrary $x \in X$, $y \in Y$ we have that $a_k \to f^{-1}(y)$ for the sequence $(a_k)$ computed by (5.16). We have to show that $f^{-1}$ is monotone increasing. This is the case iff $b_1 \leqq b_2$ follows from $f(b_1) \leqq f(b_2)$. In order to prove that this holds, we choose

$$a_0 = b_1, \quad y = f(b_2)$$

in (5.16).

Then

$$a_k \leqq a_{k+1}, \; f(a_k) \leqq y, \quad k = 0, 1, 2, \ldots$$

hold. For $k=0$ this can be seen as follows:

The inequality $f(b_1) \leqq f(b_2)$ is equivalent to

$$f(a_0) = f(b_1) \leqq f(b_2) = y.$$

Furthermore by (M1) and (M2) it follows that

$$a_1 = g(a_0, y) = g(b_1, f(b_2)) \geqq g(b_1, f(b_1)) = b_1 = a_0.$$

For general $k$ the assertion follows from (5.2).

Since $a_k \uparrow f^{-1}(y) = b_2$ we have by (C3) that $a_k \leqq b_2$, $k = 0, 1, 2, \ldots$. Therefore we have $b_1 = a_0 \leqq b_2$. □

## 6. Comparison Results

Suppose that for a given $f: X \to Y$ we have $g, h \in \mathrm{MI}(f)$. For $x \in X$, $y \in Y$ we consider besides of the sequence $(a_k)$ computed by (5.1) the sequence $(b_k)$ computed by

$$\begin{aligned} &b_0 = x, \\ &b_{k+1} = h(b_k, y) \quad k = 0, 1, 2, \ldots. \end{aligned} \tag{6.1}$$

The analogous statements corresponding to (5.2) and (5.3) read

$$f(x) \leqq y \Rightarrow f(b_k) \leqq y, \quad k = 0, 1, 2, \ldots, \; b_k \uparrow \tag{6.2}$$

and

$$f(x) \geqq y \Rightarrow f(b_k) \geqq y, \quad k = 0, 1, 2, \ldots, \; b_k \downarrow, \tag{6.3}$$

respectively.

The statements (5.1), (5.2) and (6.2), (6.3), respectively, justify the following

*Definition.* The iteration method (6.1) is at least as fast as the iteration method (5.1) if for arbitrary $x \in X$, $y \in Y$ the statements

$$f(x) \leqq y \Rightarrow b_k \geqq a_k, \quad k=0,1,2,\ldots, \tag{6.4}$$

$$f(x) \geqq y \Rightarrow b_k \leqq a_k, \quad k=0,1,2,\ldots \tag{6.5}$$

hold.

If (6.1) is at least as fast as the iteration method (5.1) then we write $h \succeq g$.

**Theorem 4.** *Suppose that $f: X \to Y$ is given and let $g, h \in \mathrm{MI}(f)$. Then $h \succeq g$ iff for $x \in X$, $y \in Y$*

$$f(x) \leqq y \Rightarrow h(x,y) \geqq g(x,y), \tag{6.6}$$

$$f(x) \geqq y \Rightarrow h(x,y) \leqq g(x,y). \tag{6.7}$$

*Proof.* a) Assume that $h \succeq g$. If $f(x) \leqq y$ then by (6.4) $b_1 \geqq a_1$ which means that $h(x,y) \geqq g(x,y)$. Therefore (6.6) holds. Analogously (6.5) implies that (6.7) holds.

b) Assume now that (6.6) holds. We show that (6.4) is true. If

$$f(x) \leqq y \tag{6.8}$$

then we have to show that

$$b_k \geqq a_k, \quad k=0,1,2,\ldots. \tag{6.9}$$

This is obviously true for $k=0$.

By Theorem 1, (6.8) implies that

$$f(a_k) \leqq y, \quad k=0,1,2,\ldots.$$

Therefore, replacing $x$ by $a_k$, we have from (6.6) that

$$h(a_k, y) \geqq g(a_k, y), \quad k=0,1,2,\ldots. \tag{6.10}$$

Assume now that (6.9) is true for some $k \geqq 0$. Then by (M1) and (6.10),

$$b_{k+1} = h(b_k, y) \geqq h(a_k, y) \geqq g(a_k, y) = a_{k+1}.$$

Therefore (6.9) is true and the statement (6.4) holds. Analogously we can prove (6.5) by using (6.7). □

## 7. Regular Splittings

Assume that a function $f: X \to Y$ is given. Then a function $r: X \times X \to Y$ is called a *regular splitting* of $f$ if the following properties (R1)–(R5) hold:

(R1) $r(x,x) = f(x)$, $x \in X$.

(R2) $r(a,x) \leqq r(b,x) \Rightarrow a \leqq b$.

(R3) The function $r(.,x): X \to Y$ is for all $x \in X$ bijective.

(R4) $a \leqq b \Rightarrow r(x,a) \geqq r(x,b)$.

(R5) If, for the sequences $(a_k)$ and $(b_k)$ in $X$, in addition to

$$r(a_k, b_k) = y, \quad k=0,1,2,\ldots,$$

either

$$a_k \uparrow a, \quad b_k \uparrow b$$

or

$$a_k \downarrow a, \quad b_k \downarrow b$$

hold, then $r(a,b)=y$.

Regular splittings for nonlinear mappings were introduced in [1] as a generalization of regular splittings of matrices which were introduced by Varga in [7]. See Sect. 8.

**Theorem 5.** *Assume that $r: X \times X \to Y$ is a regular splitting of some given function $f: X \to Y$. If $g: X \times Y \to X$ is defined via*

$$g(x,y)=a \quad \text{if } r(a,x)=y \tag{7.1}$$

*then $g \in \mathrm{MI}(f)$.*

*Proof.* We first note that because of (R3) the function $g$ is welldefined by (7.1). We now have to show that (M1)–(M4) hold. The proof consists of several steps.

a) Using (R1) it follows from (7.1) that

$$g(x, f(x))=x$$

holds. *Hence* (M2) *is proved.*

b) We show that $g$ is increasing with respect to the second variable. To prove this, we note that (R2) can be written as

$$y_1=r(a,x) \leqq r(b,x)=y_2 \Rightarrow a \leqq b.$$

By (7.1)

$$a=g(x,y_1), \quad b=g(x,y_2).$$

Therefore we have shown that $y_1 \leqq y_2$ implies $g(x,y_1) \leqq g(x,y_2)$.

c) The mapping

$$g(x,.): Y \to X$$

is bijective. This is an easy consequence of (R3).

d) If $g(x_1,y_1)=g(x_2,y_2)$ and $x_1 \leqq x_2$ then $y_1 \geqq y_2$. This can be seen as follows:

By (7.1) the equation $x=g(x_1,y_1)=g(x_2,y_2)$ can be written as

$$r(x,x_1)=y_1, \quad r(x,x_2)=y_2.$$

If $x_1 \leqq x_2$ then by (R4)

$$y_1=r(x,x_1) \geqq r(x,x_2)=y_2.$$

e) From $x_1 \leqq x_2$, $y \in Y$ it follows that $g(x_1,y) \leqq g(x_2,y)$. The proof is as follows:

Given $y \in Y$ then c) implies that there exists a $y_1 \in Y$ such that

$$g(x_1,y_1)=g(x_2,y). \tag{7.2}$$

By d) $y_1 \geqq y$. Applying b) it follows together with (7.2) that

$$g(x_1,y) \leqq g(x_1,y_1)=g(x_2,y).$$

b) *and* e) *therefore show that* (M1) *holds for the mapping g defined by* (7.1).

f) (M3) *holds, that is*

$$\left.\begin{aligned} x \leqq g(x,y) \Rightarrow f(g(x,y)) \leqq y \\ x \geqq g(x,y) \Rightarrow f(g(x,y)) \geqq y \end{aligned}\right\} \text{ for } x \in X,\ y \in Y.$$

To see this, define $a = g(x, y)$. Then by (M2)

$$g(a, f(a)) = a = g(x, y).$$

By assumption $x \leqq a$. Hence applying d) we conclude that $f(a) \leqq y$, that is $f(g(x, y)) \leqq y$.

This is the first part of (M3). The second part can be proven similarly.

g) *We show that* (M4) *holds.*

Assume first that $x_k \uparrow x$.

We define the elements of the sequence $(a_k)$ via (7.1):

$$a_k = g(x_k, y) \quad \text{ if } r(a_k, x_k) = y.$$

Since $x_k \uparrow$ we have by (M1)

$$a_k = g(x_k, y) \leqq g(x_{k+1}, y) = a_{k+1}.$$

Hence $a_k \uparrow$. By (C3) we have $x_k \leqq x$ and therefore by (M1)

$$a_k = g(x_k, y) \leqq g(x, y).$$

Hence $a_k \uparrow a$ for some $a \in X$.

We now set $b_k = x_k$ and $b = x$. Then (R5) implies that $r(a, b) = y$, or by (7.1) that

$$a = g(b, y).$$

We have therefore proved that $x_k \uparrow x$ implies $g(x_k, y) \to g(x, y)$ which is the first part of (M4).

The second part follows similarly. □

In view of Theorem 5 it is clear that Theorems 1–3 can be reformulated for regular splittings. These specializations have essentially been proved in [1]. The linear case deserves particular interest. It will be discussed in the next Section.

## 8. Regular Splittings of Linear Mappings in $\mathbb{R}^n$

Let $X = Y = \mathbb{R}^n$ and assume that both spaces are partially ordered by $\mathbb{R}^n_+$ (see Sect. 4). Let $f$ be a linear mapping. Then

$$f(x) = Ax, \qquad x \in \mathbb{R}^n, \tag{8.1}$$

with a real $(n, n)$ matrix $A$. If $M$ and $N$ are real $(n, n)$ matrices such that

$$A = M - N, \qquad M^{-1} \geqq 0, \quad N \geqq 0, \tag{8.2}$$

where "$\geqq$" is defined via the elements, then it is easy to see that

$$r(x_1, x_2) = Mx_1 - Nx_2, \qquad x_1, x_2 \in \mathbb{R}^n, \tag{8.3}$$

is a regular splitting of the mapping (8.1). The corresponding MI-function, defined by Theorem 5, reads

$$g(x, y) = M^{-1}Nx + M^{-1}y, \qquad x, y \in \mathbb{R}^n. \tag{8.4}$$

(8.2) is the famous definition of a regular splitting of the matrix $A$ introduced by Varga in [7]. See also [8].

Under the assumptions of this section Theorem 3 implies the following

**Corollary 2.** *Let $A = M - N$ be a regular splitting. Then $A^{-1} \geqq 0$ iff the iteration method*

$$x_{k+1} = M^{-1}Nx_k + M^{-1}y, \qquad k = 0, 1, 2, \ldots$$

*is for all $y$ and for all $x_0$ convergent to the solution $x = A^{-1}y$ of the equation $Ax = y$.* □

This is essentially the well known result on regular splittings of matrices proved by Varga. See [8], Theorem 3.13.

## 9. Comparison of MI-Functions Generated by Regular Splittings

Assume that $r, s$ are two regular splittings of the function $f: X \to Y$. Let $g$ and $h$, respectively, be the corresponding MI-functions defined by (7.1). If $h \succeq g$ we also write

$$s \succeq r$$

and call the regular splitting $s$ at least as fast as the regular splitting $r$.

**Theorem 6.** *Let $r$ and $s$ be two regular splittings of $f: X \to Y$. Then $s \succeq r$ iff for $a, b, x \in X$*

$$r(a, x) = s(b, x) \geqq f(x) \Rightarrow a \leqq b, \tag{9.1}$$

$$r(a, x) = s(b, x) \leqq f(x) \Rightarrow a \geqq b. \tag{9.2}$$

*Proof.* Let $g$ and $h$ be the MI-functions belonging to $r$ and $s$, respectively. Then by (7.1) and part c) of the proof of Theorem 5 we have

$$g(x, y) = a \Leftrightarrow r(a, x) = y \tag{9.3}$$

and correspondingly

$$h(x, y) = b \Leftrightarrow s(b, x) = y. \tag{9.4}$$

Because of Theorem 4 the proof is complete if the statements

$$(6.6) \Leftrightarrow (9.1) \quad \text{and} \quad (6.7) \Leftrightarrow (9.2)$$

are shown. Using (9.3) and (9.4) this needs no additional ideas. □

In [1], Satz 2, another condition for $s \succeq r$ was proved:

Assume that $r, s: X \times X \to Y$ are two regular splittings of $f: X \to Y$. Furthermore suppose that $t: X \times X \times X \to Y$ is a function for which

$$r(x_1, x_2) = t(x_1, x_1, x_2) \tag{9.5}$$

and

$$s(x_1, x_2) = t(x_1, x_2, x_2), \qquad x_1, x_2 \in X, \tag{9.6}$$

hold. If the function

$$f(x_1, \cdot, x_2): X \to Y \tag{9.7}$$

is for all $x_1, x_2 \in X$ monotone increasing then $s \succeq r$.

In Sect. 10 we will show that already in the finite dimensional and linear case (9.5)–(9.7) *are not necessary for* $s \succeq r$. Hence these conditions are not equivalent to (9.1), (9.2).

## 10. Additively Composed Regular Splittings

**Theorem 7.** *Let $Y$ be a Banach space which is partially ordered by a regular and reproducing (closed) cone (see Sect. 3). Assume that $r$ and $s$ are regular splittings of $f: X \to Y$ which have the form*

$$r(x_1, x_2) = \Phi(x_1) + U(x_2), \qquad x_1, x_2 \in X$$

*and*

$$s(x_1, x_2) = \Psi(x_1) + V(x_2), \qquad x_1, x_2 \in X,$$

*respectively. Then $s \succeq r$ if and only if for $x \in X$, $p \in Y$*

$$\Phi^{-1}(\Phi(x) + p) \leqq \Psi^{-1}(\Psi(x) + p), \qquad p \geqq 0, \tag{10.1}$$

*and*

$$\Phi^{-1}(\Phi(x) + p) \geqq \Psi^{-1}(\Psi(x) + p), \qquad p \leqq 0. \tag{10.2}$$

*Proof.* By (R1) we have

$$f(x) = \Phi(x) + U(x) = \Psi(x) + V(x), \qquad x \in X. \tag{10.3}$$

By (R3) we have that $\Phi, \Psi: X \to Y$ are bijective. For the MI-functions $g$ and $h$ belonging to $r$ and $s$, respectively, it therefore follows by Theorem 5 that

$$\left.\begin{aligned} &g(x, y) = \Phi^{-1}(y - U(x)) \\ \text{and}\quad &h(x, y) = \Psi^{-1}(y - V(x)), \qquad x \in X,\ y \in Y. \end{aligned}\right. \tag{10.4}$$

By Theorem 4 we have $s \succeq r$ if and only if

$$f(x) \leqq y \Rightarrow \Psi^{-1}(y - V(x)) \geqq \Phi^{-1}(y - U(x))$$
$$f(x) \geqq y \Rightarrow \Psi^{-1}(y - V(x)) \leqq \Phi^{-1}(y - U(x)).$$

Replacing $y$ by $p + f(x)$ and using (10.3) we immediately get (10.1), (10.2) and vice versa. □

In passing we note that Theorem 7 can also be proved by applying Theorem 6. In this case we don't need (10.4).

Consider now the special case of Theorem 7 in which besides of $Y$ also $X$ is a real Banach space (partially ordered as described in Sect. 3) and assume that $\Phi, \Psi: X \to Y$ are both linear. In this case (10.1) and (10.2) which together are necessary and sufficient for $s \succeq r$ can be written as a single condition:

$$\Phi^{-1}(p) \leqq \Psi^{-1}(p), \quad p \in Y, \ p \geqq 0. \tag{10.5}$$

If $X = Y = \mathbb{R}^n$, $K = \mathbb{R}^n_+$, $f(x) = Ax$ where $A$ is a real $(n, n)$ matrix, if furthermore

$$r(x_1, x_2) = M_1 x_1 - N_1 x_2 = \Phi(x_1) + U(x_2)$$

and

$$s(x_1, x_2) = M_2 x_1 - N_2 x_2 = \Psi(x_1) + V(x_2)$$

are two regular splittings of $f$, then (10.5) is equivalent to

$$M_2^{-1} \geqq M_1^{-1} \tag{10.6}$$

and we therefore have the

**Corollary 3.** *Let* $X = Y = \mathbb{R}^n$, $K = \mathbb{R}^n_+$ *and* $f(x) = Ax$ *with a real* $(n, n)$ *matrix* $A$. *If*

$$r(x_1, x_2) = M_1 x_1 - N_1 x_2 \tag{10.7}$$

*and*

$$s(x_1, x_2) = M_2 x_1 - N_2 x_2 \tag{10.8}$$

*are two regular splittings of* $f$ *then* $s \succeq r$ *if and only if* (10.6) *holds.* □

It is easy to see that under the assumptions of Corollary 3 the conditions (9.5)–(9.7) are equivalent to

$$N_2 \leqq N_1. \tag{10.9}$$

Hence (10.9) is a *sufficient condition* for $s \succeq r$ where $r$ and $s$ are defined by (10.7) and (10.8), respectively.

From the work of Woźnicki [9] and Csordas and Varga [2] it is known that for two regular splittings (10.7) and (10.8) the statement

$$M_2^{-1} \geqq M_1^{-1} \Rightarrow N_2 \leqq N_1$$

is not generally true. The example from [2] is as follows. Let

$$A = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and let

$$A = M_1 - N_1 = M_2 - N_2$$

where

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$M_2 = \frac{1}{4}\begin{pmatrix} 4 & -2 \\ -2 & 5 \end{pmatrix}, \quad N_2 = \frac{1}{4}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$M_1^{-1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \text{and} \quad M_2^{-1}=\frac{1}{4}\begin{pmatrix}5 & 2\\ 2 & 4\end{pmatrix}.$$

Hence we have two regular splittings of $A$ with $M_2^{-1} \geqq M_1^{-1}$. However, $N_1$ and $N_2$ are not comparable. Because of Corollary 3 we therefore have the result that (10.9) is *not necessary* for $s \succeq r$.

## 11. Linear MI-Functions in $\mathbb{R}^n$

In this section we consider $X=Y=\mathbb{R}^n$, $K=\mathbb{R}^n_+$, again.

**Theorem 8.** *Let* $g: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *be linear* (which means that $g$ can be represented in the form $g(x, y)=Tz$ with $z=\begin{pmatrix}x\\ y\end{pmatrix} \in \mathbb{R}^{2n}$ and $T$ an $(n, 2n)$ matrix). *Then* $g$ *is an MI-function of some function* $f: \mathbb{R}^n \to \mathbb{R}^n$ *if and only if* (8.1), (8.2) *and* (8.4) *hold. In other words: Every linear MI-function* $g: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is generated by a regular splitting* (8.3).

*Proof.* a) In Sect. 8 we have shown that (8.1), (8.2) and (8.4) imply that $g \in \mathrm{MI}(f)$.

b) Assume on the other hand that $g$ is linear and $g \in \mathrm{MI}(f)$ for some $f$. The linearity of $g$ implies that

$$g(x, y)=Rx+Sy, \qquad x, y \in \mathbb{R}^n, \tag{11.1}$$

with two real $(n, n)$ matrices $R$ and $S$. Because of (M1) it follows that

$$R \geqq 0, \qquad S \geqq 0. \tag{11.2}$$

By (M2) we have

$$Rx+Sf(x)=x, \qquad x \in \mathbb{R}^n. \tag{11.3}$$

Similarly the first part of (M3) implies the statement

$$x \leqq Rx+Sy \Rightarrow f(Rx+Sy) \leqq y, \qquad x, y \in \mathbb{R}^n. \tag{11.4}$$

Consider now a solution $u$ of the equation

$$Su=0. \tag{11.5}$$

Then for every real $\lambda$ the vector $y=\lambda u$ is also a solution. Now we choose $x=0$ and $y=\lambda u$, $\lambda \in \mathbb{R}$, in (11.4). Then it follows that

$$f(0) \leqq \lambda u, \qquad \lambda \in \mathbb{R}.$$

If $u \neq 0$, then we can choose $\lambda \in \mathbb{R}$ such that this last inequality no longer holds. Hence $u=0$ and $S$ is nonsingular.

We define

$$M=S^{-1}, \qquad N=S^{-1}R, \quad A=M-N. \tag{11.6}$$

Then, by (11.3),

$$f(x)=S^{-1}(x-Rx)=Ax, \qquad x\in\mathbb{R}^n, \tag{11.7}$$

which is (8.1). Similarly (11.1) and (11.6) imply

$$g(x,y)=Rx+Sy=M^{-1}Nx+M^{-1}y$$

which is (8.4).

From (11.2) and (11.6) it follows that $M^{-1}=S\geqq 0$. Therefore in order that (8.2) holds we have finally to show that $N=S^{-1}R\geqq 0$. In order to prove this we define for given $x, y\in\mathbb{R}^n$ the vector $p\in\mathbb{R}^n$ by

$$Rx+Sy=x+p. \tag{11.8}$$

Then (11.4) reads

$$0\leqq p\Rightarrow f(x+p)\leqq y.$$

Since by (11.7)

$$f(x+p)=S^{-1}(x+p-R(x+p))$$

and by (11.8)

$$y=S^{-1}(x+p-Rx),$$

this statement can be written as

$$p\geqq 0\Rightarrow S^{-1}Rp\geqq 0.$$

From this it follows that $N=S^{-1}R\geqq 0$. □

## 12. Example: An Iteration Method for Computing the Square Root

In this section we consider the real compact intervals $X=[0,\frac{1}{2}]$, $Y=[0,\frac{1}{4}]$. $X$ and $Y$ are assumed to be ordered in the natural way. Let

$$f\colon [0,\tfrac{1}{2}]\to[0,\tfrac{1}{4}]$$

be given by

$$f(x)=x^2, \qquad x\in[0,\tfrac{1}{2}].$$

a) If $g\colon [0,\frac{1}{2}]\times[0,\frac{1}{4}]\to[0,\frac{1}{2}]$ is defined by

$$g(x,y)=x-x^2+y, \quad x\in[0,\tfrac{1}{2}], \quad y\in[0,\tfrac{1}{4}], \tag{12.1}$$

then $g\in\mathrm{MI}(f)$. We omit the details of a proof. We have

$$f^{-1}(y)=\sqrt{y}, \qquad y\in[0,\tfrac{1}{4}].$$

Hence

$$f^{-1}\colon [0,\tfrac{1}{4}]\to[0,\tfrac{1}{2}]$$

is monotone increasing. From Theorem 3 (or in this elementary example by direct discussion) it follows therefore that the iteration method

$$a_{k+1}=a_k-a_k^2+y, \qquad k=0,1,2,\ldots,\ a_0\in[0,\tfrac{1}{2}]$$

produces for each $y \in [0, \frac{1}{4}]$ a sequence $(a_k)$ for which

$$a_k \to \sqrt{y}$$

holds.

b) Consider now the function

$$g(x,.)\colon [0, \tfrac{1}{4}] \to [0, \tfrac{1}{2}], \tag{12.2}$$

where $g(x, y)$ is given by (12.1). $g(x,.)$ *is for no* $x \in [0, \frac{1}{2}]$ *bijective*. Part c) of the proof of Theorem 5 therefore shows that $g(x, y)$ from (12.1) cannot be defined by a regular splitting of $f$ via (7.1).

*This example shows that the concept of an MI-function is more general than the concept of a regular splitting*. Note, however, that for $X = Y = \mathbb{R}^n$, $K = \mathbb{R}^n_+$, we have shown in Theorem 8 that a *linear* MI-function is essentially the same as a linear regular splitting.

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