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ON THE CONVERGENCE OF HALLEY'S METHOD

G. ALEFELD
Fachbereich Mathematik, Technische Universität Berlin
Strasse des 17. Juni 135, 1 Berlin 12, West Germany

1. Introduction. A number of papers have been written about Halley's method, a third-order method for the solution of a nonlinear equation. (See, for example, [**8**].) For real-valued functions, this method is usually written as

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k) - \frac{1}{2}f''(x_k)\dfrac{f(x_k)}{f'(x_k)}}, \qquad k \geq 0. \tag{0}$$

This method is also called the method of tangent hyperbolas, as in [**3**], because x_{k+1} as given by (0) is the intercept with the x-axis of a hyperbola that is osculatory to the curve $y = f(x)$ at $x = x_k$. Construction of the appropriate hyperbola, given $f(x_k)$, $f'(x_k)$, and $f''(x_k)$, is an interesting exercise.

Many of the authors writing on Halley's method have, in particular, been concerned with developing a convergence theorem similar to the so-called Newton-Kantorovich theorem. (See, for example, [**5**, p. 421 ff].) The most far-reaching results can undoubtedly be found in [**3**], where also a comprehensive list of references is given. Error bounds, also, are given in [**3**]. These reflect for the first time the correct order of the error.

In this note we add some new results on Halley's method for real functions. From a remark by G. H. Brown, Jr. [**2**], Halley's method can be derived by applying Newton's method to the function $g(x) = f(x)/\sqrt{f'(x)}$. The adaptation of Theorem 7.1 of Ostrowski [**7**] to Halley's method gives us results on the existence and uniqueness of a zero and on the convergence to this zero. In particular, we get error bounds that reflect correctly the order of the error. This method is demonstrated by a simple well-tried numerical example.

2. Theoretical Results.

THEOREM. *Let $f(x)$ be a real-valued function of the real variable x, and let $f(x_0)f'(x_0) \neq 0$ for some x_0. Furthermore let*

$$f'(x_0) - \tfrac{1}{2}f''(x_0)\frac{f(x_0)}{f'(x_0)} \neq 0.$$

Define

$$h_0 = -\frac{f(x_0)}{f'(x_0) - \tfrac{1}{2}f''(x_0)\dfrac{f(x_0)}{f'(x_0)}}, \qquad x_1 = x_0 + h_0,$$

and set

$$J_0 = \begin{cases} [x_0, x_0 + 2h_0], & h_0 > 0 \\ [x_0 + 2h_0, x_0], & h_0 < 0. \end{cases}$$

For $x \in J_0$ let f have a continuous third derivative. Suppose that f' doesn't change sign in J_0 and that with

$$g(x) = \frac{f(x)}{\sqrt{f'(x)}}$$

we have

$$|g''(x)| \leq M_0 \tag{1}$$

and

$$2|h_0|M_0 \leq |g'(x_0)|. \tag{2}$$

Then starting with x_0 the feasibility of Halley's method is guaranteed. All x_k are contained in J_0, and the sequence $\{x_k\}$ converges to a zero x^* of f (which is unique in J_0).

Defining

$$h_k = -\frac{f(x_k)}{f'(x_k) - \tfrac{1}{2}f''(x_k)\dfrac{f(x_k)}{f'(x_k)}}, \qquad k \geq 0,$$

$$J_k = \begin{cases} [x_k, x_k + 2h_k], & h_k > 0 \\ [x_k + 2h_k, x_k], & h_k < 0 \end{cases}, \qquad k \geq 0,$$

$$|g''(x)| \leq M_k, \qquad x \in J_k, \quad k \geq 0,$$

we have the error estimates

$$|x_{k+1} - x_k| \leq \tfrac{1}{2}\frac{M_{k-1}}{|g'(x_k)|}|x_k - x_{k-1}|^2, \qquad k \geq 1, \tag{3}$$

$$|x^* - x_{k+1}| \leq \tfrac{1}{2}\frac{M_{k-1}}{|g'(x_k)|}|x_k - x_{k-1}|^2, \qquad k \geq 1, \tag{4}$$

$$|x^* - x_k| \leq \frac{M_{k-1}}{|g'(x_k)|}|x_k - x_{k-1}|^2, \qquad k \geq 1. \tag{5}$$

If

$$|f'(x)| \leqslant N, \qquad x \in J_0, \tag{6}$$

and if

$$\frac{1}{2}\frac{1}{\sqrt{f'(x)}}\left|-\frac{f'''(x)}{f'(x)}+\frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2\right| \leqslant M, \qquad x \in J_0, \tag{7}$$

then we have the coarser estimates

$$|x_{k+1}-x_k| \leqslant \frac{NM}{|g'(x_k)|}|x_k-x_{k-1}|^3, \qquad k \geqslant 1, \tag{3'}$$

$$|x^*-x_{k+1}| \leqslant \frac{NM}{|g'(x_k)|}|x_k-x_{k-1}|^3, \qquad k \geqslant 1, \tag{4'}$$

$$|x^*-x_k| \leqslant \frac{2NM}{|g'(x_k)|}|x_k-x_{k-1}|^3, \qquad k \geqslant 1. \tag{5'}$$

Proof. In some essential parts the proof is similar to that of Theorem 7.1 in [7]. Because of $f'(x_0) \neq 0$, we can assume that $f'(x_0) > 0$. We then consider the function

$$g(x) = \frac{f(x)}{\sqrt{f'(x)}}, \qquad x \in J_0.$$

For $x \in J_0$, g has a continuous second derivative and we have

$$g'(x) = \sqrt{f'(x)} - \frac{1}{2}\frac{f''(x)f(x)}{\sqrt{f'(x)^3}},$$

from which

$$g''(x) = \frac{1}{2}g(x)\left[-\frac{f'''(x)}{f'(x)}+\frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2\right]. \tag{8}$$

Furthermore

$$|g'(x)-g'(x_0)| = \left|\int_{x_0}^{x} g''(t)\,dt\right| \leqslant |x-x_0|M_0,$$

and, using (2), we have

$$|g'(x_1)-g'(x_0)| \leqslant |x_1-x_0|M_0 = |h_0|M_0 \leqslant \frac{|g'(x_0)|}{2}, \tag{9}$$

from which we may estimate

$$|g'(x_1)| \geqslant |g'(x_0)| - |g'(x_0)-g'(x_1)| \geqslant \frac{|g'(x_0)|}{2}. \tag{10}$$

Integrating by parts and using

$$h_0 = -\frac{g(x_0)}{g'(x_0)},$$

we get

$$\int_{x_0}^{x_1}(x_1-x)g''(x)\,dx = g(x_1)$$

and therefore

$$|g(x_1)| \leqslant \frac{1}{2}|h_0|^2 M_0. \tag{11}$$

Because of (10), the feasibility of Halley's method (0) is guaranteed for $k = 1$ and we have

$$x_2 = x_1 + h_1.$$

Since

$$h_1 = -\frac{g(x_1)}{g'(x_1)}$$

we have from (10) and (11)

$$|h_1| \leqslant \frac{|h_0|^2 M_0}{|g'(x_0)|}.$$

Finally in a way similar to that in [7]

$$2|h_1|M_0 \leqslant |g'(x_1)|, \tag{12}$$

and

$$|h_1| \leqslant \frac{1}{2}|h_0|.$$

From these inequalities it follows that x_2 lies in J_0 and J_1 is contained in J_0; that is, we have $M_1 \leqslant M_0$. Therefore (12) can be replaced by

$$2|h_1|M_1 \leqslant |g'(x_1)|. \tag{13}$$

For the sequence

$$x_{k+1} = x_k + h_k \tag{14}$$

as computed by Halley's method it holds in general that

$$h_k = -\frac{g(x_k)}{g'(x_k)}. \tag{15}$$

Therefore (13) shows that our assumptions remain true if we replace x_0 by x_1 and h_0 by h_1, and by x_k and h_k, respectively, in general. The convergence to a zero x^* that is unique in J_0 can now be proved as in [7]. To prove the error estimates we start with (11): We have

$$|g(x_1)| \leqslant \frac{1}{2}|h_0|^2 M_0$$

and, therefore, using (15),

$$|x_2 - x_1| = |h_1| = \left|\frac{g(x_1)}{g'(x_1)}\right| \leqslant \frac{M_0}{2|g'(x_1)|}|x_1 - x_0|^2.$$

This is (3) for $k = 1$. For $k > 1$ the assertion is proved by mathematical induction. We omit the details. Since $|x^* - x_{k+1}| \leqslant h_k$ we immediately get (4) from (3).

Finally we have

$$|x^* - x_k| \leqslant |x_{k+1} - x_k| + |x_{k+1} - x^*|,$$

and (5) is therefore proved by using (3) and (4). Applying the mean-value theorem we have, for $x \in J_{k-1}$,

$$|g''(x)| = \frac{1}{2}\left|\frac{f(x)}{\sqrt{f'(x)}}\left[-\frac{f'''(x)}{f'(x)} + \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2\right]\right| \leqslant NM|x - x^*| \leqslant 2NM|x_k - x_{k-1}|.$$

Replacing, therefore, in (3), (4), and (5) M_{k-1} by the upper bound $2NM|x_k - x_{k-1}|$, we establish (3′), (4′) and (5′). □

Without going into details of the proof we remark that the estimation (5) can further be improved. If we define

$$t_k = \frac{1}{|g'(x_k)|}|h_k|M_k, \qquad k \geq 0,$$

then

$$|x^* - x_k| \leq \frac{1}{1+\sqrt{1-2t_k}} \frac{M_{k-1}}{|g'(x_k)|}|x_k - x_{k-1}|^2, \qquad k \geq 1. \tag{5''}$$

This last can be proved in exactly the same manner as the corresponding inequality for Newton's method (see [**4**, p. 34 ff]).

Since one can easily show that

$$t_k \leq \frac{1}{2}\frac{t_{k-1}}{1-t_{k-1}}, \qquad k \geq 1,$$

(5″) can be replaced by

$$|x^* - x_k| \leq \frac{1}{1+\sqrt{1-\dfrac{t_{k-1}}{1-t_{k-1}}}} \frac{M_{k-1}}{|g'(x_k)|}|x_k - x_{k-1}|^2. \tag{5'''}$$

Since $t_k \to 0$, (5″) and (5‴) are asymptotically better than (5) by a factor of $\frac{1}{2}$.

3. Numerical Example. In order to compare our theorem with other results, we consider the simple well-tried example (see, for example, [**3**, p. 453, Tabelle 2])

$$f(x) = x^3 - 10.$$

We have

$$f'(x) = 3x^2, \quad f''(x) = 6x, \quad f'''(x) = 6.$$

As in [**3**], we choose $x_0 = 2$ and get

$$h_0 = -\frac{g(x_0)}{g'(x_0)} = 2.153\,846\,154,^{\dagger}$$

$$x_1 = x_0 + h_0 = 2.153\,846\,154, \quad \text{and} \quad J_0 = [2, 2.307\,692\,308].$$

For $x \in J_0$ we have

$$g''(x) \leq M_0 = 0.288\,675\,135.$$

Furthermore

$$|g'(x_0)| = 3.752\,776\,750;$$

hence the main inequality (2) holds.

We have

$$|g'(x_1)| = 3.731\,590\,624.$$

Using this value in (5) for $k = 1$ we have

$$|x^* - x_1| \leq \frac{M_0}{|g'(x_1)|}|x_1 - x_0|^2 = 0.001\,831\,001.$$

† All numerical values have been computed using a HP21 pocket calculator.

The actual error is

$$|x^* - x_1| \approx 0.000\,588\,556.^{\dagger}$$

The error estimation reflects the order of the actual error and is only three times as large as the actual error.

For h_1 we get

$$h_1 = 0.000\,588\,536.$$

Therefore we have

$$\begin{aligned} J_1 &= [x_1, x_1 + 2h_1] \\ &= [2.153\,846\,154, 2.155\,023\,227], \end{aligned}$$

and, for $x \in J_1$,

$$g''(x) \in [-0.000\,946\,821, 0.000\,945\,788],$$

hence

$$M_1 = 0.000\,946\,821.$$

For x_2 we get the value

$$x_2 = 2.154\,434\,690,$$

and therefore

$$|g'(x_2)| = 3.731\,530\,346.$$

Using (5) for $k = 2$, we finally have

$$|x^* - x_2| \leq \frac{M_1}{|g'(x_k)|}|x_2 - x_1|^2 \approx 8.78 \times 10^{-11}.$$

The actual error for x_2 is

$$|x^* - x_2| \approx 2.93 \times 10^{-11}.$$

Therefore the actual error is overestimated only by a factor of 3. (5‴) gives us the estimate

$$|x^* - x_2| \approx 4.39 \times 10^{-11}.$$

None of the error estimates that are to be found in Tabelle 2 in [**3**, p. 453] give a smaller bound for the error than (5‴).

4. Conclusion. If we compare the assumptions of our theorem with other results (especially with those given in [**3**]), then it seems that the most drastic assumption is the requirement that f' doesn't change sign in J_0. We need this assumption, however, in order to define $g(x)$ in the whole interval J_0.

If one wants to use the more precise error estimate (5), then one has to compute the bound M_{k-1} for the second derivative of g. There is no essential difficulty in doing this since one can get these bounds very simply by using interval arithmetic in (8). See, for example, [**1**, p. 28 ff], or [**6**, p. 161 ff], or [**9**]. The same is true for the inequality (5′) and the bounds M and N appearing there.

† The "≈" sign means here and in the sequel that the number is rounded upwards in the usual manner.

In conclusion we remark that the most important applications of Halley's method are to nonlinear equations in Banach spaces. See, for instance, the discussion and examples in [3].

The generalization of the results of this paper to this general case and some numerical examples will be discussed in another paper.

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