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Bounding the Slope of Polynomial Operators and Some Applications

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Abstract — Zusammenfassung

Bounding the Slope of Polynomial Operators and Some Applications. We discuss several systematic possibilities for removing certain intervals when using an interval expression of the derivative in iterative methods. In general, the speed of convergence is accelerated in this way. Starting from a different point of view the same problem has been treated recently in [5].

Zur Einschließung von Steigungen bei Polynomoperatoren und einige Anwendungen. In dieser Arbeit werden mehrere systematische Möglichkeiten angegeben, die es erlauben, bei der Verwendung der intervallmäßigen Auswertung der Ableitung in Iterationsverfahren gewisse Intervalle durch reelle Zahlen zu ersetzen. Dadurch wird im allgemeinen die Konvergenz beschleunigt. Die gleiche Problemstellung wurde bereits von einem anderen Ausgangspunkt ausgehend in [5] betrachtet.

1. Introduction

Let $p: \mathbb{R} \to \mathbb{R}$ be a real polynomial and let X_0 be an interval which contains a zero x^* of p. If the difference quotient of p is included in the interval Y_0 $(0 \notin Y_0)$,

$$\frac{p(x_0) - p(x^*)}{x_0 - x^*} \in Y_0, \quad (x_0 \in X_0)$$

then we have the inclusion

$$x^* \in \left\{ x_0 - \frac{p(x_0)}{Y_0} \right\} \cap X_0 =: X_1 \subseteq X_0$$

for arbitrary $x_0 \in X_0$.

See for example [1, p. 83 ff.]. The width of the interval X_1 which includes the zero x^* is only dependent on the interval Y_0 for a given x_0. An obvious method for finding a Y_0 that encloses the difference quotient is found by applying the mean value theorem: For some $\eta \in X_0$ it holds that

$$\frac{p(x_0) - p(x^*)}{x_0 - x^*} = p'(\eta) \in p'(X_0) =: Y_0.$$

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Here $p'(X_0)$ denotes the interval arithmetic evaluation of the derivative of p. To be more precise: $p'(X_0)$ denotes the interval arithmetic evaluation of one of the infinitely many possible arithmetic representations of p'. See for example [1, p. 28 ff.]. The systematic repetition gives us a method (V) which converges quadratically to x^*:

$$\left\{\begin{array}{ll} x_i \in X_i & \\ X_{i+1} = \left\{x_i - \dfrac{p(x_i)}{p'(X_i)}\right\} \cap X_i & i=0,1,2,\ldots \end{array}\right\} \tag{V}$$

The fundamental idea of using the mean value theorem in this way stems from Sunaga [11]. The method (V) was studied systematically by R. E. Moore [7].

The applicability of (V) is not limited to polynomials. One can apply this method to every function f whose derivative has an interval arithmetic evaluation.

In a recent paper Hansen [5] has described some rules which allow us to replace certain intervals by real numbers in the interval arithmetic evaluation of f'. This can have a striking effect on the width of the enclosing intervals which are computed by using (V). Unfortunately the possibilities for replacing intervals by real numbers which have been discussed in [5] are not complete and are not in a form which allows us to make systematic use of them. This is especially the case if one wishes to program these rules on a computer.

In the following we present a finite set of possibilities for enclosing the difference quotient. A partial ordering is defined in this set and it is shown that the optimal inclusion can be computed in a systematic manner. Furthermore if this optimal inclusion is used in connection with (V) then it does not require more arithmetic operations than the interval arithmetic evaluation of the derivative. We obtain these results starting with completely different considerations than in [5]. For the examples given in [5] the optimal inclusions are identical to those given in [5]. The results of our paper cannot generally be compared with those of [5] since in [5] no general rules are given. The notation and definitions from interval arithmetic are the same as those used in [1].

2. Enclosing the Difference Quotient

Let there be given a real polynomial

$$p(x) = \sum_{i=0}^{n} a_i x^i .$$

Then we have

$$\begin{aligned} p(x) - p(y) &= \sum_{i=0}^{n} a_i (x^i - y^i) \\ &= \left(\sum_{i=1}^{n} a_i \sum_{j=1}^{i} x^{i-j} y^{j-1}\right)(x-y) \\ &= \left(\sum_{i=1}^{n} \left(\sum_{j=i}^{n} a_j y^{j-i}\right) x^{i-1}\right)(x-y) \end{aligned} \tag{1}$$

and

$$\begin{aligned}
p(x)-p(y) &= \sum_{i=0}^{n} a_i (x^i - y^i) \\
&= \left(\sum_{i=1}^{n} a_i \sum_{j=1}^{i} y^{i-j} x^{j-1} \right)(x-y) \\
&= \left(\sum_{i=1}^{n} \left(\sum_{j=i}^{n} a_j x^{j-i} \right) y^{i-1} \right)(x-y). \qquad (2)
\end{aligned}$$

For a fixed $y \in X$ and arbitrary $x \in X$ we get from (1) using the inclusion monotonicity of interval arithmetic (see [1, p. 28 ff.]) that

$$\begin{aligned}
\frac{p(x)-p(y)}{x-y} &\in \left(\sum_{i=1}^{n} c_{i-1} X^{i-1} \right)_H =: J_1 \\
&\subseteq J_2 := \sum_{i=1}^{n} c_{i-1} X^{i-1}
\end{aligned}$$

where the real numbers c_{i-1} are defined by

$$c_{i-1} := \sum_{j=i}^{n} a_j y^{j-i}, \quad i = 1\,(1)\,n.$$

The subscript H means here and in the sequel that the expression is evaluated using Horner's scheme. On the other hand in evaluating J_2 we compute recursively the powers X^r by $X^0 = 1$ and $X^r = X^{r-1} X$, $r \geq 1$, multiplying these powers by the corresponding coefficients and forming the sum of these intervals. The inclusion $J_1 \subseteq J_2$ follows from the so-called subdistributive law of interval arithmetic (see [1, p. 4]).

For a real number y and intervals A_j, $j = 0\,(1)\,n-1$, the equation

$$\sum_{i=1}^{n} A_{i-1} y^{i-1} = \left(\sum_{i=1}^{n} A_{i-1} y^{i-1} \right)_H$$

is always true.

Together with the subdistributive law of interval arithmetic it follows from (2) that

$$\begin{aligned}
\frac{p(x)-p(y)}{x-y} &\in \sum_{i=1}^{n} (C_{i-1})_H \, y^{i-1} = \left(\sum_{i=1}^{n} (C_{i-1})_H \, y^{i-1} \right)_H =: J_3 \subseteq \\
&\subseteq J_4 := \sum_{i=1}^{n} C_{i-1} y^{i-1} = \left(\sum_{i=1}^{n} C_{i-1} y^{i-1} \right)_H
\end{aligned}$$

holds for fixed y and arbitrary $x \in X$ where

$$(C_{i-1})_H = \left(\sum_{j=i}^{n} a_j X^{j-i} \right)_H, \quad i = 1\,(1)\,n$$

and

$$C_{i-1} = \sum_{j=i}^{n} a_j X^{j-i}, \quad i = 1\,(1)\,n.$$

We now prove the following

Theorem: *For the expressions defined above the following relations are valid:*

a) $J_1 \subseteq J_2 \subseteq J_4$

b) $J_1 \subseteq J_3 \subseteq J_4$

c) $J_4 \subseteq p'(X) := \sum_{v=1}^{n} v\, a_v\, X^{v-1}$

Proof: In order to simplify the notation we restrict ourselves to the case of a fourth order polynomial. In the general case the Theorem can be proved in an analogous manner.

a) and c): We only have to show that $J_2 \subseteq J_4 \subseteq p'(X)$ holds.

Using the inclusion monotonicity of interval arithmetic and the validity of $a(B+C) = a\,B + a\,C$ for a real number a and intervals B and C we obtain

$$\begin{aligned}
J_2 = \sum_{i=1}^{n} c_{i-1}\, X^{i-1} = (a_1 + a_2\, y + a_3\, y^2 + a_4\, y^3)\, X^0 + {} \\
(a_2 + a_3\, y + a_4\, y^2)\, X + {} \\
(a_3 + a_4\, y)\, X^2 + {} \\
a_4\, X^3 \subseteq {} \\
\subseteq \quad a_1 + a_2\, X + a_3\, X^2 + a_4\, X^3 + {} \\
+ a_2\, y + a_3\, y\, X + a_4\, y\, X^2 + {} \\
+ a_3\, y^2 + a_4\, y^2\, X + {} \\
+ a_4\, y^3 = {} \\
= \quad a_1 + a_2\, X + a_3\, X^2 + a_4\, X^3 + {} \\
+ (a_2 + a_3\, X + a_4\, y\, X^2)\, y + {} \\
(a_3 + a_4\, X)\, y^2 + {} \\
+ a_4\, y^3 = J_4 \\
\subseteq \quad a_1 + a_2\, X + a_3\, X^2 + a_4\, X^3 + {} \\
a_2\, X + a_3\, X^2 + a_4\, X^3 + {} \\
+ a_3\, X^2 + a_4\, X^3 + {} \\
+ a_4\, X^3 =: p'(X).
\end{aligned}$$

b) We only have to show $J_1 \subseteq J_3$:

$$\begin{aligned}
J_1 &= ((c_3\, X + c_2)\, X + c_1)\, X + c_0 \\
&= ((a_4\, X + (a_3 + a_4\, y))\, X + a_2 + a_3\, y + a_4\, y^2)\, X + a_1 + a_2\, y + a_3\, y^2 + a_4\, y^3 \\
&\subseteq ((a_4\, X + a_3)\, X + a_4\, y\, X + a_2 + a_3\, y + a_4\, y^2)\, X + a_1 + a_2\, y + a_3\, y^2 + a_4\, y^3
\end{aligned}$$

$$=(((a_4 X+a_3) X+a_2)+a_4 y X+a_3 y+a_4 y^2) X+a_1+a_2 y+a_3 y^2+a_4 y^3$$
$$=(((a_4 X+a_3) X+a_2)+(a_4 X+a_3) y+a_4 y^2) X+a_1+a_2 y+a_3 y^2+a_4 y^3$$
$$\subseteq((a_4 X+a_3) X+a_2) X+(a_4 X+a_3) y X+a_4 y^2 X+a_1+a_2 y+a_3 y^2+a_4 y^3$$
$$=(((a_4 X+a_3) X+a_2) X+a_1) y^0$$
$$+((a_4 X+a_3) X+a_2) y$$
$$+(a_4 X+a_3) y^2$$
$$+a_4 y^3=J_3. \blacksquare$$

Remark 1: J_2 and J_3 are in general not comparable with respect to inclusion. Either of the cases $J_2 \subseteq J_3$ and $J_3 \subseteq J_2$ are possible. We consider the example

$$p(x)=x^3-x^2, \; X=[-1,2], \; y=1.$$

Then we have

$$J_2=(a_1+a_2 y+a_3 y^2) X^0+(a_2+a_3 y) X+a_3 X^2=X^2=[-2,4]$$

and

$$\begin{aligned} J_3&=((a_3 X+a_2) X+a_1) y^0+(a_3 X+a_2) y+a_3 y^2\\ &=(X-1) X+(X-1)+1\\ &=[-5,4], \end{aligned}$$

and therefore $J_2 \subset J_3$.

If on the other hand we choose $y=0$ then $c_{i-1}=a_i$, $i=1\,(1)\,n$.

From this it follows that

$$J_2=\sum_{i=1}^{n} a_i X^{i-1}$$

and

$$J_3=\left(\sum_{i=1}^{n} a_i X^{i-1}\right)_H,$$

and therefore $J_3 \subseteq J_2$.

To be more specific we again consider the above example

$$p(x)=x^3-x^2 \text{ with } y=0 \text{ and } X=[0,2].$$

We now obtain

$$J_2=X^2-X=[-2,4]$$

and

$$J_3=(X-1) X=[-2,2]$$

with $J_3 \subset J_2$.

Remark 2: In order to evaluate J_1 or J_2 it is at first necessary to compute the numbers

$$c_{i-1}=\sum_{j=i}^{n} a_j y^{j-i}, \; i=1\,(1)\,n.$$

If the value of p at the point y is also required, which is for example the case if we perform method (V) from section 1 then no additional arithmetic work is necessary for getting the c_{i-1}. These numbers are intermediate results if one uses Horner's scheme for the evaluation of $p(y)$:

Let

$$p(x)=\sum_{i=0}^{n} a_i x^i.$$

Then the formulae for Horner's scheme read

$$\begin{aligned} p_n &:= a_n \\ p_{i-1} &:= p_i\, y + a_{i-1}, \quad i=n(-1)\,1 \end{aligned}$$

and we have $p_0 = p(y)$.

On the other hand we have by definition

$$\begin{aligned} c_{n-1} &= a_n \\ c_{n-2} &= a_n\, y + a_{n-1} \\ &\vdots \\ c_0 &= c_1\, y + a_1. \end{aligned}$$

Therefore we have $c_{i-1}=p_i$, $i=1\,(1)\,n$.

3. Example

$$p(x)=x^7+3x^6-4x^5-12x^4-x^3-3x^2+4x+12$$

$$X=[1.8, 3], \quad y=2.$$

We obtain

$$\begin{aligned} J_1 &= [173.2362, 2400] \\ J_2 &= [161.4762, 2411.76] \\ J_3 &= [24.72, 2400] \\ J_4 &= [-870.2933, 3443.5296] \\ (p'(X))_H &= [71.799808, 6520] \\ p'(X) &= [-2378.791292, 8970.592]. \end{aligned}$$

The polynomial p has a zero in $X_0=[1.8, 2.4]$ (see J. Herzberger: Bemerkungen zu einem Verfahren von R. E. Moore, ZAMM 53, 356—358, 1973). We compute this zero by using method (V) from section 1 choosing $x_i=m(X_i)$ (= the midpoint of the interval X_i) and replacing $p'(X_i)$ by $p'(X_i)_H$. The following table 1 contains the iterates.

Table 1

i \ X_i	
0	[1.8, 2.4]
1	[1.8, 2.072 761 807 7482]
2	[1.974 290 005 2812, 2.072 761 807 7842]
3	[1.994 875 714 7483, 2.005 921 548 2353]
4	[1.999 988 823 4200, 2.000 011 539 0070]
5	[1.999 999 999 9894, 2.000 000 000 0107]
6	[2.0, 2.0]

Method (V′) is obtained from (V) by replacing $p'(X_i)_H$ by J_1 in each iteration step (Table 2).

Table 2

i \ X_i	
0	[1.8, 2.4]
1	[1.941 953 810 8826, 2.056 696 405 0488]
2	[1.999 999 997 5872, 2.000 111 299 3369]
3	[1.999 999 997 5872, 2.000 000 002 9595]
4	[2.0, 2.0]

Table 3 contains the quotients $d_1^{(i)}/d_2^{(i)}$ where $d_1^{(i)}$ and $d_2^{(i)}$ denote the width of the iterates computed by (V) and (V′).

Table 3

i	0	1	2	3
$d_1^{(i)}/d_2^{(i)}$	1	2.37	492.35	$3.7_{10}6$

The example has been computed using the CDC 6500 computer (mantissa length 48 bit) of the Zentraleinrichtung Rechenzentrum der Technischen Universität Berlin.

In [12] we have demonstrated that the results of all examples from [5] are identical to the intervals which one gets using J_1.

4. Generalization to $\mathbf{V}_m(\mathbb{R})$

We now consider more generally a mapping

$$\mathfrak{f}: V_m(\mathbb{R}) \to V_m(\mathbb{R})$$

where $\underset{\cdot}{\mathfrak{f}}$ is a so-called polynomial operator defined by

$$\underset{\cdot}{\mathfrak{f}}(\underset{\cdot}{\mathfrak{x}})=\underset{\cdot}{\mathfrak{a}}_0+\underset{\cdot}{\mathfrak{A}}_1\underset{\cdot}{\mathfrak{x}}+\underset{\cdot}{\mathfrak{A}}_2\underset{\cdot}{\mathfrak{x}}^2+\ldots+\underset{\cdot}{\mathfrak{A}}_n\underset{\cdot}{\mathfrak{x}}^n$$

(see for example [8, sections 17 and 18]).

Here the $\underset{\cdot}{\mathfrak{A}}_i$, $i=1\,(1)\,n$, denote i-linear operators from $V_m(\mathbb{R})$ to $V_m(\mathbb{R})$ and $\underset{\cdot}{\mathfrak{a}}_0\in V_m(\mathbb{R})$ represents a real vector. Without loss of generality we can assume that the $\underset{\cdot}{\mathfrak{A}}_i$, $i=1\,(1)\,n$, are symmetric. See for example [8, p. 100 ff.]. Let $\underset{\cdot}{\mathfrak{f}}$ have a zero $\underset{\cdot}{\mathfrak{x}}^*\in\mathfrak{x}$ and let the so-called difference operator $\delta\underset{\cdot}{\mathfrak{f}}(\underset{\cdot}{\mathfrak{x}},\underset{\cdot}{\mathfrak{y}})$ — defined by

$$\underset{\cdot}{\mathfrak{f}}(\underset{\cdot}{\mathfrak{x}})-\underset{\cdot}{\mathfrak{f}}(\underset{\cdot}{\mathfrak{y}})=\delta\underset{\cdot}{\mathfrak{f}}(\underset{\cdot}{\mathfrak{x}},\underset{\cdot}{\mathfrak{y}})(\underset{\cdot}{\mathfrak{x}}-\underset{\cdot}{\mathfrak{y}})$$

(see for example [10]) — be included by an interval-matrix $\mathfrak{Y}$,

$$\delta\underset{\cdot}{\mathfrak{f}}(\underset{\cdot}{\mathfrak{x}},\underset{\cdot}{\mathfrak{y}})\in\mathfrak{Y},\quad \underset{\cdot}{\mathfrak{x}},\underset{\cdot}{\mathfrak{y}}\in\mathfrak{x}.$$

If all $\underset{\cdot}{\mathfrak{Y}}\in\mathfrak{Y}$ are nonsingular then for arbitrary $\underset{\cdot}{\mathfrak{x}}\in\mathfrak{x}$ it holds that

$$\underset{\cdot}{\mathfrak{x}}^*\in\{\underset{\cdot}{\mathfrak{x}}-\underset{\cdot}{\mathfrak{Y}}^{-1}\underset{\cdot}{\mathfrak{f}}(\underset{\cdot}{\mathfrak{x}})\mid\underset{\cdot}{\mathfrak{Y}}\in\mathfrak{Y}\}.$$

A proof of this fact can be found in [1, p. 270 ff.]. If the feasibility of Gaussian elimination $GA(\mathfrak{Y},\underset{\cdot}{\mathfrak{f}}(\underset{\cdot}{\mathfrak{x}}))$ with $\mathfrak{Y}$ as coefficient matrix and $\underset{\cdot}{\mathfrak{f}}(\underset{\cdot}{\mathfrak{x}})$ as a right hand side is guaranteed then

$$\underset{\cdot}{\mathfrak{x}}^*\in\{\underset{\cdot}{\mathfrak{x}}-GA(\mathfrak{Y},\underset{\cdot}{\mathfrak{f}}(\underset{\cdot}{\mathfrak{x}}))\}\cap\mathfrak{x}$$

(see for example [1, p. 218 ff. and 2]. The systematic repetition of this step gives a generalization of method (V) considered in section 1 to n-dimensional systems of equations:

$$\left.\begin{cases}\underset{\cdot}{\mathfrak{x}}^{(i)}\in\mathfrak{x}^{(i)}\\ \mathfrak{x}^{(i+1)}=\{\underset{\cdot}{\mathfrak{x}}^{(i)}-GA(\mathfrak{Y}^{(i)},\ \underset{\cdot}{\mathfrak{f}}(\underset{\cdot}{\mathfrak{x}}^{(i)}))\}\cap\mathfrak{x}^{(i)}\end{cases}\right\},\quad i=0,1,2,\ldots.$$

As in the case $m=1$ we can use the mean value theorem (for mappings $V_m(\mathbb{R})\to(\mathbb{R})$ to enclose $\delta\underset{\cdot}{\mathfrak{f}}(\underset{\cdot}{\mathfrak{x}},\underset{\cdot}{\mathfrak{y}})$. This gives for fixed $\underset{\cdot}{\mathfrak{y}}$ and arbitrary $\underset{\cdot}{\mathfrak{x}}\in\mathfrak{x}$

$$\delta\underset{\cdot}{\mathfrak{f}}(\underset{\cdot}{\mathfrak{x}},\underset{\cdot}{\mathfrak{y}})\in\mathfrak{Y}:=\underset{\cdot}{\mathfrak{A}}_1+2\,\underset{\cdot}{\mathfrak{A}}_2\,\mathfrak{x}+\ldots+n\,\underset{\cdot}{\mathfrak{A}}_n^n\,\mathfrak{x}^{n-1}=\underset{\cdot}{\mathfrak{f}}'(\mathfrak{x}).$$

Here $\underset{\cdot}{\mathfrak{f}}'(\mathfrak{x})$ denotes the interval arithmetic evaluation of the Frechet-derivative of $\underset{\cdot}{\mathfrak{f}}$.

In the same way as in the case $m=1$ the above inclusion of $\delta\underset{\cdot}{\mathfrak{f}}$ can be improved by several in general different interval matrices. In order to demonstrate this we first define the following interval matrices:

$$\mathfrak{I}_1=\left(\sum_{i=1}^{n}\underset{\cdot}{\mathfrak{C}}_{i-1}\,\mathfrak{x}^{i-1}\right)_H$$

and

$$\mathfrak{I}_2=\sum_{i=1}^{n}\underset{\cdot}{\mathfrak{C}}_{i-1}\,\mathfrak{x}^{i-1}$$

where

$$\underset{\cdot}{\mathfrak{C}}_{i-1}=\sum_{j=i}^{n}\underset{\cdot}{\mathfrak{A}}_j\,\underset{\cdot}{\mathfrak{y}}^{j-i},\quad i=1\,(1)\,n,$$

$$\mathfrak{I}_3=\left(\sum_{i=1}^{n}(\mathfrak{C}_{i-1})_H\,\underset{\cdot}{\mathfrak{y}}^{i-1}\right)_H\quad\text{where}\quad(\mathfrak{C}_{i-1})_H=\left(\sum_{j=i}^{n}\underset{\cdot}{\mathfrak{A}}_j\,\mathfrak{x}^{j-i}\right)_H$$

and

$$\mathfrak{J}_4=\sum_{i=1}^{n} \mathfrak{C}_{i-1}\, \underset{\cdot}{\mathfrak{y}}^{i-1} \quad \text{where} \quad \mathfrak{C}_{i-1}=\sum_{j=i}^{n} \underset{\cdot}{\mathfrak{A}}_j\, \mathfrak{x}^{j-i}.$$

The corresponding theorem to the theorem from section 2 can now be proved in a completely analogous way by using the symmetry of the operators $\underset{\cdot}{\mathfrak{A}}_i$, $i=2(1)n$. We omit the details.

5. Examples in $V_m(\mathbb{R})$

1. Let $m=2$, $\underset{\cdot}{\mathfrak{x}}=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and

$$\underset{\cdot}{\mathfrak{f}}(\underset{\cdot}{\mathfrak{x}})=\begin{pmatrix} x_1^2+x_2^2-1 \\ x_1^2-x_2 \end{pmatrix}.$$

This example has been discussed in [5]. Using

$$\underset{\cdot}{\mathfrak{a}}_0=\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \underset{\cdot}{\mathfrak{A}}_1=\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad \underset{\cdot}{\mathfrak{A}}_2=\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right)$$

$\underset{\cdot}{\mathfrak{f}}$ can be written as

$$\underset{\cdot}{\mathfrak{f}}(\underset{\cdot}{\mathfrak{x}})=\underset{\cdot}{\mathfrak{a}}_0+\underset{\cdot}{\mathfrak{A}}_1\, \underset{\cdot}{\mathfrak{x}}+\underset{\cdot}{\mathfrak{A}}_2\, \underset{\cdot}{\mathfrak{x}}^2.$$

From this it follows that

$$\mathfrak{J}_1=\underset{\cdot}{\mathfrak{C}}_0+\underset{\cdot}{\mathfrak{C}}_1\, \underset{\cdot}{\mathfrak{x}}$$

where

$$\underset{\cdot}{\mathfrak{C}}_0=\underset{\cdot}{\mathfrak{A}}_1+\underset{\cdot}{\mathfrak{A}}_2,$$

$$\underset{\cdot}{\mathfrak{C}}_1=\underset{\cdot}{\mathfrak{A}}_2.$$

A short computation gives

$$\mathfrak{J}_1=\begin{pmatrix} y_1+X_1 & x_2+X_2 \\ y_1+X_1 & -1 \end{pmatrix}$$

where

$$\underset{\cdot}{\mathfrak{y}}=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{and} \quad \underset{\cdot}{\mathfrak{x}}=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\in \mathfrak{x}=\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

(Compare this with the method used to arrive at the same result in [5].)

On the other hand the interval arithmetic evaluation of $\underset{\cdot}{\mathfrak{f}}'$ gives us the interval matrix

$$\underset{\cdot}{\mathfrak{f}}'(\mathfrak{x})=\begin{pmatrix} 2\,X_1 & 2\,X_2 \\ 2\,X_1 & -1 \end{pmatrix}.$$

We have $\mathfrak{J}_1 \subseteq \underset{\cdot}{\mathfrak{f}}'(\mathfrak{x})$ since in the representation of $\underset{\cdot}{\mathfrak{f}}'(\mathfrak{x})$ three real numbers have been replaced by intervals.

2. Let $\underset{\cdot}{\mathfrak{A}}_0, \underset{\cdot}{\mathfrak{A}}_1, \ldots, \underset{\cdot}{\mathfrak{A}}_n$ be real m by m matrices and define

$$\underset{\cdot}{\mathfrak{P}}(\underset{\cdot}{\mathfrak{X}}) = \underset{\cdot}{\mathfrak{A}}_n \underset{\cdot}{\mathfrak{X}}^n + \underset{\cdot}{\mathfrak{A}}_{n-1} \underset{\cdot}{\mathfrak{X}}^{n-1} + \ldots + \underset{\cdot}{\mathfrak{A}}_1 \underset{\cdot}{\mathfrak{X}} + \underset{\cdot}{\mathfrak{A}}_0 .$$

The problem of computing a zero of $\underset{\cdot}{\mathfrak{P}}$ is of some interest in connection with the nonlinear eigenvalue problem

$$\det(\underset{\cdot}{\mathfrak{A}}_n \lambda^n + \underset{\cdot}{\mathfrak{A}}_{n-1} \lambda^{n-1} + \ldots + \underset{\cdot}{\mathfrak{A}}_1 \lambda + \underset{\cdot}{\mathfrak{A}}_0) = 0 .$$

See for example [3, 4, 6, 7].

Let $\mathfrak{X}_0$ be an interval matrix which contains a zero $\underset{\cdot}{\mathfrak{X}}^*$ of $\underset{\cdot}{\mathfrak{P}}$. We define as difference operator of $\underset{\cdot}{\mathfrak{P}}$ an m by m matrix $\delta \underset{\cdot}{\mathfrak{P}}$ for which

$$\underset{\cdot}{\mathfrak{P}}(\underset{\cdot}{\mathfrak{X}}) - \underset{\cdot}{\mathfrak{P}}(\underset{\cdot}{\mathfrak{Y}}) = \delta \underset{\cdot}{\mathfrak{P}}(\underset{\cdot}{\mathfrak{X}}, \underset{\cdot}{\mathfrak{Y}})(\underset{\cdot}{\mathfrak{X}} - \underset{\cdot}{\mathfrak{Y}}), \quad \underset{\cdot}{\mathfrak{X}}, \underset{\cdot}{\mathfrak{Y}} \in \mathfrak{X}_0$$

holds. Let $\mathfrak{Z}$ be an m by m interval matrix with the property

$$\delta \underset{\cdot}{\mathfrak{P}}(\underset{\cdot}{\mathfrak{X}}, \underset{\cdot}{\mathfrak{Y}}) \in \mathfrak{Z}, \quad \underset{\cdot}{\mathfrak{X}}, \underset{\cdot}{\mathfrak{Y}} \in \mathfrak{X}_0 .$$

If no $\underset{\cdot}{\mathfrak{Z}} \in \mathfrak{Z}$ is singular then

$$\underset{\cdot}{\mathfrak{X}}^* \in \{\underset{\cdot}{\mathfrak{X}} - \underset{\cdot}{\mathfrak{Z}}^{-1} \underset{\cdot}{\mathfrak{P}}(\underset{\cdot}{\mathfrak{X}}) \mid \underset{\cdot}{\mathfrak{Z}} \in \mathfrak{Z}\},$$

for arbitrary $\underset{\cdot}{\mathfrak{X}} \in \mathfrak{X}_0$. The systematic repetition gives us again the iteration method

$$\left\{ \begin{array}{l} \underset{\cdot}{\mathfrak{X}}^{(i)} \in \mathfrak{X}^{(i)} \\ \mathfrak{X}^{(i+1)} = \{\underset{\cdot}{\mathfrak{X}}^{(i)} - GA\,(\mathfrak{Z}^{(i)}, \underset{\cdot}{\mathfrak{P}}(\underset{\cdot}{\mathfrak{X}}^{(i)}))\} \cap \mathfrak{X}^{(i)}, \quad i = 0, 1, 2, \ldots . \end{array} \right.$$

The inclusion of $\delta \underset{\cdot}{\mathfrak{P}}$ can again be computed by evaluation of the Frechet-derivative. We demonstrate this for the case $n = 2$:

$$\underset{\cdot}{\mathfrak{P}}(\underset{\cdot}{\mathfrak{X}}) = \underset{\cdot}{\mathfrak{A}}_2 \underset{\cdot}{\mathfrak{X}}^2 + \underset{\cdot}{\mathfrak{A}}_1 \underset{\cdot}{\mathfrak{X}} + \underset{\cdot}{\mathfrak{A}}_0$$

$$\underset{\cdot}{\mathfrak{P}}'(\underset{\cdot}{\mathfrak{X}}) \underset{\cdot}{\mathfrak{U}} = \underset{\cdot}{\mathfrak{A}}_2 (\underset{\cdot}{\mathfrak{U}} \underset{\cdot}{\mathfrak{X}} + \underset{\cdot}{\mathfrak{X}} \underset{\cdot}{\mathfrak{U}}) + \underset{\cdot}{\mathfrak{A}}_1 \underset{\cdot}{\mathfrak{U}}$$

$$\underset{\cdot}{\mathfrak{P}}'(\mathfrak{X}) \underset{\cdot}{\mathfrak{U}} \in \underset{\cdot}{\mathfrak{A}}_2 (\underset{\cdot}{\mathfrak{U}} \mathfrak{X} + \mathfrak{X} \underset{\cdot}{\mathfrak{U}}) + \underset{\cdot}{\mathfrak{A}}_1 \underset{\cdot}{\mathfrak{U}}$$

On the other hand we have

$$\underset{\cdot}{\mathfrak{P}}(\underset{\cdot}{\mathfrak{X}}) - \underset{\cdot}{\mathfrak{P}}(\underset{\cdot}{\mathfrak{Y}}) = \underset{\cdot}{\mathfrak{A}}_2 \{\underset{\cdot}{\mathfrak{X}}(\underset{\cdot}{\mathfrak{X}} - \underset{\cdot}{\mathfrak{Y}}) + (\underset{\cdot}{\mathfrak{X}} - \underset{\cdot}{\mathfrak{Y}}) \underset{\cdot}{\mathfrak{Y}}\} + \underset{\cdot}{\mathfrak{A}}_1 (\underset{\cdot}{\mathfrak{X}} - \underset{\cdot}{\mathfrak{Y}})$$

and therefore

$$\delta \underset{\cdot}{\mathfrak{P}}(\underset{\cdot}{\mathfrak{X}}, \underset{\cdot}{\mathfrak{Y}}) \underset{\cdot}{\mathfrak{U}} = \underset{\cdot}{\mathfrak{A}}_2 \{\underset{\cdot}{\mathfrak{X}} \underset{\cdot}{\mathfrak{U}} + \underset{\cdot}{\mathfrak{U}} \underset{\cdot}{\mathfrak{Y}}\} + \underset{\cdot}{\mathfrak{A}}_1 \underset{\cdot}{\mathfrak{U}}$$

and

$$\delta \underset{\cdot}{\mathfrak{P}}(\underset{\cdot}{\mathfrak{X}}, \underset{\cdot}{\mathfrak{Y}}) \underset{\cdot}{\mathfrak{U}} \in \underset{\cdot}{\mathfrak{A}}_2 \{\mathfrak{X} \underset{\cdot}{\mathfrak{U}} + \underset{\cdot}{\mathfrak{U}} \underset{\cdot}{\mathfrak{Y}}\} + \underset{\cdot}{\mathfrak{A}}_1 \underset{\cdot}{\mathfrak{U}} =: \mathfrak{I}_1 \underset{\cdot}{\mathfrak{U}} .$$

Comparing $\mathfrak{I}_1 \underset{\cdot}{\mathfrak{U}}$ and $\underset{\cdot}{\mathfrak{P}}'(\mathfrak{X}) \underset{\cdot}{\mathfrak{U}}$ we see that $\mathfrak{I}_1 \underset{\cdot}{\mathfrak{U}} \subseteq \underset{\cdot}{\mathfrak{P}}'(\mathfrak{X}) \underset{\cdot}{\mathfrak{U}}$ holds since $\underset{\cdot}{\mathfrak{Y}} \in \mathfrak{X}$.

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